Purity and state fidelity of quantum channels

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(Received 9 March 2004; published 29 July 2004)

We associate with every quantum channel T acting on a Hilbert space $\mathcal{H}$ a pair of Hermitian operators, referred to as "Hamiltonians," over the symmetric subspace of $\mathcal{H}^{\otimes 2}$. The expectation values of these Hamiltonians over symmetric product states give either the purity or the pure-state fidelity of T. This allows us to analytically compute these measures for a wide class of channels, and to identify states that are optimal with respect to these measures.

DOI: 10.1103/PhysRevA.70.012315 PACS number(s): 03.67.Hk

I. INTRODUCTION

The study of open quantum systems [1] is of interest in fields as diverse as quantum information science [2], quantum control [3], and foundations of quantum physics [4]. Such systems can be described, very generally, using the following formalism. Let $T \in \mathrm{CP}(\mathcal{H})$ be a completely positive (CP) trace-preserving quantum map, i.e., a *channel* over the finite-dimensional quantum state space $\mathcal{H}$. The channel T has a (nonunique) Kraus operator sum representation [5]

$$T(X) = \sum_i A_i X A_i^{\dagger}, \quad [X \in \mathrm{End}(H)], \tag{1}$$

where the Kraus operators A_i satisfy the constraint $\sum_i A_i^{\dagger} A_i = \mathbb{1}$, which guarantees preservation of the trace of a state (density operator) $X=\rho$. A fundamental property of a state is its purity $p[\rho]=\mathrm{Tr}(\rho^2)$. States are called pure if and only if $p=1$ and mixed if $p<1$. In the paradigmatic scenario of open quantum systems, a state starts out as pure, $\rho=|\psi\rangle\langle\psi|$, and is then mapped, e.g., via the interaction with an environment, to a mixed state by the action of a channel T: $p[T(\rho)]=\mathrm{Tr}[T(|\psi\rangle\langle\psi|)^2]<1$. In this case we say that the state $|\psi\rangle$ has been *decohered* by the channel. A typical goal of, e.g., quantum information processing, is to maximize the purity of a state that is transmitted via some channel T. To this end, a variety of decoherence-reduction techniques have been developed, such as quantum error correcting codes (QECCs) [6–9] and decoherence-free subspaces (DFSs) [10,11]. In this work we are interested in the intrinsic purity of quantum channels: In the following, unless otherwise specified, all state vectors $|\psi\rangle$'s (and $|\Psi\rangle$'s) will be normalized.

Definition 1. The *purity* of the channel T over the subspace $\mathcal{C} \subset \mathcal{H}$ is

$$P(T,\mathcal{C}) \coloneqq \min_{|\psi\rangle \in \mathcal{C}} \mathrm{Tr}[T(|\psi\rangle\langle\psi|)^2]. \tag{2}$$

Minimization is required since we must consider the worst-case scenario. We invoke subspaces in our definition since we know from the theory of QECCs and DFSs that it is possible to encode quantum information in a manner that maximizes purity by restricting to a subspace. In particular:

Definition 2. If $P(T,\mathcal{C})=1$ we say that that $\mathcal{C}$ is a decoherence-free subspace with respect to T, in short, a T-DFS.

In many cases it will not be possible to find a T-DFS. A central question we shall be concerned with here is the characterization of those states that optimally approximate a T-DFS, i.e., those states for which $P(T,\mathcal{C})$ is as large as possible. Thus:

Definition 3. The optimal purity of T is

$$P(T) \coloneqq \max_{\mathcal{C} \subset \mathcal{H}} P(T,\mathcal{C}). \tag{3}$$

Note that $P(T)=1 \Leftrightarrow$ the set of T-DFSs is nonempty. However, this situation is rather rare and generally requires that there be a *symmetry* in the system-environment interaction. Associated with this symmetry is a conserved quantity: quantum coherence. This in turn leads to the preservation of quantum information. Here we wish to depart from the notion of a strict symmetry and explicitly consider the situation where one can only expect optimal, as opposed to ideal, purity. However, the optimization problem defined by $P(T)$ is a hard one, since it involves a search over all possible subspaces $\mathcal{C} \subset \mathcal{H}$; the number of such subspaces grows quite rapidly in the dimension of $\mathcal{H}$, which itself may be exponential in the number of particles, in a typical quantum information processing application. Moreover, even if one restricts the problem to the computation of $P(T,\mathcal{C})$ (for a given, fixed subspace), one is still faced with a complicated-looking functional.

In this work we focus on the the computation of $P(T,\mathcal{C})$ and we associate a Hamiltonian with each channel. This "channel Hamiltonian" is a mathematical trick, rather than a physical Hamiltonian. But, as we shall show, this has the advantage in that it allows us to cast the purity problem into the familiar framework of computing eigenvalues of Hermitian operators. In addition, we show that our channel Hamiltonian leads to an elegant physical (re-) interpretation of the channel purity in terms of the expectation value of the SWAP operator.

Our work is also related to questions about channel capacity; indeed recently it has been shown that multiplicativity of generalized maximal purities implies additivity of the minimal output entropy of the quantum channel. The latter, in turn, is equivalent to the additivity of the Holevo channel capacity [12].

We introduce the first channel Hamiltonian in Sec. II. We then derive a number of properties and bounds on the purity based on this formalism in Sec. III. We then devote Sec. IV to a number of examples designed to illustrate our formalism, and derive some interesting properties for a class of channels. In Sec. V we derive an alternative interpretation of the expression for the channel purity, in terms of a dual map. It turns out that the same methods we introduce for the channel purity also apply to the pure-state fidelity of the channel. In particular, we can introduce a second-channel Hamiltonian to this end. This is addressed in Sec. VI. We conclude in Sec. VII.

