Creating Decoherence-Free Subspaces Using Strong and Fast Pulses

L.-A. Wu and D. A. Lidar
Chemical Physics Theory Group, University of Toronto, 80 St. George Street, Toronto, Ontario M5S 3H6, Canada
(Received 22 December 2001; published 6 May 2002)

A decoherence-free subspace (DFS) isolates quantum information from deleterious environmental interactions. We give explicit sequences of strong and fast ["bang-bang" (BB)] pulses that create the conditions allowing for the existence of DFSs that support scalable, universal quantum computation. One such example is the creation of the conditions for collective decoherence, wherein all system particles are coupled in an identical manner to their environment. The BB pulses needed for this are generated using only the Heisenberg exchange interaction. In conjunction with previous results, this shows that Heisenberg exchange is by itself an enabler of universal fault-tolerant quantum computation on DFSs.

DOI: 10.1103/PhysRevLett.88.207902 PACS numbers: 03.67.Lx, 03.65.Fd, 03.65.Ud, 03.65.Yz

Since the discovery of quantum error correcting codes (QECCs) [1], an arsenal of powerful methods has been developed for overcoming the problem of decoherence that plagues quantum computers (QCs). A QECC is a closed-loop procedure, that involves frequent error identification via nondestructive measurements, and concomitant recovery steps. Alternatively, decoherence-free subspaces (DFSs) [2–4] and subsystems [5], and dynamical decoupling, or "bang-bang" (BB) [6–11], are open-loop methods. A DFS is a subspace of the system Hilbert space which is isolated, by virtue of a dynamical symmetry, from the system-bath interaction. The BB method is a close cousin of the spin-echo effect. All decoherence-reduction methods make assumptions about the system (S)-bath (B) coupling embodied in a Hamiltonian of the general form $H = H_S \otimes I_B + H_{SB} + I_S \otimes H_B$. Here I is an identity operator and H_{SB} is the system-bath interaction term, which can be expanded as a sum over linear, bilinear, and higher order coupling terms:

$$H_{SB} = \sum_i H_i + \sum_{i<j} H_{ij} + \cdots + \sum_{i_1<\cdots<i_p} H_{i_1 i_2 \ldots i_p}. \quad (1)$$

Restricting our attention to qubits, a typical assumption is $p = 1$,

$$H_i = \vec{\sigma}_i \cdot \vec{B}_i = \sum_{\alpha=x,y,z} \sigma_i^\alpha \otimes B_i^\alpha, \quad (2)$$

where $\vec{\sigma}_i = (\sigma_i^x, \sigma_i^y, \sigma_i^z) \equiv (X_i, Y_i, Z_i)$ are the Pauli matrices acting on the ith qubit, and $\vec{B}_i = (B_i^x, B_i^y, B_i^z)$ are arbitrary bath operators. This includes the *independent errors model* (all B_i^α different, $\alpha = x, y, z$), and the *collective decoherence model* ($B_i^\alpha \equiv B^\alpha \ \forall i$), important, respectively, for the QECC and DFS methods. The bilinear terms are in general descibed by a second-rank tensor $\mathbf{G}_{ij}$, so that $H_{ij} = \vec{\sigma}_i \cdot \mathbf{G}_{ij} \cdot \vec{\sigma}_j$. One of the main problems in applying the various decoherence-countering strategies is that, typically, the conditions under which they apply are not wholly satisfied experimentally [12]. This problem is particularly severe for the DFS method, since it demands a high degree of symmetry in the system-bath interaction. Two main cases are known that admit *scalable* DFSs (i.e., subspaces that occupy a finite fraction of the system Hilbert space): collective decoherence [2–4] and the model of "multiple qubit errors" (MQE) [13]. Collective decoherence, as defined above, assumes qubit-permutation-invariant system-bath coupling. This may be satisfied at ultralow temperatures in solid-state QC implementations, provided the dominant decoherence mechanism is due to coupling to a long-wavelength reservoir, e.g., phonons [14,15]. MQE assumes that the system terms appearing in H_{SB} generate an Abelian group under multiplication (referred to below as the "error group"). This is a somewhat artificial model that typically imposes severe restrictions on $\vec{B}_i$ and $\mathbf{G}_{ij}$ (examples are given below). On the other hand, a two-dimensional (2D) DFS (encoding one logical qubit) can be constructed using as few as three qubits under collective decoherence conditions [5], and, unlike QECCs, requires no active intervention other than the initial encoding and final decoding steps. One is thus faced with a rather frustrating situation: the attractively simple DFS method imposes symmetry demands that are likely to be perturbed in practice. Even though DFSs are robust with respect to such symmetry-breaking perturbations [4,16], and can be further stabilized by concatenation with QECCs [17], it is highly desirable to be able to artificially engineer conditions under which scalable DFSs may exist. Here we show how this can be accomplished, for both the collective decoherence and MQE models, by combining the DFS encoding with the BB method. While such "environment engineering" methods have been proposed before [10,18], we show here how this can be accomplished for DFSs, assuming only physically reasonable resources. In particular, we show that, by using decoupling pulses that are generated using only the isotropic Heisenberg exchange interaction, one can transform the general linear system-bath term $\sum_i H_i$ into a purely collective decoherence term. Since the Heisenberg interaction is by itself universal on the DFS-encoded qubits [15,19], this result has direct implications for the promising QC proposals that make use of the Heisenberg interaction to couple qubits [20–22], and in which the use of single-qubit operations is preferably avoided [23–25].

