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Combined error correction techniques for quantum computing architectures

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Abstract. Proposals for quantum computing devices are many and varied. They each have unique noise processes that make none of them fully reliable at this time. There are several error correction/avoidance techniques which are valuable for reducing or eliminating errors, but not one, alone, will serve as a panacea. One must therefore take advantage of the strength of each of these techniques so that we may extend the coherence times of the quantum systems and create more reliable computing devices. To this end we give a general strategy for using dynamical decoupling operations on encoded subspaces. These encodings may be of any form; of particular importance are decoherence-free subspaces and quantum error correction codes. We then give means for empirically determining an appropriate set of dynamical decoupling operations for a given experiment. Using these techniques, we then propose a comprehensive encoding solution to many of the problems of quantum computing proposals which use exchange-type interactions. This uses a decoherence-free subspace and an efficient set of dynamical decoupling operations. It also addresses the problem of controllability in solid-state quantum dot devices.

1. Quantum error correction strategies

The main obstacle to building a quantum computing device is noise and decoherence in the quantum system due to the inevitable interaction with the environment. There are several error correction/avoidance strategies for treating this problem. They can be divided into three broad categories. Quantum error correction codes (QECCs) [1–4] (for a review see [5]) use redundancy and an active measurement and recovery scheme to correct errors that occur during a computation (we include topological quantum codes in this category; see [6] and references therein). Decoherence-free subspaces (DFSs) [7–12], rely on symmetric system–bath interactions to find encodings that are immune to decoherence effects. Dynamical decoupling, or ''bang-bang'' (BB) operations [13–24] are strong fast pulses which suppress errors by averaging them away. QECCs use extra qubits, which, at this time are a scarce resource. They require at least a 5 physical qubit to 1 logical qubit encoding [4, 25] (neglecting ancillas required for fault-tolerant recovery) in order to correct a general single qubit error [3]. DFSs also require extra qubits and are most effective for collective errors, or errors where multiple qubits are coupled to the same bath mode [12]. The minimal encoding for a single qubit is 3 physical qubits to one logical qubit [10]. Finally, the BB control method

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requires a complete set of pulses to be implemented within the correlation time of the bath [13]. It does not, however, require extra qubits.

In this article we discuss the combination of BB operations with other encoding techniques to conserve qubit resources while making quantum computing devices more robust. An experiment combining BB with quantum error correction has recently been reported [26]. We begin with a brief review of the BB control formalism before presenting an empirical formula for the determination of BB operations from a set of quantum process tomography measurements [27]. We then give a general theorem which provides sufficient conditions for the elimination of errors via BB controls on logically encoded subspaces. This is used to discuss the application of the BB controls in conjunction with QECCs. Our results are then used to determine a combined effective encoding, recoupling [28], and decoupling (BB) strategy for quantum computing devices which rely on exchange-type interactions [29]. We give estimates for the number of BB operations which can be performed in experiments using spin-coupled quantum dots in GaAs. The estimates are based upon models of the underlying mechanisms of decoherence in these systems. However, the empirical method for determining BB operations proposed in [29, 30], circumvents the need for a detailed understanding of the underlying decoherence processes.

1.1. *Bang–bang operations*

Let us briefly review some important aspects of the method of BB controls. BB controls are strong and fast pulses, applied cyclically, which average out the environment-induced noise [13]. In the limit of infinitely fast pulsing, BB controls have been shown to completely remove decoherence. The simplest example of BB is the 'parity-kick' sequence [13, 17]. Suppose that an error E (an operator in the system–bath Hamiltonian) acts on the system, and that we can find a pulse U (unitary operator) which anticommutes with E, and therefore changes the sign of this error:

$$\{E, U\} = 0, \quad \Rightarrow U^{\dagger} E U = -E. \tag{1}$$

Allowing the system to repeatedly undergo the sequence: {free evolution under E (for time Δt), application of U, free evolution, application of U^{-1}}, will cause the error to be averaged out ('symmetrized' [15, 17]), thus *decoupling* system and bath. The parity kick (whose origins can be traced to the well-known Carr-Purcell sequence of NMR [31]) and its generalizations have been the subject of several recent publications [13–24]. In reality, for decoupling to work the time taken for a complete cycle of pulses, T_c, must be significantly shorter than the fastest bath correlation time τ_c:

