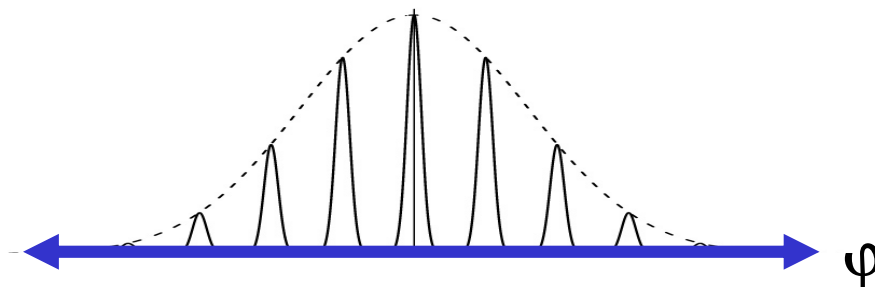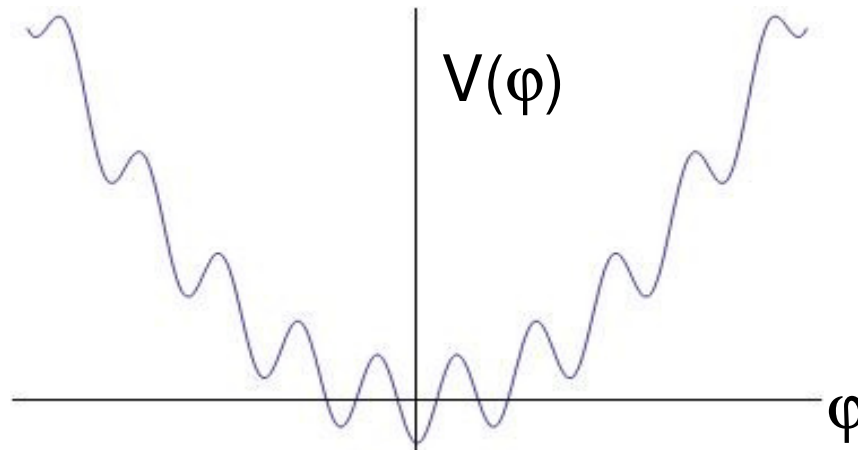# Protected gates for superconducting qubits

John Preskill
with Peter Brooks
and Alexei Kitaev
QEC 2011
5 December 2011

Institute for Quantum
Information and Matter

Quantum fault tolerance

• Error correction and fault tolerance will be essential for operating large-scale quantum computers.

• In the standard "software" approach to fault-tolerant quantum computing, the deficiencies of quantum hardware are overcome (if the hardware is not *too* noisy) through clever circuit design.

• In the alternative "topological" approach, the hardware itself is intrinsically resistant to decoherence (if operated at a temperature well below the mass gap).

• Both approaches exploit the idea that logical qubits can be stored and processed reliably when encoded in a quantum system with many degrees of freedom.

• Even if topological quantum computing achieves quantum gates with a low error rate, we may still need to combine topological processing with the software approach to perform long computations with an acceptably low error probability.

• Meanwhile, it is important to search for new ways to build quantum hardware with intinsic robustness resulting from the physical encoding.

Protected superconducting qubit

Feigel'man & Ioffe
Doucot & Vidal
Kitaev

Physically robust encodings have been proposed using superconducting circuits containing Josephson junctions, for example the "0-Pi qubit". The circuit's energy $E(\theta)$, as a function of the superconducting phase difference θ between its leads, is a periodic function with period π to an excellent approximation.
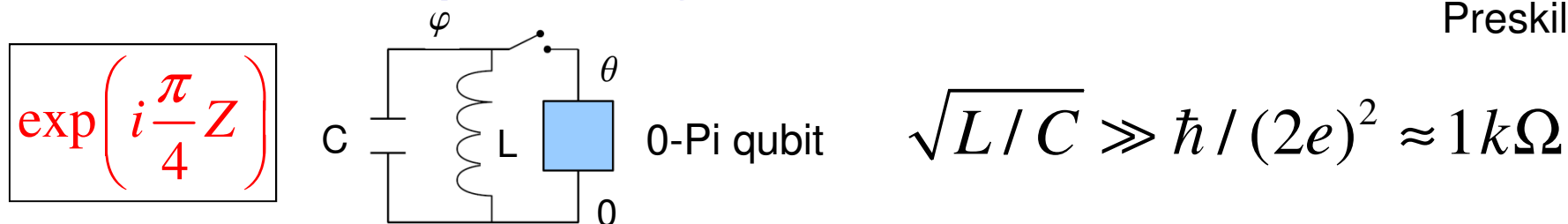
"0-Pi qubit":
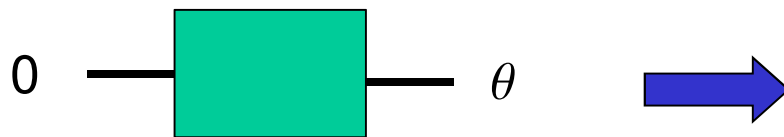
$$E \approx f(2\theta) + O\left(\exp\left(-c(\text{size})\right)\right)$$
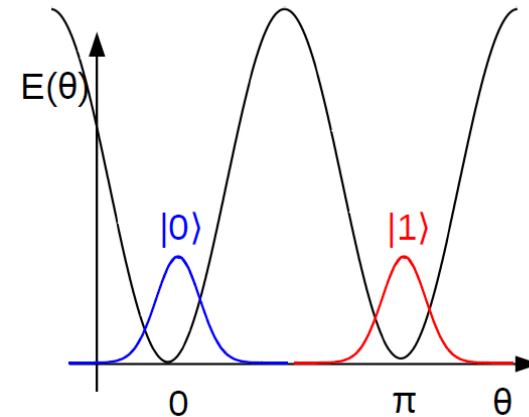
Two states localized near $\theta=0$ and $\theta=\pi$ are the basis states of a protected qubit. The barrier is high enough to suppress bit flips, and the stable degeneracy suppresses phase errors. Protection arises because the encoding of quantum information is highly nonlocal, and splitting of degeneracy scales exponentially with size of the device.

Protected phase gate

Brooks,
Kitaev,
Preskill

$$\exp\left(i\frac{\pi}{4} Z \right)$$

φ

θ

C   L   0-Pi qubit

0

$$\sqrt{L/C} \gg \hbar/(2e)^2 \approx 1 k\Omega$$

For reliable quantum computing, we need not just very stable qubits, but also the ability to apply very accurate nontrivial quantum gates to the qubits.

In this talk, I'll describe how accurate (Clifford group) phase gates can be applied to 0-Pi qubits by turning on and off the coupling between a qubit (or pair of qubits) and a harmonic oscillator (an LC circuit whose inductance is large in natural units). In principle the gate error becomes exponentially small as the inductance grows.

The reliability of the gate arises from a continuous-variable quantum error-correcting code underlying its operation, in which a qubit is embedded in the infinite-dimensional Hilbert space of a harmonic oscillator. Coupling the 0-Pi qubit to the oscillator sends the oscillator on a state-dependent phase space excursion during which it acquires a geometric phase that is protected by the code.