II. A HAMILTONIAN OPERATOR FOR QUANTUM CHANNELS

Associated with the channel T we define an operator over $\mathcal{H}^{\otimes 2}$:

Definition 4. The channel purity Hamiltonian is

$$\Omega(T) := \sum_{ij} \Omega_{ij}^{\dagger} \otimes \Omega_{ij}, \quad \Omega_{ij} = A_i^{\dagger} A_j. \tag{4}$$

[We shall refer to $\Omega(T)$ simply as the "channel Hamiltonian" until our discussion of the pure-state fidelity in Sec. VI.] It follows immediately from $\Omega_{ij}^{\dagger}=\Omega_{ji}$ that $\Omega(T)$ is a symmetric, Hermitian operator. Thus, $\Omega(T)$ has the status of a Hamiltonian over $\mathcal{H}^{\otimes 2}$. Moreover, $\Omega(T)$ is independent of the particular Kraus operator-sum representation chosen for T: all possible operator-sum representations of T are obtained by considering new Kraus operators of the form $A_i' = \Sigma_j U_{ij} A_j$, where the U_{ij}'s are the entries of unitary matrix. By inserting this expression into the definition (4) one can explicitly check that $\Omega(T)$ is invariant.

We now come to our key result: a representation of the purity of quantum channels as the expectation value of the channel Hamiltonian. Let $|\psi^{\otimes 2}\rangle \equiv |\psi^{\otimes}\rangle^2$ (we will use both notations interchangeably).

Proposition 0. For every quantum channel T and subspace $\mathcal{C}$, one has the identity

$$P(T,\mathcal{C}) = \min_{|\psi\rangle\in\mathcal{C}} \langle \psi^{\otimes 2}|\Omega(T)|\psi^{\otimes 2}\rangle. \tag{5}$$

Proof. One has

$$\begin{aligned}\mathrm{Tr}\, T^2(|\psi\rangle\langle\psi|) &= \sum_{ij} \langle\psi|A_j^{\dagger}A_i|\psi\rangle\langle\psi|A_i^{\dagger}A_j|\psi\rangle = \sum_{ij} |\langle\psi|A_j^{\dagger}A_i|\psi\rangle|^2 \\ &= \sum_{ij} \mathrm{Tr}[|\psi\rangle\langle\psi|^{\otimes 2} A_j^{\dagger}A_i \otimes A_i^{\dagger}A_j] \\ &= \mathrm{Tr}[|\psi\rangle\langle\psi|^{\otimes 2}\Omega(T)]. \end{aligned} \tag{6}$$

Equation (5) now follows by taking the minimum over $|\psi\rangle\in\mathcal{C}$. ■

Note that $\Omega(T)$ is a formal Hamiltonian over the "double" Hilbert space $\mathcal{H}^{\otimes 2}$, and is therefore unrelated to the physical Hamiltonian for the original problem. However, as we show below, there does exist an attractive physical interpretation of Eq. (5), in terms of the expected value of the SWAP operator.

III. BOUNDS AND OTHER CHANNEL PROPERTIES

We now derive upper and lower bounds on the purity and then give a characterization of T-DFSs.

Proposition 1. Let $\omega_0^{+}(A)$ denote the minimum eigenvalue of the symmetric operator A in the symmetric subspace of $\mathcal{H}^{\otimes 2}$, and let $\Pi^{+}(\mathcal{C})$ denote the normalized projector over the symmetric part of $\mathcal{C}^{\otimes 2}$. Then the following bounds hold:

$$\mathrm{Tr}[\Pi^{+}(C)\Omega(T)] \geqslant P(T,C) \geqslant \omega_0^{+}[\Omega(T)]. \tag{7}$$

Proof. Note that since $\Omega(T)$ is a symmetric operator, the symmetric subspace of $\mathcal{H}^{\otimes 2}$ is $\Omega(T)$ invariant. Therefore, the minimum expectation value of $\Omega(T)$ in this subspace coincides with the minimum eigenvalue ω_0^{+}. The lower bound in Eq. (7) is simply due to the fact that minimization over the symmetric subspace of $\mathcal{H}^{\otimes 2}$ includes the minimization over the $|\psi\rangle^{\otimes 2}\in\mathcal{C}^{\otimes 2}$. The upper bound in Eq. (7) derives from the identity $\int_{\mathcal{C}}|\psi\rangle\langle\psi|^{\otimes 2}=\Pi^{+}(\mathcal{C})$ (integration over the uniform distribution over $\mathcal{C}$ [13]) and from the obvious fact that the average value of a function is no smaller than its minimum value. ■

Lemma 1. Let T be unital $[T(\mathbb{1})=\mathbb{1}]$. Then $\forall|\psi\rangle\in\mathcal{H}$:

$$\|\Omega(T)|\psi\rangle^{\otimes 2}\| \leqslant 1. \tag{8}$$

Proof. Let $p_{ij} := \|A_j^{\dagger}A_i|\psi\rangle\|$. One has the following normalization condition

$$\sum_{ij} p_{ij}^2 = \sum_{ij} \langle\psi|A_i^{\dagger}A_jA_j^{\dagger}A_i|\psi\rangle = \sum_i \langle\psi|A_i^{\dagger}A_i|\psi\rangle = 1,$$

where in the first (second) equality we used the unitality (CP map) condition $\Sigma_j A_j A_j^{\dagger}=\mathbb{1}$ ($\Sigma_i A_i^{\dagger}A_i=\mathbb{1}$). Now