$$\Delta t \leqslant T_c \ll \tau_c. \tag{2}$$

Even in the case which the time scales are close, one can achieve some noise reduction [3, 17, 20, 23, 32]. Knowledge of τ_c, the inverse of the bath spectral density frequency cut-off, is clearly desirable for determining the success of the BB procedure, and will be discussed in detail below for quantum dots. Given that pulses have finite durations, the ratio τ_c/T_c imposes further constraints on the length of the experimentally implementable pulse sequences. However the empirical method for the determination of the BB operations, outlined below, takes these constraints into account.

1.2. *Empirically determined BB controls*

Previous analyses of BB controls have typically assumed model system–bath Hamiltonians [13–24]. However, the total system–bath Hamiltonian is often not known. As an alternative to this model-based approach we review here a procedure for finding BB operations from experimental data [29, 30]. This empirical determination requires neither a detailed understanding of the fundamental processes nor a detailed experimental analysis of each of the decoherence processes in the system. It requires only a set of quantum process tomography measurements [21] on the logical qubits to determine the *types* of errors that occur. With this, one may empirically determine the set of *required* corrective pulses and the efficacy of the *experimentally available* pulse set [18].

Empirical BB is based on the following set of observations. Very generally, the evolution of an open quantum system, described by a density matrix ρ, satisfies the (completely positive [33]) map

$$\rho(t) = \sum_{\alpha,\beta} \chi_{\alpha\beta}(t) K_\alpha \rho(0) K_\beta^\dagger, \tag{3}$$

where the matrix $\chi_{\alpha\beta}(t)$ is hermitian and $\{K_\alpha\}$ is a system operator basis [34, 35]. The χ matrix can be determined from a quantum process tomography measurement [34]. It can be shown that equation (3) can be transformed into [36] (using $\hbar = 1$)

$$\rho(t) = -\mathrm{i}[S(t), \rho(0)] + \frac{1}{2} \sum_{\alpha,\beta=1} \chi_{\alpha\beta}(t)([K_\alpha, \rho(0)K_\beta^\dagger] + [K_\alpha \rho(0), K_\beta^\dagger]), \tag{4}$$

where

$$S(t) = \frac{\mathrm{i}}{2} \sum_{\alpha=1} [\chi_{\alpha 0}(t) K_\alpha - \chi_{0\alpha}(t) K_\alpha^\dagger]. \tag{5}$$

For BB operations, a short-time expansion of equation (4) is relevant. Choosing a hermitian operator basis $\{K_\alpha\}$, to first order in τ

$$\rho(\tau) \approx \mathrm{i}[S(\tau), \rho(0)], \tag{6}$$

where $S(\tau) = \sum_{\alpha \geqslant 1} \mathrm{Im}\,(\chi_{\alpha 0}^{(1)}(\tau)) K_\alpha$, $\chi_{\alpha 0}^{(1)}(\tau) = \tau(\mathrm{d}(\chi_{\alpha 0})/\mathrm{d}t)_{t=0}$ and $K_0 \equiv \mathbb{1}$ [35]. Note that $S(\tau)$ behaves as a Hamiltonian. Thus, using the abbreviation $\bar{\chi}_\alpha \equiv \mathrm{Im}\,(\chi_{\alpha 0}^{(1)}(\tau))$, under the action of a group $\mathcal{G} = \{U_k\}_{k=1}^N$ of unitary BB controls $S(\tau)$ transforms as

$$S(\tau) \to \sum_k U_k S(\tau) U_k^\dagger. \tag{7}$$

Therefore the operator basis transforms as

$$\begin{aligned} \sum_\alpha \bar{\chi}_\alpha K_\alpha &\to \frac{1}{N} \sum_\alpha \bar{\chi}_\alpha \sum_k U_k^\dagger K_\alpha U_k \\ &= \frac{1}{N} \sum_{\alpha\beta} \sum_k \bar{\chi}_\alpha R_{\alpha\beta}^{(k)} K_\beta. \end{aligned} \tag{8}$$