Josephson junction

Superconducting island contains Q Cooper pairs, each with electrical charge 2e. A Cooper pair can tunnel through the junction, increasing the charge from n to n+1. Tunneling Hamiltonian:

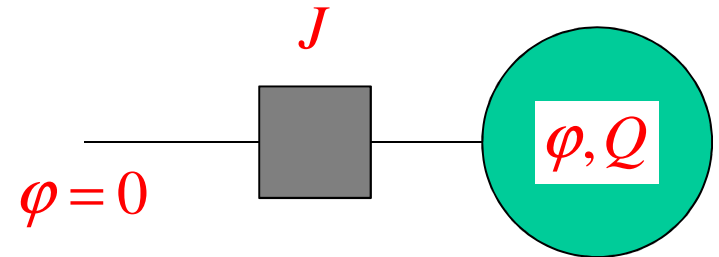$$H = -\frac{1}{2} J \sum_n \left(|Q+1\rangle\langle Q| + h.c. \right)$$

The charge Q (in units of 2e), defined relative to a background charge, can be positive or negative (and large). Conjugate basis:

$$|\varphi\rangle = \frac{1}{\sqrt{2\pi}} \sum_{Q=-\infty}^{\infty} e^{-iQ\varphi} |Q\rangle, \quad \langle \varphi | \varphi' \rangle = \delta(\varphi - \varphi').$$

Then $|Q+1\rangle\langle Q| = e^{i\varphi}$ and $H = -J\cos\varphi$

Note that Q and φ are noncommuting canonical variables (Cf., momentum and position):

$$Q = -i\frac{d}{d\varphi}, \quad [\varphi, Q] = i$$

Flux quantization

Inside a superconductor, we have the freedom to change our conventions for defining phase at each point in space. The electromagnetic vector potential is a "connection" defining a notion of parallel transport of the phase from one point to another.

$$\varphi(y) = \exp\left(-i\frac{2e}{\hbar}\int_x^y A \cdot dx\right)\varphi(x)$$

If the magnetic field is nonzero, then transport is path dependent, i.e. there is *curvature*. It is energetically unfavorable for the transport around a close path to produce a nontrivial phase. Persistent current flows to augment the applied B field enclosed by the ring, so

$$\Phi = \oint A \cdot dx = 2\pi m \frac{\hbar}{2e} = m\Phi_0, \quad \Phi_0 = \frac{h}{2e}$$

For a superconducting ring with a Josephson junction, if magnetic flux does not leak out (no "phase slips"), then we may think of the variable φ (the phase difference across the junction) as a real variable rather than a periodic variable with period 2π ; when φ winds by 2π , the enclosed flux increases by one flux quantum.
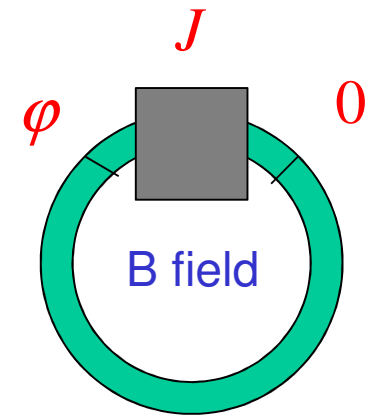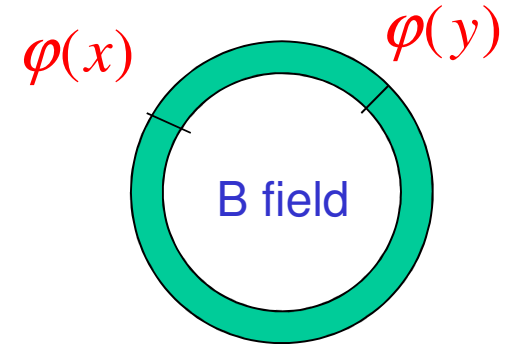
Capacitance and inductance

φ

A (superconducting) circuit has capacitance (Coulomb energy) and inductance (magnetic field energy).

$$H = \frac{q^2}{2C_{\text{conv}}} + \frac{\Phi^2}{2L_{\text{conv}}} = \frac{Q^2}{2C} + \frac{\varphi^2}{2L}$$
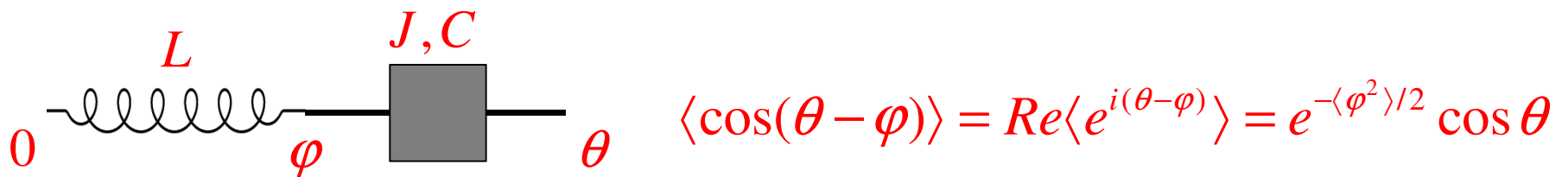
C L

0
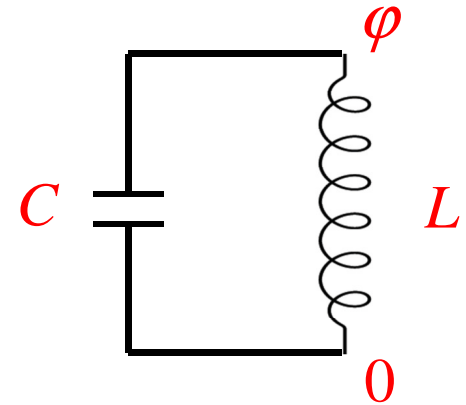
where

$$\sqrt{L/C} = \sqrt{L_{\text{conv}}/C_{\text{conv}}}/(\hbar/4e^2) = \sqrt{L_{\text{conv}}/C_{\text{conv}}}/1.03 \ k\Omega$$

A harmonic oscillator with Gaussian ground state:

$$|\psi_{\text{gr}}(\varphi)|^2 \propto e^{-\varphi^2/2\langle\varphi^2\rangle}, \quad \langle\varphi^2\rangle = \frac{1}{2}\sqrt{\frac{L}{C}}$$

For a "superinductor" with (L/C)$^{1/2}$ >> 1 (which is hard to achieve experimentally), the phase φ has large fluctuations in the ground state.

L J, C

0 φ θ

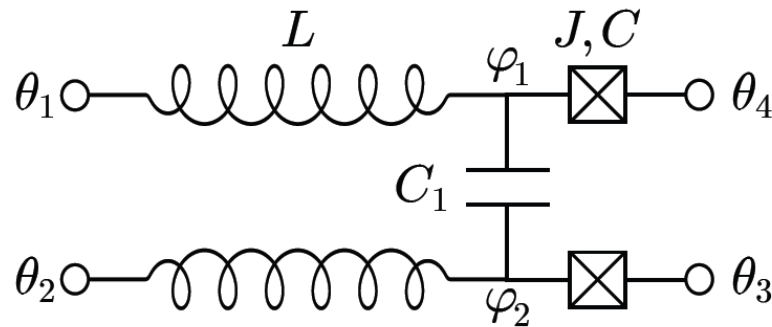$$\langle\cos(\theta - \varphi)\rangle = Re\langle e^{i(\theta-\varphi)}\rangle = e^{-\langle\varphi^2\rangle/2}\cos\theta$$

Exponentially weak sensitivity to the phase difference, due to averaging over many wiggles of the cosine Josephson energy.