$$\begin{aligned}\|\Omega(T)|\psi\rangle^{\otimes 2}\| &= \|\sum_{ij} A_j^{\dagger}A_i|\psi\rangle \otimes A_i^{\dagger}A_j|\psi\rangle\| \leqslant \sum_{ij} \|A_j^{\dagger}A_i|\psi\rangle\| \\ &\quad \times \|A_i^{\dagger}A_j|\psi\rangle\| = \sum_{ij} p_{ij}p_{ji} \leqslant \sum_{ij} p_{ij}^2 = 1, \end{aligned} \tag{9}$$

where in the last line we used the Cauchy-Schwartz inequality for the Hilbert-Schmidt product of matrices,

$$\sum_{ij} p_{ij}p_{ji} = \mathrm{Tr}\, P^2 = \langle P, P^{\dagger}\rangle \leqslant \|P\|\,\|P^{\dagger}\| = \|P\|^2 = \sum_{ij} p_{ij}^2.$$

■

We now proceed to characterize T-DFSs. To this end we introduce a special subspace:

Definition 5. The subspace $\mathcal{H}^{\Omega}$ of Ω-invariant states $(\mathcal{H}^{\Omega}\subset\mathcal{H}^{\otimes 2})$ is the eigenspace of Ω with eigenvalue one.

Proposition 2.

(i) If $\forall|\psi\rangle\in\mathcal{C}$ and $\forall i$, it holds that $A_i|\psi\rangle=\alpha_i U|\psi\rangle$, $A_i^{\dagger}|\psi\rangle=\alpha_i^{*}U^{\dagger}|\psi\rangle$, where U is unitary, then $|\psi\rangle^{\otimes 2}\in H^{\Omega}$.

(ii) Let T be unital. Then $\mathcal{C}$ is a T-DFS$\Leftrightarrow\mathcal{C}^{\otimes 2}\subset\mathcal{H}^{\Omega}$.

(iii) T-DFS $\Leftrightarrow$ the first inequality in Eq. (7) is an equality.

Proof.

(i) Notice first that from the CP map condition, $\Sigma_i A_i^\dagger A_i=\mathbb{1}$, it follows that $\Sigma_i|\alpha_i|^2=1$. Now for $|\psi\rangle\in\mathcal{C}$, one has that $\langle\psi^{\otimes 2}|\Omega(T)|\psi^{\otimes 2}\rangle=\Sigma_{ij}|\alpha_i\alpha_j|^2=(\Sigma_i|\alpha_i|^2)^2=1$

(ii) ($\Rightarrow$) If $\mathcal{C}$ is a T-DFS then $\min_{|\psi\rangle\in\mathcal{C}}\langle\psi^{\otimes 2}|\Omega(T)|\psi^{\otimes 2}\rangle=1$. But from the Cauchy-Schwartz inequality and Lemma 1 above, one has that $\langle\psi^{\otimes 2}|\Omega(T)|\psi^{\otimes 2}\rangle\leqslant 1$ $(\forall|\psi\rangle)$, and the equality holds if and only if $\Omega(T)|\psi^{\otimes 2}\rangle=|\psi^{\otimes 2}\rangle$ $(\forall|\psi\rangle\in\mathcal{C})$. Now, if $|\Psi\rangle$ is in the symmetric part of $\mathcal{C}^{\otimes 2}$, one has that $|\Psi\rangle=\Pi_+(\mathcal{C})|\Psi\rangle=\alpha(\mathcal{C})\int_{\mathcal{C}}|\psi^{\otimes 2}\rangle\langle\psi^{\otimes 2}|\Psi\rangle$ [where $\alpha(\mathcal{C})\coloneqq\dim\mathcal{C}(\dim\mathcal{C}+1)$]. Therefore,

$$\Omega(T)|\Psi\rangle=\alpha(\mathcal{C})\int_{\mathcal{C}}\Omega(T)|\psi^{\otimes 2}\rangle\langle\psi^{\otimes 2}|\Psi\rangle$$

$$=\alpha(\mathcal{C})\int_{\mathcal{C}}|\psi^{\otimes 2}\rangle\langle\psi^{\otimes 2}|\Psi\rangle. \tag{10}$$

It follows that $|\Psi\rangle\in H^{\Omega}$.

($\Leftarrow$) If $\forall|\psi\rangle\in\mathcal{C}$ it holds that $|\psi\rangle^{\otimes 2}\in\mathcal{H}_1^{\Omega}$, then C is clearly a T-DFS, i.e., $P(T,\mathcal{C})=1$. *A fortiori*, this holds if all the elements of the symmetric part of $\mathcal{C}^{\otimes 2}$ are in $\mathcal{H}^{\Omega}$.

(iii) We have just seen that $\forall|\Psi\rangle=|\psi\rangle^{\otimes 2}$ such that $|\psi\rangle\in\mathcal{C}$, one has $\langle\Psi|\Omega(T)|\Psi\rangle=1$. By integrating over $|\psi\rangle$, one obtains that the average [leftmost part of Eq. (7)] coincides with the minimum [middle term in Eq. (7)]. ■

IV. EXAMPLES

We now present a variety of examples to illustrate our formalism, to actually compute the purity of a number of interesting channels, and to find the corresponding optimally pure states.