The last expression implies that S and therefore χ transform according to the adjoint representation of $\mathcal{G}$, defined by $\sum_\beta R_{\alpha\beta}^{(k)} K_\beta = U_k^\dagger K_\alpha U_k$. For example

$R \in SO(3)$ for $U \in SU(2)$, which leads to a geometric description of the result [21]. Specifically, we have under BB that $\sum_{\alpha\geq 1} \bar{\chi}_\alpha K_\alpha \rightarrow \sum_{\beta\geq 1} \tilde{\chi}_\beta K_\beta$, where

$$\tilde{\chi}_\beta = \frac{1}{N} \sum_k \sum_{\alpha\geq 1} \bar{\chi}_\alpha R^{(k)}_{\alpha\beta}. \tag{9}$$

The coefficients $\hat{\chi}_\beta$ can be viewed as the expansion coefficients of a 'desired' Hamiltonian and the coefficients $\tilde{\chi}_\beta$, the BB-modified evolution. For example, for storage the target evolution would be one for which all $\hat{\chi}_\beta$ vanish. For computation we would have a set of non-vanishing $\hat{\chi}_\beta$ describing the Hamiltonian we would wish to implement [21]. *The key idea of empirical BB is to use the experimentally determined* $\bar{\chi}_\alpha$, *together with a specified set of* $\hat{\chi}_\beta$ (*corresponding to a desired evolution*), *to solve equation* (9) *for the rotation matrices* $R^{(k)}_{\alpha\beta}$, such that $\tilde{\chi}_\beta = \hat{\chi}_\beta$. These, in turn, determine a set of BB operations [18]. Thus, using the empirical BB method, *one may determine the required BB operations directly from experimental data*. In practice one would wish to minimize the difference between the target and BB-modified evolutions. This difference can be described by any of the standard measures of distance including the Euclidean distance between the corresponding vector fields [21]. Repeatedly performing the BB procedure determines the optimal BB process, given the available controls and accounting for constraints, through a control loop [37]. In this manner only the experimentally relevant errors are ever addressed, thus potentially reducing the size of the set of BB operations.

1.3. *Bang–bang operations on encoded spaces*

Since the introduction of QECC [1], encoding techniques have become extremely important. They have been used for DFSs [7–9] and universality considerations [11, 28, 38–46], in some cases combining DFS and QECC ideas [47–49]. Here we wish to take advantage of the benefits of encoding techniques while reducing noise in quantum systems using BB operations (see also [26, 50, 51] for related results and ideas). Indeed, it will be shown that BB operations on encoded operations can be very advantageous for the BB requirements as well. We believe that the methods for universal quantum computation and BB controls using the $\{|01\rangle, |10\rangle\}$ code (below) are of immediate value to solid-state QC implementations. Let us first present a generally applicable result which gives sufficient conditions for the elimination of errors on encoded spaces using BB operations. Let $\bar{\mathcal{G}}$ denote the generators of the group of logical operations (e.g. $\bar{\mathcal{G}} = \{\bar{X}, \bar{Y}, \bar{Z}\}$, acting as *gates* on a single encoded qubit). In analogy to standard BB theory [13–24] we define 'symmetrization of a Hamiltonian H with respect to $\bar{\mathcal{G}}$' as: $H \mapsto \sum_{U\in\bar{\mathcal{G}}} U^\dagger H U$. We then have the following result, which is a straightforward generalization of the BB condition for unencoded qubits [15, 52]:

Theorem 1: Symmetrization with respect to $\bar{\mathcal{G}}$ suffices to completely decouple the dynamics of the encoded subspace.

Proof: Symmetrization takes any system–bath Hamiltonian and projects it onto the centralizer of the group generated by $\bar{\mathcal{G}}$ (i.e. the set of elements that commutes with all elements of this group). By irreducibility of the representation of $\bar{\mathcal{G}}$, it follows, from Shur's Lemma, that the BB-modified system–bath Hamiltonian is proportional to identity on the code space. That is, the code space dynamics will be decoupled.