Dynamical decoupling and DFS.—Let us start by briefly reviewing the decoupling technique, as it pertains to our problem (for a thorough review, see, e.g., [10]). Decoupling relies on the ability to apply strong and fast (BB) pulses [6], in a manner which effectively averages H_{SB} to zero. Since the pulses are strong, one ignores the evolution under H_{SB} while the pulses are on, and, since the pulses are fast, one makes the short-time approximation, i.e., $\exp[(A + B)t] \approx \exp(At)\exp(Bt)$ for $[A, B] \neq 0$. Systematic corrections are known [6]. The simplest example of eliminating an undesired unitary evolution $U = \exp(-itH_{SB})$ is the *parity-kick sequence* [6,7]. Suppose we have at our disposal a fully controllable interaction generating a gate R such that "*R conjugates U*": $R^\dagger U R = U^\dagger$. Then the sequence $UR^\dagger UR = I$ serves to eliminate U.

Now, turning on the single-qubit Hamiltonian $\epsilon_i^x X_i$ for a time $t = \pi/2\epsilon_i^x$ generates the single-qubit gate $X_i = i\exp(-i\frac{\pi}{2}X_i)$. Each term in H_{SB} either commutes or anticommutes with X_i since each term contains at least one factor of X_i, Y_i, Z_i. *We call a term A "even" with respect to B if $[A, B] = 0$, "odd" if $\{A, B\} = 0$.* If a term A in H_{SB} is odd with respect to X_i then the evolution under it will be conjugated by the gate X_i: $X_i \exp(-iA\tau)X_i = \exp(iA\tau)$. This allows for selectively removing this term using the parity-kick cycle, which we write as $[\tau, X_i, \tau, X_i]$. Reading from right to left, this notation means the following: apply X_i pulse, free evolution for time τ, repeat. Since every system factor in H_{SB} contains a single-qubit operator, it follows that we can selectively keep or remove each term in H_{SB} by using the parity-kick cycle. Note, however, that in general we have to use a short-time approximation since $[U, R^\dagger UR] \neq 0$ [7]. Further, without additional symmetry assumptions restricting p of Eq. (1), this procedure, if used to eliminate *all errors*, requires a number of pulses that are exponential in N [8]. The reason is that without symmetry we will need at least two noncommuting single-qubit operators per qubit (e.g., X_i, Y_i).

The DFS method uses a very different idea for overcoming decoherence [2–4]. Suppose that there exist states $\{|\psi_i\rangle\}$ that are degenerate under the action of all system operators S_α in the interaction Hamiltonian $H_{SB} = \sum_\alpha S_\alpha \otimes B_\alpha$, i.e., $S_\alpha|\psi_i\rangle = c_\alpha|\psi_i\rangle \; \forall\, \alpha, i$, where c_α are constants (more general conditions can be found [5]). Such a collection of states forms a subspace that acquires only an overall phase under the action of H_{SB}, and is therefore decoherence-free. The requisite degeneracy arises from a symmetry in H_{SB}, such as collective decoherence.