Example 1. Single qubit anisotropic depolarizing channel. Let T be the one-qubit channel given by $\rho\rightarrow\Sigma_{i=0}^3 p_i\sigma^i\rho\sigma^i$, where the σ^i's are the Pauli matrices ($\sigma^0=\mathbb{1}$) and the p_i's a probability distribution. One finds $\Omega_{01}=\Omega_{10}=\sqrt{p_0p_i}\sigma^0\sigma^i=\sqrt{p_0p_i}\sigma^i$; $\Omega_{ij}=-\Omega_{ji}=i\sqrt{p_jp_i}\epsilon_{ijk}\sigma^k$ $(i=1,2,3)$; $\Omega_{ii}=p_i\mathbb{1}$, $(i=0,\ldots,3)$. It then follows that

$$\Omega(T)=\sum_{i=0}^{3}\alpha_i\sigma^i\otimes\sigma^i, \tag{11}$$

where $\alpha_0=\Sigma_{i=0}^3 p_i^2$, $\alpha_k=2(p_0p_k+p_ip_j)$ $(i\neq j\neq k, k=1,2,3)$. Note that $\Sigma_{i=0}^3\alpha_i=(\Sigma_{i=0}^3 p_i)^2=1$, that $\alpha_0\in[1/4,1]$, and $\alpha_k\in[0,1/2]$, $(k=1,2,3)$. The eigenstates of $\Omega(T)$ are the Bell states $|\psi^-\rangle=(|01\rangle-|10\rangle)/\sqrt{2}$ (singlet) and $\{|\phi^-\rangle=(|00\rangle-|11\rangle)/\sqrt{2},|\psi^+\rangle=(|01\rangle+|10\rangle)/\sqrt{2},|\phi^+\rangle=(|00\rangle+|11\rangle)/\sqrt{2}\}$ (triplet). Their respective eigenvalues are $2\alpha_0-1$ and $\{1-2\alpha_1,1-2\alpha_3,1-2\alpha_2\}$. Furthermore, note that $\mathrm{Spec}\,\Omega(T)\subset[-1/2,1]$ and that $\Omega(T)$ in the triplet sector is a positive operator. The triplet sector is symmetric, while the singlet is antisymmetric. From Eq. (7), we thus know that the minimal eigenvalue in the triplet sector provides a lower bound on the purity

$$P(T)\geqslant 1-2\max_{i=1,2,3}\alpha_i\geqslant 0 \tag{12}$$

(of course in this single-qubit example there are no nontrivial subspaces: $\mathcal{C}=\mathcal{H}$). In this general case we cannot directly determine the actual purity or find the corresponding maximally robust state(s), since the Bell triplet states are not product states. To find the optimal purity states in such a case, one has to resort to other optimization techniques. However, in certain special cases the eigenstates of $\Omega(T)$ will be product states, whence our method directly yields the optimally robust states. For instance, consider the case $p_0=p_1=1/2$ and $p_2=p_3=0$; one finds $\Omega(T)=(\mathbb{1}+\sigma_x\otimes\sigma_x)/2$. In this case $|\psi^-\rangle,|\phi^-\rangle$ are degenerate, as are $|\psi^+\rangle,|\phi^+\rangle$. We then find, respectively, the symmetric product eigenstates $[(|0\rangle-|1\rangle)/\sqrt{2}]^{\otimes 2}$ and $[(|0\rangle+|1\rangle)/\sqrt{2}]^{\otimes 2}$, both with eigenvalue 1. The states $|\pm\rangle:=(|0\rangle\pm|1\rangle)/\sqrt{2}$ are thus both T-DFSs. This is intuitively clear, as the channel in this case is simply $T(\rho)=p_o\rho+p_1\sigma^x\rho\sigma^x$, and the states $|\pm\rangle$ lie on the Bloch sphere x axis, which is invariant.

As another example, consider the fully depolarizing channel with $p_i=(1-p_0)/3$, $(i=1,2,3)$. Then the following (antiferromagnetic Heisenberg exchange) Hamiltonian is obtained: $\Omega(T)=\alpha_0\mathbb{1}+\alpha\Sigma_{i=1}^3\sigma^i\otimes\sigma^i$, where $\alpha_0=p_0^2+(1-p_0)^2/3$, $\alpha=(2/3)[p_0(1-p_0)+(1-p_0)^2/3]\geqslant 0$. We can rewrite this as $\Omega(T)=\alpha_0\mathbb{1}+\alpha(2S-\mathbb{1})$, where the SWAP operator S is defined by its action on basis states $S|i\rangle\otimes|j\rangle=|j\rangle\otimes|i\rangle$. In this case, clearly every symmetric product state is an eigenstate of $\Omega(T)$, with eigenvalue $\alpha_0+\alpha$, which equals the channel purity. Thus all single-qubit states are equally (and optimally) robust. Again, this is intuitively clear: the fully depolarizing channel isotropically shrinks the Bloch sphere.

Example 2. Correlated two-qubit anistropic depolarizing channel.

Consider the correlated map

$$T(\rho)=\sum p_\alpha(\sigma_\alpha\otimes\sigma_\alpha)\rho(\sigma_\alpha\otimes\sigma_\alpha). \tag{13}$$

Then $\Omega_{\alpha\beta}=\sqrt{p_\alpha p_\beta}\sigma_\alpha\sigma_\beta\otimes\sigma_\alpha\sigma_\beta$ and

$$\Omega=\sum_{\alpha=0,x,y,z}p_\alpha^2(I\otimes I)^{\otimes 2}+\sum_{\alpha\neq\beta\neq\gamma}p_\alpha p_\beta(\sigma_\gamma\otimes\sigma_\gamma)^{\otimes 2}. \tag{14}$$

This example can be solved directly by observing that each of the Bell states has eigenvalue +1 or −1 under the action of $\sigma_\gamma\otimes\sigma_\gamma$, when the purity is one. Thus, the Bell states are T-DFSs. Notice that this result appears to be related to the communication problem for channels with correlated noise studied in Ref. [17].