This theorem shows that encoded BB operations may be combined with *any encoding*. Of particular interest are DFSs and QECCs. In addition, the sufficiency of the logical operations is important since they are assumed to be available in experiments. Later in this article we will discuss in detail a physically applicable case in which BB operations may be combined with a DFS. Here we briefly comment on how they may be combined with a QECC (see also the experiment [26]).

An obvious way in which BB operations may be used in conjunction with QECCs is the following: one may simply apply BB operations to each individual qubit. This may well reduce the error rate and thus make an error correction code feasible when it would not be otherwise. However, there are less obvious, but still beneficial techniques for combining these methods.

QECCs can often be described by a stabilizer $\mathcal{S} = \{S_i\}$ [3], which is a group that has all codewords as eigenstates with eigenvalue 1. The errors $\mathcal{E} = \{E_j\}$ that a stabilizer code can detect are exactly the operators which *anticommute* with at least one element of $\mathcal{S}$ [3]. To every stabilizer QECC there also corresponds a set at logical operations (the normalizer), that is composed of operators that commute with stabilizer, and thus preserve the code space. Every E_j also anticommutes with at least one element of the normalizer: $\{\bar{g}_i, E_i\} = 0$. This immediately implies that the logical operations can be used as elements of an encoded BB control scheme. The same is true for the elements of the stabilizer. Thus, *to suppress $\mathcal{E}$, apply the generators of $\mathcal{S}$ or of the normalizer as a set of BB operations*. Furthermore, since syndrome measurement for a stabilizer code corresponds to measuring the elements of the stabilizer, BB operations can be applied during the measurement procedure. The final component of a QECC loop are recovery operations, which typically correspond to applying the inverse of the error operators. It is clear therefore that BB operations cannot be applied during recovery, as they anticommute with the recovery operations. Thus BB operations can be applied during the entire QECC procedure with the exception of the recovery operations, without loss of the desired interaction. (See [21] for a geometric explanation of this).

For a demonstration of this latter point, consider the following simple, but important example of trying to protect against all single qubit errors. The smallest QECC uses 5 physical qubits per logical qubit [25]. Instead, we could start by encoding 1 logical qubit into 3: $|0\rangle_{\mathrm{L}} = |000\rangle$, $|1\rangle_{\mathrm{L}} = |111\rangle$, in order to protect just against independent bit flip errors $\mathcal{E}_X = \{X_1, X_2, X_3\}$ [27] (X_i represents the Pauli matrix σ_x acting on the i^{th} qubit, etc.) Using the theorem we require that the logical operations come from the set $\{\bar{X} = X_1X_2X_3,\ \bar{Y} = -Y_1Y_2Y_3,\ \bar{Z} = Z_1Z_2Z_3\}$. The three qubit code leaves independent phase flip errors $\mathcal{E}_Z = \{Z_1, Z_2, Z_3\}$. We can suppress these using the following BB operations on the encoded qubits. The stabilizer for the 3 qubit code for phase flips is $\mathcal{S}_X = \{X_1X_2, X_2X_3, X_1X_3\}$, which clearly anticommutes with $\mathcal{E}_Z$. Note that here X_iX_j are gates, not Hamiltonians, and are therefore implemented using simultaneous application of the single-body Hamiltonians X_i and X_j. Thus, frequent application of the stabilizer elements as parity kick operators will suppress the $\mathcal{E}_Z$ errors. Since they are elements of the stabilizer, they will commute with the logical operations and thus, in principle, can be simultaneously applied. These stabilizer operations will leave no component of error in the Y_i or Z_i directions when implemented as BB. When one measures for X_i errors, they will be projected onto the eigenbasis in which the measurement is performed. This will not affect the Y_i or Z_i directions. The advantage of these schemes, compared to the 5-qubit code [25], is in the conservation of qubit resources. Of course, this comes at the expense of additional gate operations which must be included in the QECC circuitry, but this may well be a worthwhile trade-off in situations where qubits are scarce. We now turn to an explicit demonstration of combining DFS and BB to QC proposals based on exchange-type interactions.