Symmetrization.—We now turn to showing how decoupling may be used to create the conditions for DFSs. General group-theoretic arguments for using BB pulses for "symmetrizing" system-bath interactions, thus creating DFS conditions, were given in [10]. However, these proposals did not consider the MQE model and did not give an explicit Hamiltonian realization for the collective decoherence model. Here we focus on the MQE and collective decoherence models, and give explicit pulse sequences that respect the constraints imposed by physically available resources.

Generation of the MQE model.—The MQE model assumes that the system operators appearing in H_{SB} form an Abelian group $\mathcal{G}$ under multiplication [13]. E.g., $H_{SB} = \sum_{i=1}^{N-1} Z_i Z_{i+1} \otimes B_i$, or $H_{SB} = \sum_{i=1}^{N/2} \sum_{\alpha=x,y,z} \sigma_{2i-1}^\alpha \sigma_{2i}^\alpha \otimes B_i^\alpha$. The dimension of the DFS supported by $\mathcal{G}$ is $2^N/|\mathcal{G}|$, where $|\mathcal{G}|$ is the order of $\mathcal{G}$, which also counts the number of independent errors the DFS is immune to [13]. An Abelian group with M generators has order 2^M. Universal, fault-tolerant quantum computation can be performed on the MQE class of DFSs using the method of [26]. Briefly, this method involves a hybrid DFS-QECC approach, wherein logic gates acting on DFS states are supplemented with fault-tolerant error detection and recovery. This active QECC intervention is needed since, unlike the collective decoherence case treated below, in the MQE case logic gates take encoded states on a trajectory that begins inside the DFS, leaves it, and then returns (as is also the case for computation using QECCs). Let us now show how to generate the MQE conditions starting from a $p = 2$ system-bath Hamiltonian.

We assume that *single-qubit gates are available*. In this case, it has been shown that the linear term, $\sum_i H_i$, can be eliminated using four pulses, each acting simultaneously on all qubits [8,9]. We reproduce this result and show further how single-qubit gates can efficiently decouple bilinear Hamiltonians H_{ij} with nearest-neighbor interactions. Let $H_{\rm NN} = \sum_{i=1}^N H_i + H_{i,i+1}$ and $U_{\rm NN} = \exp(-i\tau H_{\rm NN})$. Define collective rotation operators

$$R = R_1 R_2 \ldots R_N, \qquad R_O = R_1 R_3 \ldots R_{N/2-1}, \qquad (3)$$

where N is even and R can be X, Y, or Z. First note that $U'_{\rm NN} = U_{\rm NN}(XU_{\rm NN}X)$ leaves only those linear terms containing X_i, and all bilinear terms of the form $\sigma_i^\alpha \sigma_j^\alpha$, $Y_i Z_j$. Let us apply Z to the outcome, i.e., $U''_{\rm NN} = U'_{\rm NN}(ZU'_{\rm NN}Z)$. This eliminates all linear terms in four pulses: $[\tau, Y, \tau, X, \tau, Y, \tau, X]$ (where we have used $Y = -iXZ$). It also eliminates all $Y_i Z_j \otimes B_{ij}^{yz}$, leaving just $\sum_j \sigma_j^\alpha \sigma_{j+1}^\alpha \otimes B_{j,j+1}^{\alpha\alpha}$ ($\alpha = x, y, z$). We can rewrite this as $\sum_j \sigma_j^\alpha \sigma_{j+1}^\alpha \otimes B_{j,j+1}^{\alpha\alpha} = \sum_{j=\rm odd} \vec{\sigma}_j \cdot \vec{B}'_j$, i.e., the even-numbered qubits act as baths for the odd-numbered qubits. Now let $U'''_{\rm NN} = U''_{\rm NN}(X_O U''_{\rm NN} X_O)$, which requires eight pulses. At this point we are left with only errors of the form $X_j X_{j+1}$. These generate an Abelian group denoted Q_{2X} in [26]. Since Q_{2X} has $N - 1$ generators its order is $|Q_{2X}| = 2^{N-1}$, so that the DFS is $2^N/2^{N-1} = 2$ dimensional, i.e., supports a single encoded qubit, which of course is not scalable. A larger number of encoded qubits can be supported by reducing the dimension of the error group. This, in turn, requires a few more pulses. E.g., consider applying the BB pulse $Z_3Z_4Z_7Z_8\ldots$ to $U'''_{\rm NN}$. This eliminates $X_2X_3, X_4X_5, X_6X_7, \ldots$. What is left, after 16 pulses, is the error group generated by $\{X_{2j-1}X_{2j}\}_{j=1}^{N/2}$, denoted Q_X in [13]. It has order $|Q_X| = 2^{N/2}$, thus supporting a $2^N/2^{N/2} = 2^{N/2}$ dimensional DFS. This DFS encodes $N/2$ qubits, so it is scalable. The methods of [26] now apply for the purpose of fault-tolerant universal quantum computation using the hybrid DFS-QECC method.