One can also find the *minimal* purity states by differentiating $\langle\Omega\rangle\coloneqq{}^{\otimes 2}\langle\psi|\Omega(T)|\psi\rangle^{\otimes 2}$ as a function of expansion parameters of $|\psi\rangle$ over the Bell states. This yields $\langle\Omega\rangle_{\min}=\Sigma_{\alpha=0,x,y,z}p_\alpha^2$, and the corresponding minimally robust set of states are superpositions of pairs of Bell states with arbitrary phases

$$|\psi_1\rangle=\frac{e^{i\alpha_1}}{2}(|00\rangle+|11\rangle)+\frac{e^{i\beta_1}}{2}(|01\rangle-|10\rangle),$$

$$|\psi_2\rangle = \frac{e^{i\alpha_2}}{2}(|00\rangle - |11\rangle) + \frac{e^{i\beta_2}}{2}(|01\rangle + |10\rangle).$$

This channel thus has the interesting property that the maximally entangled Bell states are more robust than any separable (pure) state.

Example 3. Amplitude damping.

Let $T(\rho)=|0\rangle\langle 0|$, $\forall\ \rho\in S(\mathbb{C}^d)$. A set of Kraus operators is given by $A_i=|0\rangle\langle i|$, $(i=1,\ldots,d)$. Note that the channel is nonunital: $\Sigma_i A_i A_i^\dagger = d|0\rangle\langle 0| > \mathbb{1}$. One has $A_j^\dagger A_i=|j\rangle\langle i|$, so that $\Omega(T)=\Sigma_{ij}|j\rangle\langle i|\otimes|i\rangle\langle j|=\Sigma_{ij}|ji\rangle\langle ij|=S$ (the SWAP operator). Here all states are mapped onto a pure one and $\Omega(T)$ is identically $\mathbb{1}$ in the symmetric subspace. A slight generalization is given by $T(\rho)=(1-p)\rho+p|0\rangle\langle 0|$, $(p\in[0,1])$. In this case one finds

$$\Omega(T) = (1-p)^2\mathbb{1} + p^2 S + p(1-p)(|0\rangle\langle 0|\otimes\mathbb{1} + \mathbb{1}\otimes|0\rangle\langle 0|)S. \tag{15}$$

Note that $\|\Omega(T)\|\leqslant(1-p)^2+p^2+2p(1-p)=1$. The only T-DFS is $\mathbb{C}|0\rangle$.

Example 4. Projective measurements.

Let $T(\rho)=\Sigma_i\Pi_i\rho\Pi_i$, $\Pi_i\Pi_j=\delta_{ij}\Pi_i$, $\Sigma_i\Pi_i=\mathbb{1}$. Then $\Omega_{ij}=\delta_{ij}\Pi_i$, from which

$$\Omega(T) = \sum_i \Pi_i\otimes\Pi_i. \tag{16}$$

If $\mathcal{H}_i := \mathrm{Im}\,\Pi_i$ then $\mathcal{H}_i^{\otimes 2}\subset\mathcal{H}^\Omega$, i.e., from Proposition 2 all the eigenvectors of the Π_i's are one-dimensional (1D) T-DFSs. The maximum eigenvalue of $\Omega(T)$ is 1; this follows from $\langle\Psi|\Omega(T)|\Psi\rangle\leqslant 1(\forall|\Psi\rangle)$. The latter inequality results from the following argument: $\mathbb{1}^{\otimes 2}=(\Sigma_i\Pi_i)^{\otimes 2}=\Sigma_{ij}\Pi_i\otimes\Pi_j=\Omega(T)+\Sigma_{i\neq j}\Pi_i\otimes\Pi_j$. The last term is a non-negative operator (sum of products of non-negative operators), so that $\mathbb{1}-\Omega(T)\geqslant 0$. Taking the expectation value of the last inequality with respect to $|\Psi\rangle$ proves the bound above.

In the following, we use the operator norm $\|A\|_\infty := \max_{\||\psi\rangle\|=1}\|A|\psi\rangle\|$. We shall write $\|A\|$ for simplicity.

Example 5. Unitary mixture of a group representation.

This is a rather general and quite important example, which includes Examples 1 and 2 above. Let

$$T(\rho) = \sum_g p_g U_g \rho U_g^\dagger, \tag{17}$$

where $g\mapsto p_g$ is a probability distribution over the group $\mathcal{G}=\{g\}$ and $g\mapsto U_g$ is a unitary representation of $\mathcal{G}$. One finds $\Omega_{gh}=\sqrt{p_g p_h}U_g^\dagger U_h=\sqrt{p_g p_h}U_{g^{-1}h}$; thus

$$\Omega(T) = \sum_{k\in\mathcal{G}} q_k U_k\otimes U_{k^{-1}}, \tag{18}$$

where $q_k := \Sigma_g p_g p_{gk^{-1}}$ is also a probability distribution. If $|\psi\rangle\in\mathcal{H}$ is a $\mathcal{G}$ singlet, i.e., $U_g|\psi\rangle=|\psi\rangle$ $(\forall\ g\in\mathcal{G})$ then $|\psi\rangle^{\otimes 2}\in\mathcal{H}^\Omega$, i.e., all the $\mathcal{G}$ singlets are 1D T-DFSs. Here again, the maximum eigenvalue of $\Omega(T)$ is 1: Indeed, it is easy to see that $\|\Omega(T)\|\leqslant 1 : \|\Omega(T)\|\leqslant\Sigma_k q_k\|U_k\otimes U_{k^{-1}}\|=\Sigma_k q_k=1$.