2. QC in solid-state devices

We now wish to discuss the application of the aforementioned techniques to quantum computing (QC) devices which use a form of the exchange interaction with particular emphasis on solid state proposals. Essentially all promising solid-state QC proposals [53] are based on either direct or effective exchange interactions between qubits, with a Hamiltonian of the form

$$H_{\text{ex}} = \sum_{i<j} J_{ij}^x X_i X_j + J_{ij}^y Y_i Y_j + J_{ij}^z Z_i Z_j. \tag{10}$$

(X_i represent the Pauli matrix σ_x acting on the ith qubit, etc.) Representative examples are quantum dots [54–59], nuclear [60] or electron [61] spins of donor atoms in silicon [43, 44], quantum Hall systems [62] and electrons on helium [63]. These implementations combine scalability with a clear route to controllability of qubit interactions via tunable exchange couplings J_{ij}^{α}. At the same time two major problems arise in these proposals. Problem I: this, inherent problem, is shared by all other QC proposals, and concerns the inevitable coupling to the environment (lattice, impurities and other degrees of freedom). This coupling leads to decoherence, which introduces computational errors that must either be prevented in the first place [7–9], frequently corrected [1–3], or suppressed [13–24]. Problem II: this, technological problem, is to some extent unique to solid-state QC architectures, and concerns the fact that different constraints are involved in implementing single-qubit versus two-qubit operations, for a variety of reasons detailed, e.g. in [43]. In fact the single-qubit operations often involve significantly more demanding constraints. A large body of literature has been devoted to overcoming the decoherence problem (for a review see [27]), some pertaining directly to quantum dots [64]. A number of recent papers have proposed solutions to the different constraints imposed by single and two qubit operations [11, 28, 38–46]. Here, we propose a comprehensive and realistic solution to both problems.

2.1. *Encoding*

We use a well-known code first proposed in a quantum information context in [65]. Blocks of two qubits encode single logical qubits as follows:

$$|0_{\text{L}}\rangle_i \equiv |0\rangle_{2i-1} \otimes |1\rangle_{2i}, \quad |1_{\text{L}}\rangle \equiv |1\rangle_{2i-1} \otimes |0\rangle_{2i}. \tag{11}$$

Here $i = 1, \ldots, N/2$ indexes logical qubits and N is the total number of physical qubits. It is simple to see how logic operations can be performed on this code. Let us denote encoded logical operations by a bar; they act on the encoded qubits in the same manner as the unencoded operations act on physical qubits. For example, $\bar{X}|0_{\text{L}}\rangle = |1_{\text{L}}\rangle$ and $\bar{X}|1_{\text{L}}\rangle = |0_{\text{L}}\rangle$. Then, the single-encoded-qubit logic operations, defined by $\bar{X}_i = (X_{2i-1}X_{2i} + Y_{2i-1}Y_{2i})/2$ and $\bar{Z}_i = (Z_{2i-1} - Z_{2i})/2$, viewed as controllable Hamiltonians, can be used to generate all encoded-qubit $SU(2)$ transformations. Together with the two-encoded-qubits operation $\overline{Z_i Z_{i+1}} = Z_{2i}Z_{2i+1}$ which couples qubits in two neighbouring blocks, and which can be used to implement a controlled-phase transformation [27] between encoded qubits $i, i+1$, they form a *universal set of Hamiltonians* on the space of encoded qubits [47]. Universality means that by selectively turning the Hamiltonians $\{\bar{X}_i, \bar{Z}_i, \overline{Z_i Z_{i+1}}\}$ on/off it is possible to generate the Lie group $U(2^{N/2})$ of all possible transformations on the encoded qubits. Let us assume that the single-qubit spectrum is non-degenerate, but not necessarily controllable, i.e. the free Hamiltonian of the qubit system is $\sum_i \epsilon_i \sigma_i^z$, with $\epsilon_i \neq \epsilon_j$, but the ϵ_i are not separately tunable. As shown in [22] it is then in fact sufficient to actively control *only* $\bar{X}_i$ in order to achieve (encoded) universality, in the Heisenberg ($J_{ij}^x = J_{ij}^y = J_{ij}^z$), XXZ ($J_{ij}^x = J_{ij}^y \neq J_{ij}^z$), and XY ($J_{ij}^x = J_{ij}^y$, $J_{ij}^z = 0$) instances of the general exchange Hamiltonian, equation (10). The 'encoded recoupling' method introduced to this end in [22] generalizes the standard NMR selective recoupling method by applying pulses not to 'bare' (physical) qubits, but instead to encoded (logical) qubits. Encoded recoupling eliminates the need for single-qubit control in exchange-based quantum computer architectures and thus solves Problem II.