Note that we can also go further and eliminate all second order coupling terms: $U'''_{\rm NN}(Z_O U'''_{\rm NN} Z_O) = I$ and also uses a total of 16 collective pulses. If there is a next-nearest-neighbor system-bath interaction, it can similarly be removed using the collective operator $R_{OO} = R_1R_5R_9R_{13}\ldots$, etc. for longer-range interactions. These manipulations will leave higher order MQE models; which option to choose will depend on which pulse sequences are most easily implementable. It should be clear that this method of generating MQE models is quite general: given a system-bath Hamiltonian, one can design a set of BB pulses that will transform this Hamiltonian into a desired Abelian error group. The number of pulses will scale with the system-bath coupling order p [Eq. (1)] and the interaction range r ($r = 1$ for nearest neighbors, etc.), and we have given explicit sequences for the case $p, r \leq 2$. We note that this combination of decoupling with a hybrid DFS-QECC strategy is, as far as we know, the first time that *all three methods for combatting decoherence have been used together*.

As a final comment on generating MQE models, we note that the analysis above applies also to the case where it may be preferable to control two-qubit "product" Hamiltonians of the form X_iX_j and Y_iY_j. Similar to the case of controllable single-qubit gates, we now have the gates $X_iX_j = i\exp(-i\pi X_iX_j/2)$ and $Y_iY_j = i\exp(-i\pi Y_iY_j/2)$. Such gates could be implemented naturally, e.g., in certain superconducting QC implementations [27]. It is simple to check that the product Hamiltonians can be used to decouple any linear system-bath Hamiltonian and any bilinear term other than $\sigma_i^\alpha\sigma_j^\alpha$. In fact, we can construct the R gates [Eq. (3)] by simply turning on all nearest-neighbor gates $\sigma_j^\alpha\sigma_{j+1}^\alpha$, and the R_O gates by simultaneously turning on all next-nearest-neighbor gates, $\sigma_j^\alpha\sigma_{j+2}^\alpha$. Therefore the methods used above for the case of single-qubit gates apply directly.

Generating collective dephasing—Let us consider permutation-symmetric system-bath interactions. We focus first on the simplified case of a linear dephasing model, $H_{SB}^{\rm deph} = \sum_{i=1}^N Z_i \otimes B_i^z$. We show how to generate from it a block-collective dephasing model, $B_{2i-1}^z = B_{2i}^z \equiv \tilde{B}_i^z$, wherein each block supports a DFS of the form Span$[\{|0_{2i-1}1_{2i}\rangle, |1_{2i-1}0_{2i}\rangle\}]$ [2,19]. A controllable *XY*- or *XXZ*-type system Hamiltonian can provide a universal set of encoded logic operations for the DFS-encoded qubits, where the encoded σ^x generator is [24] $\overline{X}_j = (X_{2j-1}X_{2j} + Y_{2j-1}Y_{2j})/2$. Now $\exp[-i(B_1^zZ_1 + B_2^zZ_2)\tau]\exp(i\frac{\pi}{2}\overline{X}_1)\exp[-i(B_1^zZ_1 + B_2^zZ_2)\tau]\exp(-i\frac{\pi}{2}\overline{X}_j) = \exp[-i(B_1^z + B_2^z)(Z_1 + Z_2)\tau]$. Thus two BB pulses suffice to symmetrize $H_{SB}^{\rm deph}$ so that *only a block-collective dephasing component remains*.