As a particular instance of this kind of channel, let us consider an N-qubit case with the U_k's generating an Abelian subgroup of $\mathcal{G}$ the Pauli group (all tensor products of Pauli matrices on N qubits), as was the case in Examples 1 and 2 above. The set of $\mathcal{G}$ singlets is now given by the stabilizer of $\mathcal{G}$ [denoted $\mathcal{S}(\mathcal{G})$], i.e., the subspace generated by the $|\psi\rangle$ such that $U_k|\psi\rangle=|\psi\rangle$ $(\forall\ k)$. Since $U_k=U_k^\dagger$, one finds immediately that elements of the form $|\psi\rangle^{\otimes 2}$, where $|\psi\rangle\in\mathcal{S}(\mathcal{G})$, are eigenvectors of $\Omega(T)$ with maximum eigenvalue $\Sigma_k q_k$. These states also play the role of code words of stabilizer QECCs [8]. We thus see that, in this example, the stabilizer-QECC code words are maximally robust, though no active error correction is assumed.

In fact, the connection to quantum error correction can be made more general: the formalism developed so far allows us to establish an intriguing identity for the purity of states belonging to a QECC $\mathcal{C}$ for the CP map T: $\rho\to\Sigma_i A_i\rho A_i^\dagger$. If $|\psi_\alpha\rangle,|\psi_\beta\rangle\in\mathcal{C}$, then the error correction condition is

$$\langle\psi_\alpha|A_i^\dagger A_j|\psi_\beta\rangle = c_{ij}\delta_{\alpha\beta}, \tag{19}$$

where the matrix c_{ij} is Hermitian, non-negative, and has trace one [7]. For nondegenerate codes, c_{ij} has maximal rank. Let us now consider states of the form $|\psi_\alpha\rangle^{\otimes 2}$ $(|\psi_\alpha\rangle\in C)$. From Eq. (4) and the error correction condition, one has that

$$P(T,\mathcal{C}) = \langle\psi_\alpha^{\otimes 2}|\Omega(T)|\psi_\alpha^{\otimes 2}\rangle = \sum_{ij}|\langle\psi_\alpha|A_i^\dagger A_j|\psi_\alpha\rangle|^2 = \sum_{ij}|c_{ij}|^2 = \mathrm{Tr}(c^2). \tag{20}$$

Viewing c as a state (density operator), we have thus found that the purity of the channel acting on the code words of $\mathcal{C}$ is just the purity of the "state" c associated to the code itself. For example, for DFSs, c is simply a rank-one matrix with unit trace [14], so that $\mathrm{Tr}\,c^2=1$ and the maximum eigenvalue condition is readily recovered. As a more interesting example, consider a CP map T with $A_i=\sqrt{p_i}U_i$ with unitary U_i's (e.g., chosen from the Pauli group, as in stabilizer QEC). Recall that $c=\lambda^\dagger\lambda$, where the matrix λ is defined by the error-recovery relation $R_r A_i=\lambda_{ri}\mathbb{1}$ (restricted to $\mathcal{C}$), for each recovery operator R_r [7]. Then $R_r=\lambda_{ri}U_i^{-1}/\sqrt{p_i}$, and from the CP condition $\Sigma_r R_r^\dagger R_r=\mathbb{1}$, we find $\Sigma_r|\lambda_{ri}|^2=p_i$. Assume for simplicity that there is a unique recovery operator per error, i.e., $\lambda_{ri}=\lambda_i\delta_{ri}$, $\lambda_i\neq 0\ \forall\ i$ (this is an example of a nondegenerate code). Then $|\lambda_i|^2=p_i$ and $c=\mathrm{diag}(p_i)$; it follows that the purity over such a QECC associated to T is simply given by $\Sigma_i p_i^2$.

V. THE DUAL REPRESENTATION

We now develop an alternative representation of the channel Hamiltonian, which is useful for the derivation of several additional results, and sheds light on the physical interpretation of the channel purity.

Definition 6. The dual T_* of a CP map T [see Eq. (1)] is $T_*(X)=\Sigma_i A_i^\dagger X A_i$.

Proposition 3. Let S be the SWAP operator (defined above). Then

$$\Omega(T) = T_*^{\otimes 2}(S)S. \tag{21}$$

We give two different proofs.

Proof.

(a)

$$T_*^{\otimes 2}(S)S = \sum_{ij} (A_i^\dagger \otimes A_j^\dagger)S(A_i \otimes A_j)S = \sum_{ij} (A_i^\dagger \otimes A_j^\dagger)(A_j \otimes A_i)$$
$$= \sum_{ij} A_i^\dagger A_j \otimes A_j^\dagger A_i = \Omega(T). \tag{22}$$

(b) By writing the SWAP operator explicitly as $S = \Sigma_{lm} |m\rangle\langle l| \otimes |l\rangle\langle m|$ and applying $T_*^{\otimes 2}$, one obtains $\Sigma_{lm,ij} A_i^\dagger |m\rangle\langle l| A_i \otimes A_j^\dagger |l\rangle\langle m| A_j$. Then the proof follows by explicitly comparing the matrix elements of the latter operator times S, with the ones of $\Omega(T)$. ■

We remark that one is led to consider the operator $T_*^{\otimes 2}(S)$ by the following argument:

$$\mathrm{Tr}[T^2(\rho)] = \mathrm{Tr}[S\{T(\rho) \otimes T(\rho)\}] = \mathrm{Tr}[ST^{\otimes 2}(\rho \otimes \rho)]$$
$$= \mathrm{Tr}[T_*^{\otimes 2}(S)\rho \otimes \rho],$$

where in the first step we used the identity

$$\mathrm{Tr}[AB] = \mathrm{Tr}[SA \otimes B], \tag{23}$$

which is valid for general operators A, B [15], and in the last step we "dualized" the map. Then for pure inputs $\rho = |\psi\rangle\langle\psi|$ one has $\langle\psi^{\otimes 2}|\Omega(T)|\psi^{\otimes 2}\rangle = \langle\psi^{\otimes 2}|T_*^{\otimes 2}(S)|\psi^{\otimes 2}\rangle$. This dualization is quite useful since it moves the burden of calculation of the channel action away from the entire *set* of states ρ to the *single* observable S.