The second advantage of the above encoding is that it is a DFS with regard to collective phase errors [8, 11, 65–67]. Suppose the system is affected by a system–bath interaction Hamiltonian $H_{\rm I} = S_z \otimes B_z$, where $S_z = \sum_i Z_i$ is the collective dephasing operator. For logical qubit states $|\psi_{\rm L}\rangle = a|0_{\rm L}\rangle + b|1_{\rm L}\rangle$, it is simple to check that $S_z|\psi_{\rm L}\rangle = 0$, so that $H_{\rm I}$ does not affect the code. The collective errors are expected to be particularly relevant for solid-state systems at low temperatures and dephasing is one of the main problems in this class of quantum computing devices. The DFS property of the above encoding is therefore a partial solution to Problem I. However, collective dephasing is by necessity an approximation. In realistic solid-state devices there are other types of errors arising from a variety of sources. It is our goal in this paper to show how the methods reviewed thus far can be extended in a simple and realistic manner, to deal with these other sources of decoherence. In particular, we now turn to the combination of these techniques with the method of BB controls, except that, in the spirit of encoded recoupling [28], we apply these controls on the space of *encoded* qubits (see also [50, 51]). The DFS encoding together with BB operations on the encoded qubits will serve to counter decoherence, while the method of encoded recoupling will allow for universal quantum computation on the encoded qubits. The result of combining these three techniques is the basis for our claim of a comprehensive solution to problems of noise and design in solid-state quantum computing.

2.2. *Applying bang–bang operations on a decoherence-free subspace*

As noted above, the logical qubits of equation (11) are immune to collective dephasing errors $Z_{2i-1} + Z_{2i}$. Let us focus on the first encoded qubit ($i = 1$), and consider which other errors can act on it. A basis for all possible errors are the 2^4 different tensor products of all Pauli matrices (including the identity I) acting on two qubits. Now, in general, four types of operations that affect a DFS can be identified [48]: (i) the set of 2 operations to which the DFS is invariant, $(I, Z_1 + Z_2)$; (ii) the set of 3 operations that take states outside of the DFS to other states which are also outside of the DFS. Both (i) and (ii) have no effect on the DFS. (iii) The set of 3 logical operations, $[\bar{X} = (X_1X_2 + Y_1Y_2)/2$,

$\bar{Y} = (X_1 Y_2 - Y_1 X_2)/2, \bar{Z} = (Z_1 - Z_2)/2$]. When acting uncontrollably, these operations can cause *logical* errors. (iv) The set of 8 operations which mix DFS states with states out of the DFS, see equation (12). These operations are responsible for leakage from and into the DFS. Sets (iii) and (iv) are those that damage the encoding. Both can cause decoherence by entangling the encoded information with uncontrollable bath degrees of freedom. Let us now apply this classification to our code. A basis for the leakage errors (iv) is represented by the following set of operators:

$$\{X_1, X_2, Y_1, Y_2, X_1 Z_2, Z_1 X_2, Y_1 Z_2, Z_1 Y_2\}. \tag{12}$$

This error set can clearly be seen to take the encoded states of equation (11) out of the DFS (and vice versa) since it involves single bit flips, or bit and phase flips on individual physical qubits.