Protected superconducting qubit
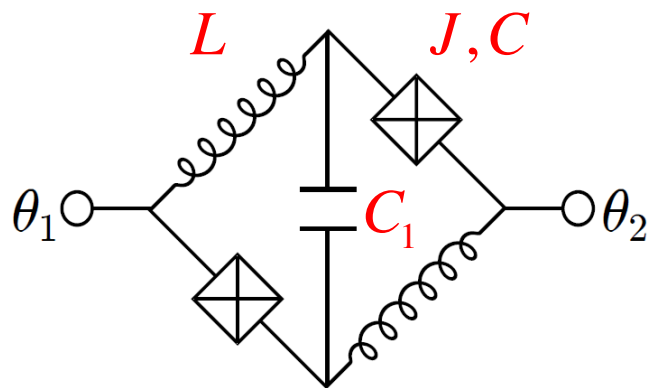
Feigel'man & Ioffe
Doucot & Vidal
Kitaev

Inductance is large:

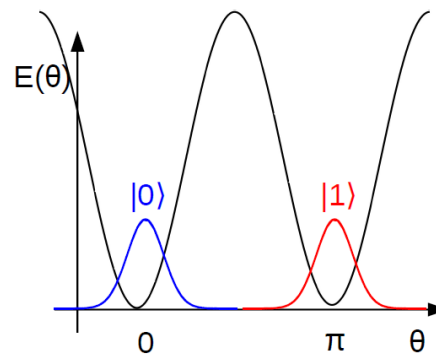$$\sqrt{L/C} \gg \hbar/(2e)^2 \approx 1 k\Omega$$

$$C_1 \gg C$$

The phase $\varphi_+ = (\varphi_1 + \varphi_2)/2$ is "light" (has large fluctuations) but the difference $\varphi_- = \varphi_1 - \varphi_2$ is "heavy" and locks to external phase

$$(\theta_4 - \theta_1) - (\theta_3 - \theta_2) = (\theta_4 + \theta_2) - (\theta_1 + \theta_3) \longrightarrow 2(\theta_2 - \theta_1)$$

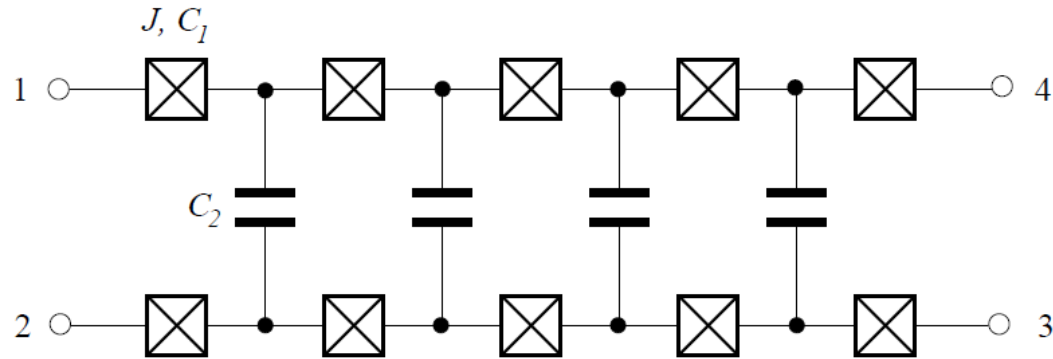$$E \approx f(2(\theta_2 - \theta_1)) + O\left(\langle \cos \varphi_+ \rangle\right)$$

$$\approx f(2(\theta_2 - \theta_1)) + O\left(\exp(-(1/8)\sqrt{L/C})\right)$$

"0-Pi qubit"

Protected superconducting qubit

How is this scheme related to topological protection? Kitaev proposed to realize a large inductance using a long chain of Josephson junctions.

In this case, the phase change along the chain is distributed among many devices, and the information that distinguishes the basis states of the qubit is not locally accessible, because of phase fluctuations along the chain.

Protection arises because the encoding of quantum information is highly nonlocal. Splitting of the degeneracy, associated with quantum tunneling from one end of the chain to the other, scales exponentially with the size of the device.
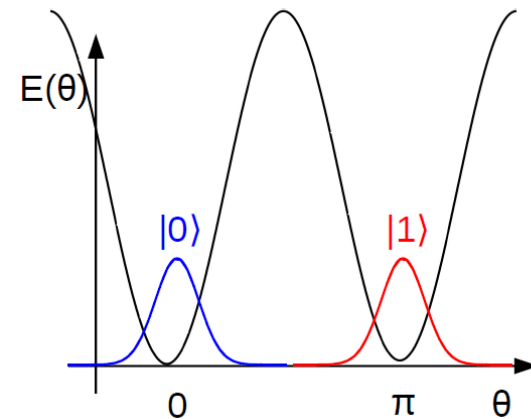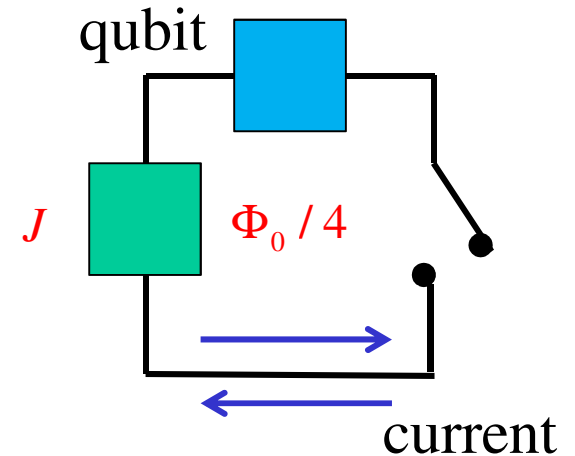

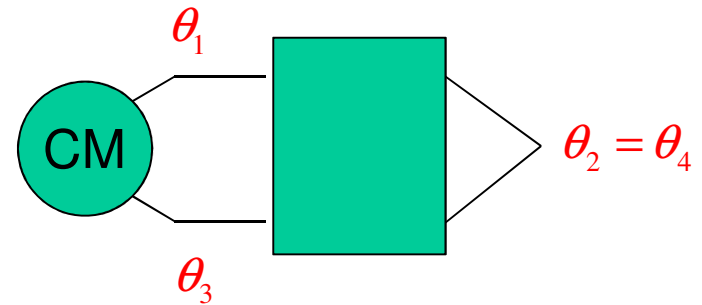$$E \approx f(2\theta) + O\left(\exp\left(-c(\text{size})\right)\right)$$

Measurement

To measure in the Z basis (distinguish phase difference 0 and π across the qubit), couple to a junction, with ¼ of a flux quantum linking loop. Observe direction of current flow.

qubit

J $\Phi_0 / 4$

current

$$I \propto \sin(\theta), \quad \theta = \frac{\pi}{2} + \{0 \text{ or } \pi\}$$

Measurement of X is a charge measurement. Break the connection between θ_1 and θ_3, and measure the charge dual to θ_1 - θ_3.

θ_1

CM $\theta_2 = \theta_4$

θ_3

$$\boxed{E \approx f(\theta_1 + \theta_3 - 2\theta_2)}$$

As θ_1 winds from 0 to 2π with θ_3 fixed, θ_2 winds by π. Thus either the wave function is invariant (for X = 1), or it changes sign (for X = -1). Correspondingly, the dual charge is either an even or odd multiple of ½.