Generating collective decoherence.—We now turn to collective decoherence. To do so, we consider the important case of a controllable Heisenberg exchange Hamiltonian $J_{ij}\vec{\sigma}_i \cdot \vec{\sigma}_j$, crucial for the operation of, e.g., quantum dot QCs [20] and donor atom nuclear [21] or electron [22] spin QCs. We will show that, *by using a few collective pulses generated by the Heisenberg interaction alone, we can symmetrize any linear system-bath interaction* $\sum_i \vec{\sigma}_i \cdot \vec{B}_i$, *such that only a block-collective component remains*. This collective decoherence can then be avoided by encoding into a four-qubit DFS [2] or a three-qubit DFS/noiseless subsystem (NS) [5]. Let

$$O_{ij} \equiv \exp(-i\pi\vec{\sigma}_i \cdot \vec{\sigma}_j/4).$$

A simple calculation shows that O_{ij} is a SWAP gate for the Pauli matrices: $O_{ij}^\dagger\vec{\sigma}_iO_{ij} = \vec{\sigma}_j$. From this follows $O_{ij}^\dagger(\vec{\sigma}_i \pm \vec{\sigma}_j)O_{ij} = \pm(\vec{\sigma}_i \pm \vec{\sigma}_j)$, i.e., all *differences* $\vec{\sigma}_i - \vec{\sigma}_j$ are odd with respect to O_{ij}, and hence can be eliminated. Let us then rewrite $\sum_{i=1}^N \vec{\sigma}_i \cdot \vec{B}_i = \sum_{\beta=\pm}\sum_{j=1}^{N/2}(\vec{\sigma}_{2j} + \beta\vec{\sigma}_{2j-1}) \cdot \vec{B}_{2j}^\beta$, where $\vec{B}_{2j}^\pm \equiv (\vec{B}_{2j} \pm \vec{B}_{2j-1})/2$. To eliminate the nearest-neighbor differences $(\vec{\sigma}_{2j} - \vec{\sigma}_{2j-1})$ we can use the collective BB pulse $O = \bigotimes_{j=1}^{N/2} O_{2j-1,2j}$. This leaves only the (block-)collective decoherence term $\sum_{j=1}^{N/2}(\vec{\sigma}_{2j} + \vec{\sigma}_{2j-1}) \cdot \vec{B}_j^+$, which in turn we can rewrite as $\sum_{\beta=\pm}\sum_{j=1}^{N/2}[\vec{\sigma}_{2j+2} + \vec{\sigma}_{2j+1} + \beta(\vec{\sigma}_{2j} + \vec{\sigma}_{2j-1})] \cdot \vec{B}_{2j}^{+,\beta}$, where $\vec{B}_{2j}^{+,\pm} \equiv (\vec{B}_{2j+2}^+ \pm \vec{B}_{2j}^+)/2$. We can now eliminate the next-nearest-neighbor differences $(\vec{\sigma}_{2j+2} - \vec{\sigma}_{2j})$ and $(\vec{\sigma}_{2j+1} - \vec{\sigma}_{2j-1})$ using a second collective pulse $O_O = \bigotimes_{j=1}^{N/2-1} O_{2j-1,2j+1}O_{2j,2j+2}$. At this point we are left with just the collective decoherence terms on blocks of four qubits, and the encoding into the four-qubit DFS [2] becomes relevant. This scheme uses a total of six collective BB pulses: $[O_O, O, \tau, O^\dagger, \tau, O_O^\dagger, O, \tau, O^\dagger]$. The apparent drawback of needing next-nearest-neighbor interactions (in the O_O pulse) can be avoided by swapping local gates, e.g., $O_{i,i+2} = O_{i+1,i+2}^\dagger O_{i,i+1}O_{i+1,i+2}$, at the expense of more pulses. Note that in a 2D hexagonal arrangement, $i, i+1$ and $i, i+2$ can all be nearest neighbors.