We now come to a crucial observation first made in [29]. Let $U_{\bar{X}}(\phi) \equiv \exp(-i\phi\bar{X})$. Then, *a single BB pulse of the form*

$$U_{\bar{X}}(\pi) = \exp(-i\pi(X_1 X_2 + Y_1 Y_2)/2) = -Z_1 Z_2, \tag{13}$$

can eliminate all type (iv) leakage errors. That this is so follows since $U_{\bar{X}}(\pi)$ anticommutes with all of the errors in equation (12). As noted above (equation (1)), this is the condition for the parity kick version of BB controls. Thus, all type (iv) leakage errors can be eliminated by a *single* pair of BB pulses per cycle. This single pulse pair aspect is extremely important given the severe time constraints under which BB must operate.

In order to implement the BB operation $U_{\bar{X}}(\pi)$ it is necessary to be able to switch on the Hamiltonian $J(X_1 X_2 + Y_1 Y_2)$ for a time $t = \pi/2J$. This (XY) Hamiltonian is directly available in a number of QC proposals (quantum dots/atoms in cavities [57, 68], quantum Hall systems [62]). In dealing with systems that are governed by the Heisenberg or XXZ Hamiltonians, the encoded selective recoupling method can be used make these Hamiltonians simulate the XY type [28]. The Heisenberg case applies to the spin-coupled quantum dots and donor-spin proposals of [54, 60]. The XXZ case applies directly to the electrons-on-helium proposal [63], and to the XY and Heisenberg proposals if symmetry breaking mechanisms are taken into account [28, 43]. We note that spin–orbit coupling induces anisotropic terms which appear as corrections to equation (10) [61]. Methods for treating these have recently been suggested [70–72]. Thus the method using $U_{\bar{X}}(\pi)$ for eliminating leakage is applicable to a wide range of solid-state QC proposals.

The elimination of all leakage errors by a single pair of BB operations per cycle is a rather drastic alternative to the severe, factor of 5, qubit overhead incurred by attempting to do the same using a concatenation of DFS and QECC encoding [48]. The advantage is somewhat diminished if one is also worried about the type (iii), logical, errors, which the DFS-QECC concatenation method is capable of correcting at no extra cost [48]. Note first that $U_{\bar{X}}(\pi/2) = -i\bar{X}$ anticommutes with both $\bar{Y}$ and $\bar{Z}$. Thus in fact *all but one error* ($\bar{X}$ itself) *can be eliminated using just the single BB control Hamiltonian* $\bar{X}$. In order to eliminate $\bar{X}$ as a logical error we must introduce other BB controls, $\bar{Z} = (Z_1 - Z_2)/2$, and $\bar{Y} = i[\bar{Z}, \bar{X}]$ which, by the theorem above, can then be used to eliminate other errors. To the extent that these operations are available, this is a reasonable proposition. However, since one of our goals was to avoid needing to directly control single qubits, it is reassuring that the encoded recoupling method [28] can be used here again, in order to switch on/off the Hamiltonian term $\bar{Z}$ by controlling $\bar{X}$ alone.

To summarize thus far, we have shown that the DFS encoding $|0\rangle_L = |01\rangle$, $|1\rangle_L = |10\rangle$, which is immune to collective dephasing errors, can be made robust against all leakage errors in conjunction with the *single* BB pulse $\exp(-i\pi\bar{X})$. To further eliminate all logical errors it is necessary to introduce two more BB pulses, which can also be obtained from pulsing the XY Hamiltonian $\bar{X} = (X_1X_2 + Y_1Y_2)/2$. This seems like a modest set of requirements for the elimination of *all* decoherence errors on a single logical qubit, provided that the BB cycle time can indeed be made much smaller than the bath time scale, as in equation (2). We turn to an evaluation of this issue next, in the context of quantum dots.

2.3. *Estimation of bath cut-off frequency in quantum dots*

Here we are primarily concerned with the spin-based GaAs quantum dots QC proposals [54–57]. The main spin relaxation and dephasing channels for electron-spin qubits in GaAs have been recently thoroughly reviewed in [73]. The dominant low temperature mechanisms are related to spin–orbit coupling, which couples spins to impurities and the lattice. Nevertheless, a lack of detailed understanding of the various decoherence mechanisms persists. It is noteworthy that our approach to error suppression does not rely on a detailed microscopic understanding of these mechanisms. In the case of GaAs quantum dots, experimental estimates for the spin dephasing time T_2 are $\sim$100 ns [74]. We are not aware of direct measurements or theoretical calculations of the bath cut-off frequency $1/\tau_c$ in these systems. Nevertheless, we can provide positive evidence for the ability to achieve the required BB pulse rates.