Corollary 1. Upon restriction to the symmetric subspace of $\mathcal{H}^{\otimes 2}$, one can write $\Omega(T) = T_*^{\otimes 2}(S)$.

Proof. Immediate. ■

The following corollary contains a general derivation, based on the dual representation of $\Omega(T)$, of a fact that was already proved for specific examples in Sec. IV.

Corollary 2. $\|\Omega(T)\| \leq 1$.

Proof. One has $\|\Omega(T)\| = \|T_*^{\otimes 2}(S)S\| \leq \|T_*^{\otimes 2}(S)\| \, \|S\| \leq \|T_*^{\otimes 2}(S)\|$. Since $T_*^{\otimes 2}$ is the dual of a *CP* map, elements smaller (greater) than the identity (minus the identity) are mapped onto elements smaller than the identity. Since $-\mathbb{1} \leq S \leq \mathbb{1}$, one has $-\mathbb{1} \leq T_*^{\otimes 2}(S) \leq \mathbb{1}$. This relation implies in particular that the maximum eigenvalue of the Hermitian operator $T_*^{\otimes 2}(S)$ is smaller than one. Since this maximum eigenvalue coincides with the $\|\cdot\|_\infty$ norm of $T_*^{\otimes 2}(S)$, the inequality is proved. ■

We now present a result that allows one to directly compute the *average* purity of a quantum channel.

Proposition 4. The Haar average purity of the *CP* map T is given by

$$\overline{\mathrm{Tr}[T^{\otimes 2}(|\psi\rangle\langle\psi|)]}^{\psi} = \frac{1}{d(d+1)}\mathrm{Tr}[ST^{\otimes 2}(\mathbb{1}) + \Omega(T)]. \tag{24}$$

Proof. Using the fact that $\int d\psi |\psi\rangle\langle\psi|^{\otimes 2}$ is the normalized projector over the symmetric subspace of $\mathcal{H}^{\otimes 2}$, i.e., $(\mathbb{1} + S)/[d(d+1)]$ [13], one has

$$\int d\psi\, \mathrm{Tr}[T_*^{\otimes 2}(S)|\psi\rangle\langle\psi|^{\otimes 2}] = \frac{\mathrm{Tr}[T_*^{\otimes 2}(S)(\mathbb{1} + S)]}{d(d+1)}$$
$$= \frac{1}{d(d+1)}\mathrm{Tr}[ST^{\otimes 2}(\mathbb{1}) + T_*^{\otimes 2}(S)S]. \tag{25}$$

■

In other words, the Haar average purity of a channel T is given by the expectation value of $\Omega(T)$ over the maximally mixed state $\Pi_+(\mathcal{H}) = (\mathbb{1} + S)/[d(d+1)]$ over the symmetric subspace of $\mathcal{H}^{\otimes 2}$.

Corollary 3. Using Eq. (24) one can get the Haar averaged purities of the channels considered above:

(i) One qubit depolarizing channel: $(1 + 2\alpha_0)/3$.

(ii) Amplitude damping channel: $(1-p)^2 + p^2 + 2p(1-p)/d$.

(iii) Projective measurements: $[d + \Sigma_i (\mathrm{Tr}\, \Pi_i)^2]/[d(d+1)]$.

(iv) Unitary mixing: $[d + \Sigma_{g \in \mathcal{G}}\, q_g |\mathrm{Tr}\, U_g|^2]/[d(d+1)]$.

From (iii) and (iv), it follows that:

(a) One-dimensional projective measurements achieve the minimal average purity, of $2/(d+1)$.

(b) For unitary mixing and assuming a Haar uniform distribution (all q_g equal, i.e., the fully depolarizing channel), minimal purity is obtained for U_g's in a $\mathcal{G}$ irrep. Indeed, one has in general that $(1/|\mathcal{G}|)\Sigma_{g \in \mathcal{G}} |\mathrm{Tr}\, U_g|^2 = \Sigma_J n_J^2$, where n_J is the multiplicity of the Jth $\mathcal{G}$ irrep [16]. The minimum is clearly achieved when just one irrep appears, i.e., the irreducible case.

Before concluding this section we would like to point out that the formula $\langle\Psi|\Omega(T)|\Psi\rangle = \mathrm{Tr}[ST^{\otimes 2}(|\Psi\rangle\langle\Psi|)]$ allows us to give an operational meaning to the operator Ω, and in the particular case in which $|\Psi\rangle = |\psi\rangle^{\otimes 2}$, to the purity of $T(|\psi\rangle\langle\psi|)$. Indeed, this expectation value of $\Omega(T)$ is nothing but the expectation value of the observable S in the state $T^{\otimes 2}(|\Psi\rangle\langle\Psi|)$. The latter state can in turn be viewed as the result of an action of the channel on a pair of, possibly entangled, input states from $\mathcal{H}$.

VI. PURE STATE FIDELITY OF A CHANNEL

We now show how many of the techniques introduced above for the channel purity carry over to the (simpler) problem of calculating the

Definition 7. Pure-state fidelity

$$F(T, |\psi\rangle) := \langle\psi|T(|\psi\rangle\langle\psi|)|\psi\rangle. \tag{26}$$

Proposition 5.