We can also symmetrize into blocks of three, which can be used for the three-qubit DFS/NS [5]. Let us rewrite $H_3 = \sum_{i=1}^3 \vec{\sigma}_i \cdot \vec{B}_i = (\vec{\sigma}_1 + \vec{\sigma}_2 + \vec{\sigma}_3) \cdot \vec{A}^+ + (\vec{\sigma}_2 - \vec{\sigma}_1) \cdot \vec{A}^- + \vec{\sigma}_3 \cdot \vec{C}$, where $\vec{A}^\pm \equiv (\vec{B}_2 \pm \vec{B}_1)/2$ and $\vec{C} \equiv \vec{B}_3 - \vec{A}^+$. We can eliminate $\vec{\sigma}_2 - \vec{\sigma}_1$ using O_{12}: $U_1(\tau) = \exp(-iH_3\tau)O_{12}^\dagger\exp(-iH_3\tau)O_{12} = \exp[-2i\tau((\vec{\sigma}_1 + \vec{\sigma}_2 + \vec{\sigma}_3) \cdot \vec{A}^+ + \vec{\sigma}_3 \cdot \vec{C})]$. Next consider

$$U_2(\tau) = U_1(\tau/2)O_{23}^{\dagger}U_1(\tau)O_{23} = e^{-i\tau[3(\vec{\sigma}_1+\vec{\sigma}_2+\vec{\sigma}_3)\cdot\vec{A}^{+}+(2\vec{\sigma}_2+\vec{\sigma}_3)\cdot\vec{C}]}.$$

Finally,

$$U_2(\tau)O_{12}^{\dagger}U_2(\tau)O_{12} = e^{-2i\tau(\vec{\sigma}_1+\vec{\sigma}_2+\vec{\sigma}_3)\cdot(\vec{B}_1+\vec{B}_2+\vec{B}_3)},$$

leaving only the collective component. This scheme uses a total of 14 collective BB pulses.

To quantum compute universally on DFSs it is necessary to couple blocks of DFS qubits, in order to implement a controlled-logic gate [15,19]. An extra symmetrization step is thus required, creating collective decoherence conditions over blocks of six or eight qubits. This is a simple extension of the procedure above. E.g., for two coupled four-qubit DFS blocks, we need collective pulses of the form $O_{OO} = \bigotimes_{j=1}^{N/2-3} O_{2j-1,2j+3}O_{2j,2j+4}O_{2j+1,2j+5}O_{2j+2,2j+6}$. By swapping local gates we can again avoid direct control over long-range interactions. The corresponding increase in the number of gates may well be a worthwhile trade-off. Similar pulse sequences can be found for creating block-collective decoherence conditions over six qubits, for computation using the three-qubit NS.

Discussion and conclusions.—The prospect of decoherence-free quantum computation is very attractive, but, so far, ideas for obtaining the conditions enabling the existence of scalable decoherence-free subspaces focused mostly on lowering the temperature and neglecting other sources of decoherence [14,15]. The exception is previous existential results showing how decoupling methods can be used for symmetrization of system-bath interactions [10]. In this Letter we showed explicitly how conditions for the two most important examples of DFSs (the models of collective decoherence [2–4] and multiple qubit errors [13]) can be actively generated using symmetrizing cycles of fast and strong decoupling (bang-bang) pulses. In the MQE case a cycle of 16 pulses suffices to symmetrize a system-bath Hamiltonian with arbitrary linear and nearest-neighbor bilinear couplings, such that conditions enabling the existence of a scalable DFS are established. This result is applicable for quantum computer proposals where single-qubit gates are easily tunable. In this case a hybrid DFS-active quantum error correction scheme, developed in [26], can be used for universal, fault-tolerant quantum computation.