The measurements may be noisy, but can be made more robust by repeating or by coding (more later).
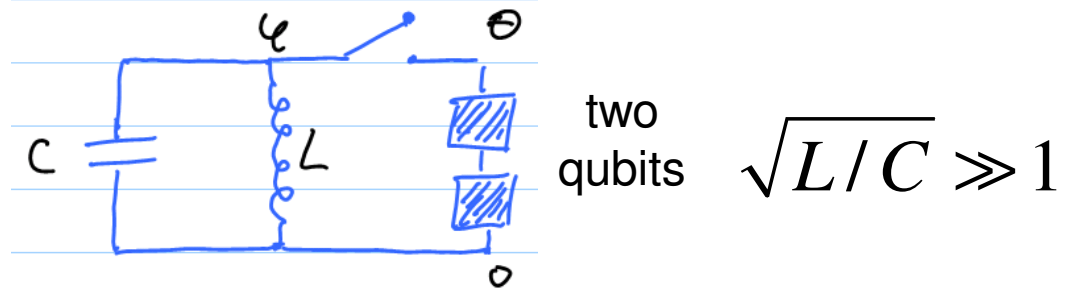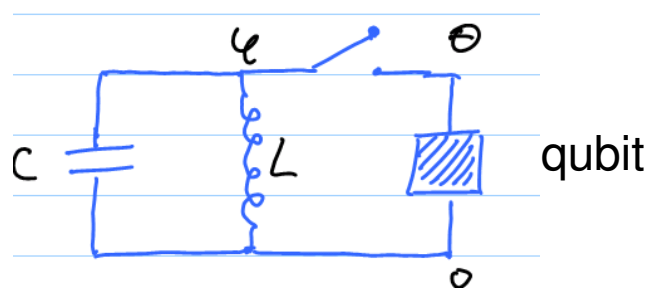
Protected superconducting qubit

Kitaev,
Brooks,
Preskill

Some gates are also protected: we can execute

$$\exp\left(i\frac{\pi}{4}Z\right) \text{ and } \exp\left(i\frac{\pi}{4}Z_1 \otimes Z_2\right)$$

with exponential precision. This is achieved by coupling a qubit or a pair of qubits to a "superinductor" with large phase fluctuations:

qubit

two qubits $\sqrt{L/C} \gg 1$

To execute the gate, we (1) close the switch, (2) keep it closed for awhile, (3) open the switch. This procedure alters the relative phase of the two basis states of the qubit:

$$\left(a|0\rangle + b|1\rangle\right) \otimes |\text{init}\rangle \rightarrow \left(a|0\rangle + be^{-i\alpha}|1\rangle\right) \otimes |\text{final}\rangle$$

time switch
Is closed
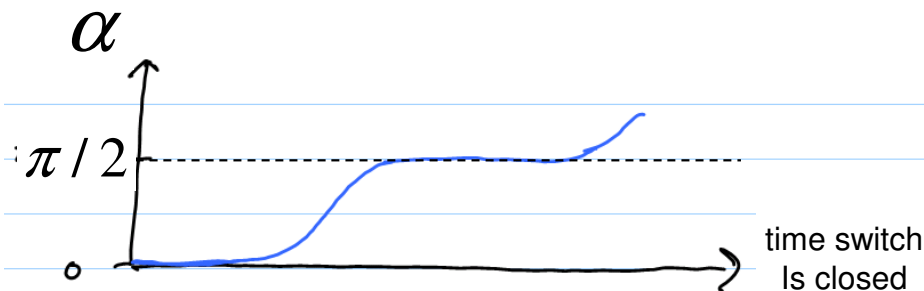
The relative phase induced by the gate "locks" at $\pi/2$. For $\sqrt{L/C} \approx 80$ phase error ~ 10^{-8} is achieved for timing error of order 1 percent. Why?

A qubit encoded in an oscillator   Gottesman, Kitaev, Preskill

This is a stabilizer code, generated by: $M_Z = e^{2i\varphi}, \quad M_X = e^{-2\pi iQ}$

With logical operators: $\bar{Z} = e^{i\varphi}, \quad \bar{X} = e^{-i\pi Q}, \quad [\varphi, Q] = i$

Note $e^A e^B = e^{[A,B]} e^B e^A$ Hence, logical ops commute with stabilizer.
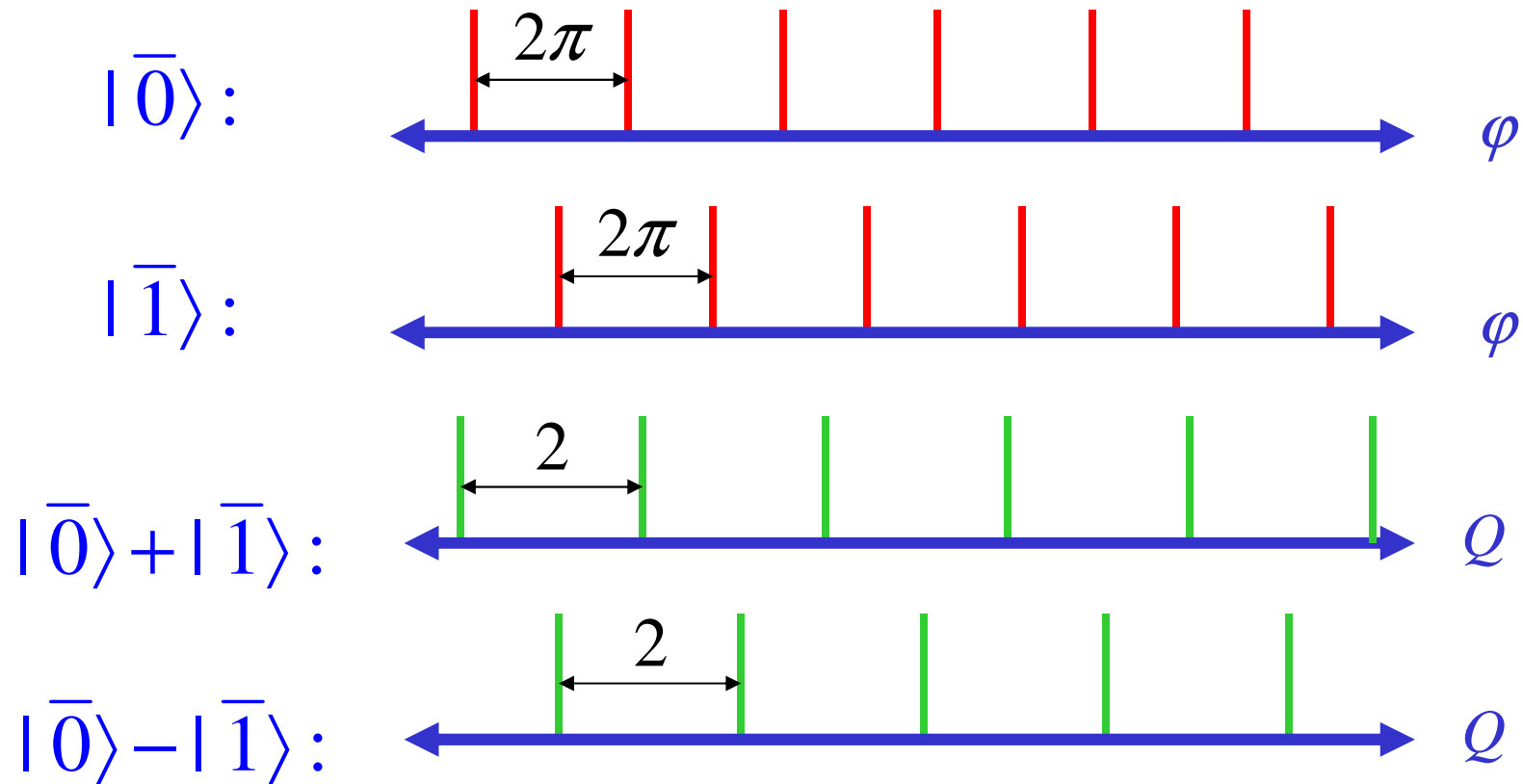
This code can correct all shifts that satisfy: $|\Delta\varphi| < \dfrac{\pi}{2}, \quad |\Delta Q| < \dfrac{1}{2}.$

Finitely squeezed codewords

Realistic codewords are *normalizable, finitely squeezed* in φ and Q:

Not

φ

but

φ

Δ

κ^{-1}
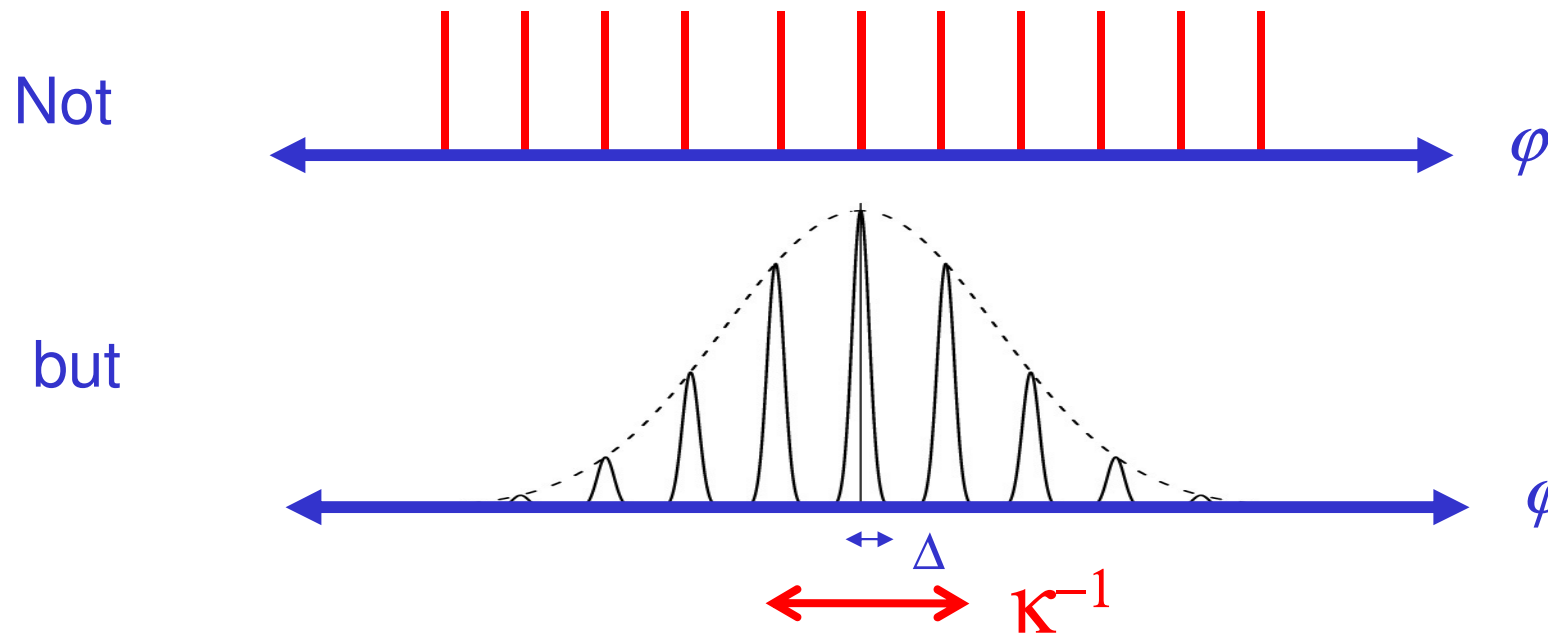
Approximate code word in φ–space is a coherent superposition of squeezed Gaussians, each with width Δ, governed by a Gaussian envelope with width κ^{-1}. In Q-space, the sharp peaks have width κ, and the envelope has width Δ^{-1}.

Small shifts in φ and Q can still be corrected with high probability. For Δ and κ smaller than the largest correctable shifts by a factor of 5, the intrinsic failure probability due to the finite squeezing is less than 10^{-6}.

(The peak functions and envelope functions need not actually be Gaussian.)

Protected phase gate

$$\exp\left(i\frac{\pi}{4}Z\right)$$

0-Pi qubit
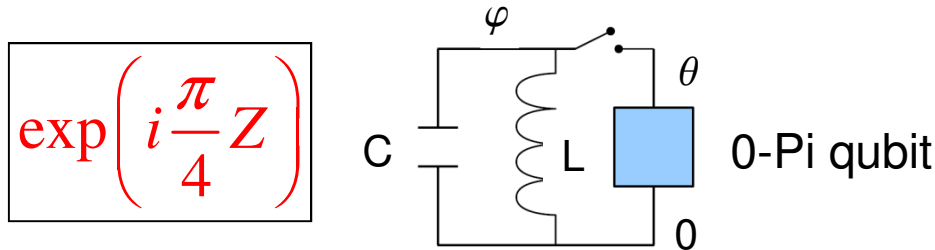
$$\sqrt{L/C} \gg \hbar/(2e)^2 \approx 1 k\Omega$$

Switch is really a tunable Josephson junction:

$$H = \frac{Q^2}{2C} + \frac{\varphi^2}{2L} - J(t)\cos(\varphi - \theta)$$

Under suitable adiabaticity conditions, closing the switch transforms a broad oscillator state (e.g. the ground state) into a grid state (approximate codeword).

V(φ)

φ

φ

Δ

κ^{-1}
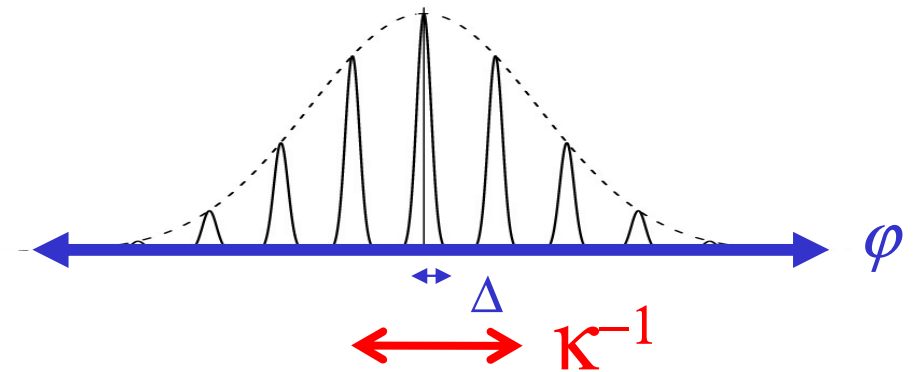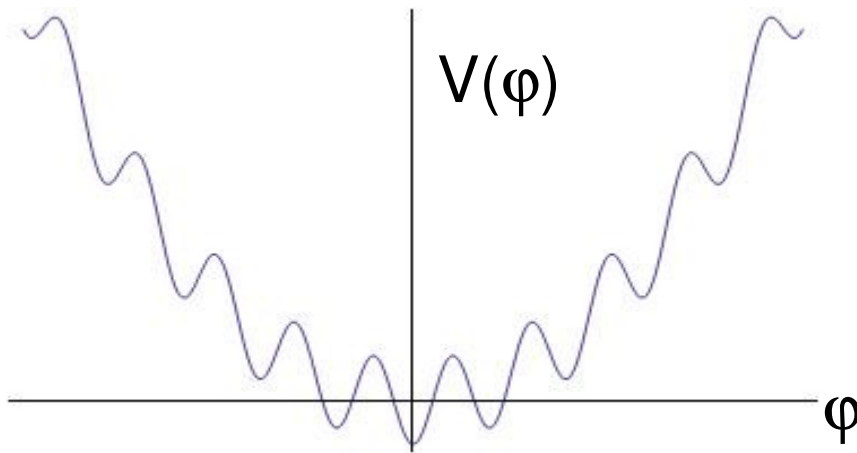
Peaks are at even or odd multiples of π depending on whether θ is 0 or π, i.e. on whether qubit is 0 or 1. Inner width squared is (JC)$^{-1/2}$ and outer width is (L/C)$^{1/2}$

$$\omega_J^{-1} = \sqrt{C/J} \ll \text{switching time} \ll \omega^{-1} = \sqrt{LC} \gg 1$$