We consider the spin-bath and spin-boson models, which are rather general models of low energy effective Hamiltonians, adaptable to a surprisingly wide range of problems, including ours. The spin-boson model describes dephasing due to coupling to delocalized modes (lattice vibrations), while the spin-bath model captures the coupling to localized modes, such as nuclear and paramagnetic spins, and defects [75]. In both models it can be shown that the characteristic decay time of coherence, $T_2 = f(\tau_c, T)$ (τ_c is the inverse of the bath spectral density high-frequency cut-off, T is the temperature) and the function f can be analytically determined in various cases [13, 35, 65, 75, 76]. Note that exponential coherence decay is rigorously valid only in the Markovian limit: e.g. in the spin-boson model at $T = 0$ with Ohmic damping, coherence decays polynomially as $1/(1 + (t/\tau_c)^2)$ [76], in which case one can identify $T_2 = \tau_c$. In fact, since τ_c is the primary time scale describing the bath, it is not unreasonable to quite generally identify $T_2 = c(T)\tau_c$, where c is a function that depends only on T. This is supported by a variety of instances of the spin-boson and spin-bath models, differing by the specific form of the bath spectral density. Furthermore, at low temperature $c(T) \approx 1$. Given $T_2 \sim 100$ ns [74], we thus conservatively estimate $\tau_c \sim 1 - 100$ ns for spin-coupled GaAs quantum dots. The gate operation time in these systems is of the order of 50 ps [73], and cannot be made much shorter because of induced spin–orbit excitations [55]. Thus a range of 20–2000 BB parity-kick pulses seems attainable. The first-order correction to the ideal limit of infinitely fast and strong BB operations is $O((T_c/\tau_c)^2)$ [13], which, for parity kicks, in our case therefore translates to a correction of $O(10^{-2})$–$O(10^{-6})$.

3. Conclusions

To reduce noise and improve the reliability of quantum computing devices, new methods will have to be employed which take into account the constraints on current experiments. In particular, qubits are scarce resources today and will be in the near future. In order to reduce qubit overhead in different error correction/avoidance encodings, we have presented two results for making the recently introduced BB operations more practical in present-day experiments. The first is the theorem which gives necessary conditions for the removal of all errors on encoded spaces using logical operations alone. The importance of this result lies in its generality; *Theorem 1 gives sufficient conditions, in terms of logical operations, for the removal of all errors on an encoded subspace via bang–bang controls using logical operations only*. Particularly important is the potential for *combining the error suppression methods of BB operations* [13–21], *with DFSs* [7–9] *and QECCs* [1–3]. The second result emphasizes the practical concerns of the experimentalist. *Without relying on a particular model Hamiltonian we may, using quantum process tomography* [27], *determine an appropriate and efficient set of BB controls for physical and/or logical quantum computational states*.

Application of our methods results in a rather comprehensive solution to problems of decoherence and gate implementation in quantum computer proposals governed by exchange Hamiltonians. Our solution combines ideas from the theory of decoherence-free subspaces [7–9] and bang–bang (BB) controls [13–21], and the recently proposed method of encoded selective recoupling [28]. By encoding logical qubits into pairs of physical qubits a first level of protection against collective decoherence is obtained, which can be further significantly enhanced using a *single* type of BB operation, that can eliminate all leakage errors from the DFS. Two more BB operations are required to suppress *all* other decoherence errors. We have estimated that 10–1000 parity-kick cycles can realistically be implemented in the case of GaAs spin-coupled quantum dots within the bath correlation time. In conjunction with the elimination of the need for difficult-to-implement single qubit operations enabled by the encoded recoupling method [28], we believe that *our methods offer a realistic and comprehensive solution to some of the major difficulties associated with the design of quantum dot, and other exchange-based solid-state quantum computers*.

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