In the case of collective decoherence a very attractive general picture is emerging, from a combination of previous studies and this work. The collective decoherence model was first proposed as an example allowing the existence of DFSs with the property of scalable encoding [2–4]. It was later realized that universal quantum computation is possible on these DFSs using the Heisenberg exchange interaction alone [15,19]. Our present result shows how *collective decoherence conditions can be actively created with a few pulses generated using only Heisenberg exchange*. Since our method relies on BB pulses, some degree of leakage out of the DFS (due to imperfect symmetrization) is inevitable. Fortunately, such leakage errors can also be reduced using Heisenberg-generated BB pulses [28], or can be detected with a circuit that utilizes, again, only Heisenberg exchange [25]. The combination of all these results shows that *Heisenberg exchange is by itself an enabler of universal fault-tolerant quantum computation on decoherence-free subspaces*. This has potentially important applications for those solid-state proposals of QCs, where Heisenberg exchange is the natural qubit-qubit coupling mechanism [20–22]. The combination of the decoupling method with encoding methods developed in the quest to protect fragile quantum information thus seems to be a promising route towards robust implementations of QCs.

The present study was sponsored by the DARPA-QuIST program (managed by AFOSR under Agreement No. F49620-01-1-0468), and by PRO (to D. A. L.).

[1] P. W. Shor, Phys. Rev. A **52**, 2493 (1995); A. M. Steane, Phys. Rev. Lett. **77**, 793 (1996).
[2] P. Zanardi and M. Rasetti, Phys. Rev. Lett. **79**, 3306 (1997).
[3] L.-M Duan and G.-C. Guo, Phys. Rev. A **57**, 737 (1998).
[4] D. A. Lidar *et al.,* Phys. Rev. Lett. **81**, 2594 (1998).
[5] E. Knill *et al.,* Phys. Rev. Lett. **84**, 2525 (2000).
[6] L. Viola and S. Lloyd, Phys. Rev. A **58**, 2733 (1998).
[7] D. Vitali and P. Tombesi, Phys. Rev. A **59**, 4178 (1999).
[8] L. Viola *et al.,* Phys. Rev. Lett. **82**, 2417 (1999).
[9] L. Viola *et al.,* Phys. Rev. Lett. **83**, 4888 (1999).
[10] P. Zanardi, Phys. Lett. A **258**, 77 (1999); L. Viola *et al.,* Phys. Rev. Lett. **85**, 3520 (2000).
[11] M. S. Byrd and D. A. Lidar, quant-ph/0110121.
[12] D. G. Cory *et al.,* Phys. Rev. Lett. **81**, 2152 (1998); D. Kielpinski *et al.,* Science **291**, 1013 (2001).
[13] D. A. Lidar *et al.,* Phys. Rev. A **63**, 022306 (2001).
[14] P. Zanardi and F. Rossi, Phys. Rev. Lett. **81**, 4752 (1998).
[15] D. Bacon *et al.,* Phys. Rev. Lett. **85**, 1758 (2000).
[16] D. Bacon *et al.,* Phys. Rev. A **60**, 1944 (1999).
[17] D. A. Lidar *et al.,* Phys. Rev. Lett. **82**, 4556 (1999).
[18] J. F. Poyatos *et al.,* Phys. Rev. Lett. **77**, 4728 (1996).
[19] J. Kempe *et al.,* Phys. Rev. A **63**, 042307 (2001).
[20] D. Loss and D. P. DiVincenzo, Phys. Rev. A **57**, 120 (1998); J. Levy, Phys. Rev. A **64**, 052306 (2001).
[21] B. E. Kane, Nature (London) **393**, 133 (1998); A. Khitun *et al.,* Phys. Rev. A **64**, 062304 (2001).
[22] R. Vrijen *et al.,* Phys. Rev. A **62**, 012306 (2000).
[23] D. P. DiVincenzo *et al.,* Nature (London) **408**, 339 (2000).
[24] L.-A. Wu and D. A. Lidar, Phys. Rev. A **65**, 042318 (2002); D. A. Lidar and L.-A. Wu, Phys. Rev. Lett. **88**, 017905 (2001).
[25] J. Kempe *et al.*, quant-ph/0112013.
[26] D. A. Lidar *et al.*, Phys. Rev. A **63**, 022307 (2001).
[27] T. P. Spiller, Fortschr. Phys. **48**, 1075 (2000).
[28] L.-A. Wu *et al.*, quant-ph/0202168.