Protected phase gate

$$H = \frac{Q^2}{2C} + \frac{\varphi^2}{2L} - J(t)\cos(\varphi - \theta)$$

Codewords evolve while switch is closed.

$$\left|\varphi\right\rangle \rightarrow e^{-i\tau\varphi^2/2L}\left|\varphi\right\rangle = e^{-i\varphi^2/2\pi}\left|\varphi\right\rangle, \text{ for } \tau = L/\pi.$$

Therefore,

$$\left|0\right\rangle : \left|\varphi = 2\pi n\right\rangle \rightarrow e^{-2\pi i n^2}\left|\varphi = 2\pi n\right\rangle = \left|\varphi = 2\pi n\right\rangle, \text{ for } t=1,$$

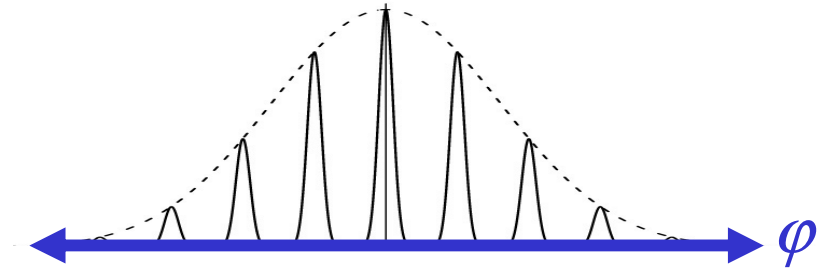$$\left|1\right\rangle : \left|\varphi = 2\pi(n+1/2)\right\rangle \rightarrow e^{-2\pi i(n+1/2)^2}\left|\varphi = 2\pi(n+1/2)\right\rangle = e^{-i\pi/2}\left|\varphi = 2\pi(n+1/2)\right\rangle$$

The codespace rotates, and when it returns to the original codespace, the codewords have undergone a nontrivial holonomy (Berry phase), resulting in a nontrivial encoded gate. (And if two qubits are connected in series to the oscillator, a two-qubit phase gate.)

This is the action of encoded operation $\exp\left(i\frac{\pi}{4}Z\right)$ or $\exp\left(i\frac{\pi}{4}Z \otimes Z\right)$

Protected phase gate

$$H = \frac{Q^2}{2C} + \frac{\varphi^2}{2L} - J(t)\cos(\varphi - \theta)$$
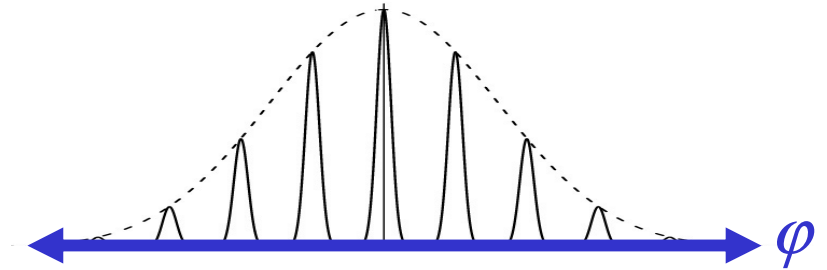
But what if the switch stays closed too long or not long enough? Then evolution is actually

$$|\varphi\rangle \to e^{-i\varphi^2(1+\varepsilon)/2\pi}|\varphi\rangle, \text{ for } \tau = (1+\varepsilon)L/\pi.$$

where ε is the timing error. The error produces broadening in Q-space, but the error is correctable if the broadening is small enough. In fact, the intrinsic error of the codeword is little affected if ε is small; the two encoded states $|0\rangle \pm |1\rangle$ of the oscillator remain highly distinguishable.

Since the operation is Gaussian, we can calculate explicitly how over-rotation augments the intrinsic phase error of the approximate codewords (for a Gaussian grid state):

$$P_{\text{error}}^{|+\rangle_C}(\varepsilon) \approx \exp\left(\frac{\varepsilon^2}{4\pi^2\kappa^6}\right) P_{\text{error}}^{|+\rangle_C}(\varepsilon = 0), \quad \kappa^{-2} = \sqrt{L/C}$$

$$P_{\text{error}}^{|+\rangle_C}(\varepsilon = 0) \approx \frac{2}{\sqrt{\pi}}\left(\frac{L}{C}\right)^{-1/4} e^{-\frac{1}{4}\sqrt{\frac{L}{C}}}$$

But … what happens when we "open the switch" – i.e. decouple qubit and oscillator?

Closing and opening the switch

$$\text{qubit} \,|0\rangle: \quad |\psi^{\text{init}}\rangle \rightarrow |\psi_0^{\text{begin}}\rangle \rightarrow |\psi_0^{\text{end}}\rangle \rightarrow |\psi_0^{\text{fin}}\rangle$$

$$\text{qubit} \,|1\rangle: \quad |\psi^{\text{init}}\rangle \rightarrow |\psi_1^{\text{begin}}\rangle \rightarrow |\psi_1^{\text{end}}\rangle \rightarrow |\psi_1^{\text{fin}}\rangle$$

switch   state   switch
closes   rotates   opens

switch closes:

$$H_0 = \frac{Q^2}{2C} - J(t)\cos(\varphi), \quad \text{qubit is } |0\rangle$$

$$H_1 = \frac{Q^2}{2C} + J(t)\cos(\varphi), \quad \text{qubit is } |1\rangle$$

Ignore the effect of the (small) $\varphi^2/2L$ term in the Hamiltonian when the switch is closing or opening.

therefore: $\quad H_1 = \bar{X}H_0\bar{X}, \quad U_1 = \bar{X}U_0\bar{X}, \quad \bar{X} = (-1)^{[Q]}$

$$|\psi_0^{\text{begin}}\rangle = U_0\,|\psi^{\text{init}}\rangle, \quad |\psi_1^{\text{begin}}\rangle = U_1\,|\psi^{\text{init}}\rangle = \bar{X}U_0\bar{X}\,|\psi^{\text{init}}\rangle \approx \bar{X}\,|\psi_0^{\text{begin}}\rangle$$

hence: $|\psi_1^{\text{begin}}\rangle \approx \bar{X}\,|\psi_0^{\text{begin}}\rangle$

if the switch closes quickly enough, and initial state narrow in Q space.

because $\quad \bar{X} = 1$ in the interval [-1/2,1/2] in Q space.

Closing and opening the switch

$$\text{qubit}\,|0\rangle:\quad |\psi^{\text{init}}\rangle \rightarrow |\psi_0^{\text{begin}}\rangle \rightarrow |\psi_0^{\text{end}}\rangle \rightarrow |\psi_0^{\text{fin}}\rangle$$

$$\text{qubit}\,|1\rangle:\quad |\psi^{\text{init}}\rangle \rightarrow |\psi_1^{\text{begin}}\rangle \rightarrow |\psi_1^{\text{end}}\rangle \rightarrow |\psi_1^{\text{fin}}\rangle$$

switch	state	switch
closes	rotates	opens

gate: $\quad |\psi_1^{\text{begin}}\rangle \approx \overline{X}\,|\psi_0^{\text{begin}}\rangle \rightarrow |\psi_1^{\text{end}}\rangle \approx \overline{X}\,|\psi_0^{\text{end}}\rangle$

because the code protects against logical phase errors.

switch opens:

$$|\psi_0^{\text{fin}}\rangle = V_0\,|\psi_0^{\text{end}}\rangle, \quad |\psi_1^{\text{fin}}\rangle = V_1\,|\psi_1^{\text{end}}\rangle = \overline{X}V_0\overline{X}\,|\psi_1^{\text{end}}\rangle \approx \overline{X}\,|\psi_0^{\text{fin}}\rangle$$

conclude:

$$|\psi_1^{\text{fin}}\rangle \approx \overline{X}\,|\psi_0^{\text{fin}}\rangle \approx |\psi_0^{\text{fin}}\rangle \qquad$$ if support outside $Q \in [-1/2, 1/2]$ is negligible in final state of oscillator (adiabaticity).