(i)

$$F(T, |\psi\rangle) = \langle\psi^{\otimes 2}|\Omega_1(T)|\psi^{\otimes 2}\rangle, \tag{27}$$

where $\Omega_1(T) := (\mathbb{1} \otimes T_*)(S)S$.

(ii)

$$\Omega_1(T) = \sum_i A_i \otimes A_i^\dagger. \tag{28}$$

(iii)

$$\overline{F(T,|\psi\rangle)}^{\psi} = \frac{1}{d(d+1)}\mathrm{Tr}[\Omega_1(T) + S(\mathbb{1}\otimes T(\mathbb{1}))]. \qquad (29)$$

In particular, for a unital map the average pure-state fidelity is given by $[d+\Sigma_i|\mathrm{Tr}\,A_i|^2]/[d(d+1)]$.

Proof.

(i)

$$\begin{aligned}F(T,|\psi\rangle) &= \langle\psi|T(|\psi\rangle\langle\psi|)|\psi\rangle = \mathrm{Tr}[|\psi\rangle\langle\psi|T(|\psi\rangle\langle\psi|)] \\ &= \mathrm{Tr}[S|\psi\rangle\langle\psi|\otimes T(|\psi\rangle\langle\psi|)] = \mathrm{Tr}[S(\mathbb{1}\otimes T)|\psi\rangle\langle\psi|^{\otimes 2}] \\ &= \mathrm{Tr}[(\mathbb{1}\otimes T_*)(S)|\psi\rangle\langle\psi|^{\otimes 2}] \\ &= \langle\psi^{\otimes 2}|(\mathbb{1}\otimes T_*)(S)S|\psi^{\otimes 2}\rangle.\end{aligned}$$

(ii)

$$\begin{aligned}(\mathbb{1}\otimes T_*)(S)S &= \sum_i (\mathbb{1}\otimes A_i^\dagger)S(\mathbb{1}\otimes A_i)S = \sum_i (\mathbb{1}\otimes A_i^\dagger)(A_i^\dagger\otimes\mathbb{1}) \\ &= \sum_i A_i\otimes A_i^\dagger.\end{aligned}$$

(iii)

$$\begin{aligned}\overline{F(T,|\psi\rangle)}^{\psi} &= \int_\psi \mathrm{Tr}[|\psi\rangle\langle\psi^{\otimes 2}\Omega_1(T)] \\ &= \frac{\mathrm{Tr}[(\mathbb{1}+S)\Omega_1(T)]}{d(d+1)} \\ &= \frac{\mathrm{Tr}[\Omega_1(T)+(\mathbb{1}\otimes T_*)(S)]}{d(d+1)}.\end{aligned}$$

Notice that the second term inside the square brackets is, for unital maps, simply $\mathrm{Tr}\,S=d$. ∎

Is important to stress that $\Omega_1(T)$ defined above is, in general, *non*-Hermitian. On the other hand, $\Omega_1(T)S=(\mathbb{1}\otimes T_*)(S)$ *is* Hermitian (image of an Hermitian operator via *CP* map) and has the same expectation values as $\Omega_1(T)$ over symmetric states in $\mathcal{H}^{\otimes 2}$. We thus associate a second channel Hamiltonian with T.

Definition 8. The channel fidelity Hamiltonian is

$$\Omega'(T) := (\mathbb{1}\otimes T_*)(S). \qquad (30)$$

We now report, as corollaries of point (iii) above, the average pure-state fidelities of a few relevant channels.

Corollary 4.

(i) Mixing of unitaries from the Pauli group (N qubits): $(1+2^N p_0)/(1+2^N)$.

(ii) Mixing of general unitaries: $[d+\Sigma_i p_i|\mathrm{Tr}\,U_i|^2]/[d(d+1)]$.

(iii) Amplitude damping: $1-p(1-1/d)$.

(iv) Projective measurements: $[d+\Sigma_i|\mathrm{Tr}\,\Pi_i|^2]/[d(d+1)]$.

As in the channel-purity case, the result (29) can be simply stated by saying that the Haar average fidelity of a channel T is given by the expectation value of $\Omega_1(T)$ over the maximally mixed state $\Pi_+(\mathcal{H})=(\mathbb{1}+S)/[d(d+1)]$ over the symmetric subspace of $\mathcal{H}^{\otimes 2}$. We note that a formula related to Eq. (29) for the average fidelity of quantum operations has been given in Ref. [18].

VII. CONCLUSIONS AND OUTLOOK

We have introduced a "Hamiltonian" operator formalism for the calculation of the channel purity and pure-state fidelity. Using this formalism we have been able to analytically compute these measures for a variety of channels of interest in the theory of open quantum systems, and quantum information theory. These analytical results are restricted to cases where the eigenstates of the Hamiltonian Ω (or Ω') are product states in the symmetric subspace of $\mathcal{H}^{\otimes 2}$. When this is not the case one may have to resort to numerical methods to compute the purity and fidelity.

A tempting generalization of our method is to consider perturbations to the channel Hamiltonian and use the well-developed tools of perturbation theory to thus study perturbations to given channels. One may further speculate about an adiabatic approximation, wherein slowly time-dependent channels can be studied using the adiabatic theorem applied to the channel Hamiltonian. We leave these as subjects for future investigations.

ACKNOWLEDGMENTS

P.Z. gratefully acknowledges financial support by Cambridge-MIT Institute Limited and by the European Union project TOPQIP (Contract No. IST-2001-39215). D.A.L. gratefully acknowledges financial support from NSERC and the Sloan Foundation.

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