$\hookrightarrow$ High gate fidelity!

The conclusion still holds if we include the $\varphi^2/2L$ term as switch opens and closes; this term causes some modest additional spreading in Q space.

Why does it work?

1) Symmetry:

$$\langle \psi_0^{\text{begin}} \mid \overline{X} \mid \psi_1^{\text{begin}} \rangle \approx \langle \psi^{\text{init}} \mid \overline{X} \mid \psi^{\text{init}} \rangle$$

(ignoring the effect of the $\varphi^2/2L$ term in the Hamiltonian when the switch is closing or opening)

$$\langle \psi_0^{\text{fin}} \mid \overline{X} \mid \psi_1^{\text{fin}} \rangle \approx \langle \psi_0^{\text{end}} \mid \overline{X} \mid \psi_1^{\text{end}} \rangle$$

2) Phase fluctuations: $\langle \psi^{\text{init}} \mid \overline{X} \mid \psi^{\text{init}} \rangle \approx 1$

(initial state of oscillator is narrow in Q space)

3) Coding:

$$\langle \psi_0^{\text{begin}} \mid \overline{X} \mid \psi_1^{\text{begin}} \rangle \approx 1 \rightarrow \langle \psi_0^{\text{end}} \mid \overline{X} \mid \psi_1^{\text{end}} \rangle \approx 1$$

(code vectors are well protected against phase errors)

4) Adiabaticity: $\langle \psi_0^{\text{fin}} \mid \overline{X} \mid \psi_1^{\text{fin}} \rangle \approx \langle \psi_0^{\text{fin}} \mid \psi_1^{\text{fin}} \rangle \approx 1$

(the final state of the oscillator is not highly excited)

5) Separation of time scales:

(so that $\varphi^2/2L$ term in the Hamiltonian causes only modest spreading in Q space while the switch closes or opens)

The oscillator is the "ancilla" whose final state records the "error syndrome"; it absorbs the entropy introduced by noise.

$(L/C)^{1/2} = 80$

$(JC)^{1/2} = 8$

$\tau_J / C = 80$
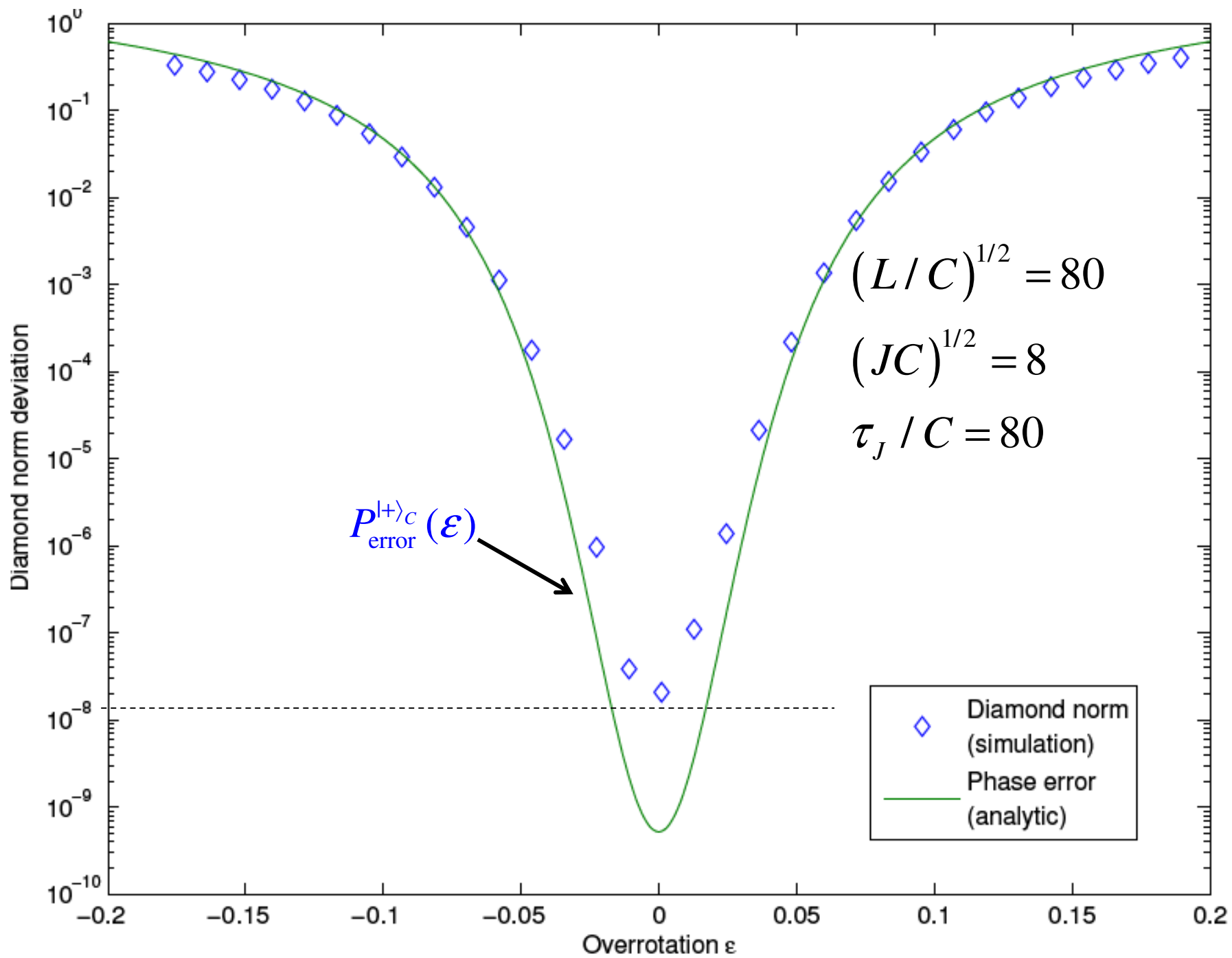
$P_{\text{error}}^{|+\rangle_C} (\varepsilon)$

Nonzero temperature

Excited states of the oscillator are also narrow in Q space if not too highly excited.

$$P_{\text{error}}^{|+\rangle_C}(n) \approx \frac{1}{2^n n!}\left(\frac{L}{C}\right)^{n/2} P_{\text{error}}^{|+\rangle_C}(0)$$

Hence

$$P_{\text{error}}^{|+\rangle_C}(\beta) = Z^{-1}\sum_{n=0}^{\infty} e^{-n\beta/\sqrt{LC}}\, P_{\text{error}}^{|+\rangle_C}(n) \approx P_{\text{error}}^{|+\rangle_C}(n=0) \approx \frac{2}{\sqrt{\pi}}\left(\frac{L}{C}\right)^{-1/4} e^{-\frac{1}{4}\sqrt{\frac{L}{C}}}$$
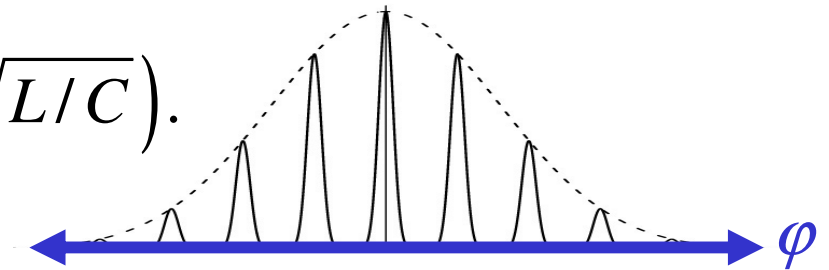
assuming

$$\sqrt{\frac{L}{C}}\, e^{-\beta/\sqrt{LC}} \ll 1$$

We can also check robustness with respect to (anharmonic) perturbations of the oscillator Hamiltonian and (periodic) perturbations to the Josephson energy.

Large inductance

The intrinsic error scales like $\exp\left(-(1/4)\sqrt{L/C}\right).$

Is $\sqrt{L/C} \approx 80$ reasonable?

Devoret et al. achieved ~ 6 with a chain of 43 Josephson junctions. The inductance scales linearly with the length of the chain, but there are potential obstacles to building very long chains. Another possible approach is to exploit the large (kinetic) inductance in amorphous superconductors.

What about universal quantum computation and measurement?

-- If we can prepare and measure in the basis $|0\rangle \pm |1\rangle$, a noisy $\pi/4$ single-qubit phase gate (F > .93) suffices for fault-tolerant universality.

-- It is also okay if measurements are noisier than gates, as we can protect measurements using repetition (or coding)

-- So if we can really do a two-qubit phase gate with high fidelity, that's worth a lot!

Measurement errors

Measurements are likely to be noisy in practice. If they can be performed nondestuctively, with a small probability of changing the eigenvalue of the measured observable, then they can be repeated multiple times to improve reliability by majority voting.

Alternatively, we could use repetition coding to protect against errors in X measurements.

For example, for a CPHASE error rate of 10^{-5} and a bias factor of 10^4, the logical CNOT gate shown has error rate below 10^{-6} if the probability of an X measurement error is below 1%.
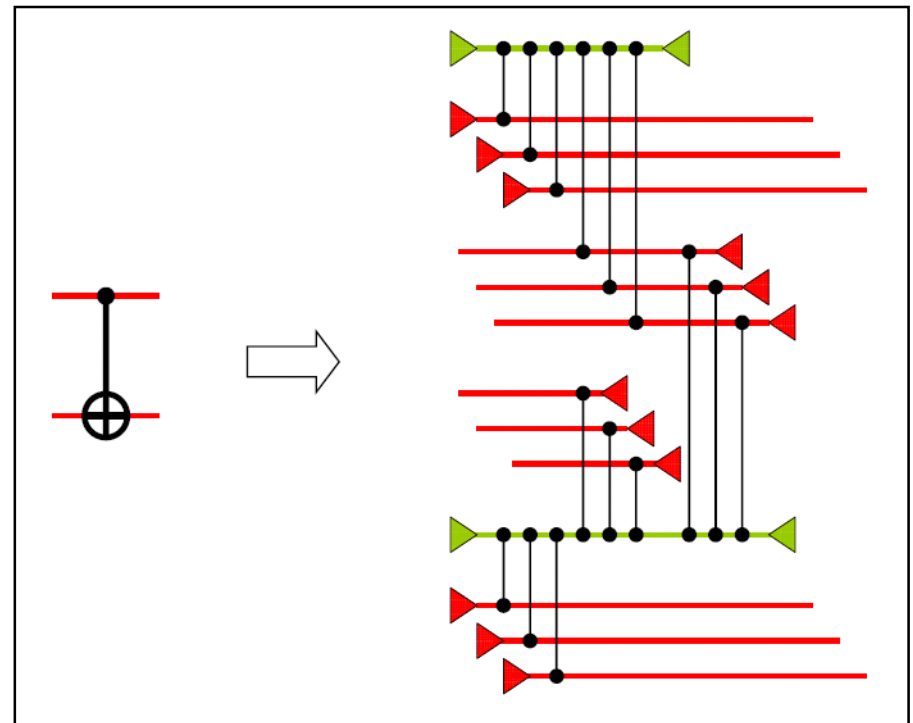
Teleported encoded CNOT constructed from CPHASE gates, $|+\rangle$ preparations, and X measurements

Aliferis and Preskill

Protected Gates

Operating a large-scale quantum computer will be a grand scientific and engineering achievement.

Judicious application of the principles of fault-tolerant quantum computing will be the key to making it happen.

A protected two-qubit Clifford phase gate is a powerful tool for fault-tolerant quantum computing, even if measurements and other gates are noisy.

For "Zero-Pi" qubits with tunable coupling to an oscillator, single-qubit or two-qubit Clifford group phase gates can be executed with fidelity exponentially close to one, as a function of system parameters.
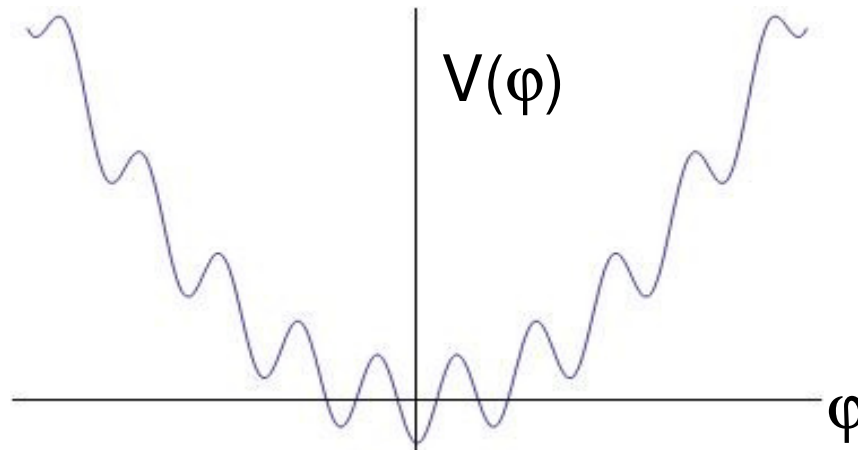
The high fidelity is achieved by exploiting a continuous variable quantum error-correcting code.

Both the Zero-Pi qubit and the protected gate make use of superconducting circuits with large inductance, $(L/C)^{1/2} \gg 1$ in natural units, which may be hard to achieve in practice.

In principle, the gate is robust against Hamiltonian noise and thermal effects.

It's important to come up with new ideas for incorporating fault-tolerance into the design and function of quantum hardware!

Protected gates for superconducting qubits
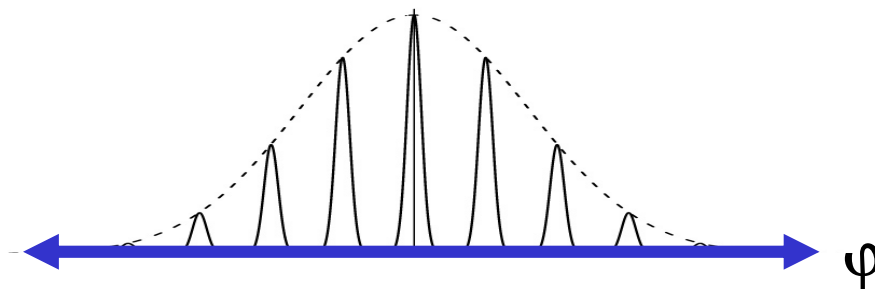
John Preskill
with Peter Brooks
and Alexei Kitaev
QEC 2011
5 December 2011

Institute for Quantum
Information and Matter

Additional Slides

Flux qubit

Frustrate the loop by applying half a flux quantum through it. The magnetic field energy wants applied flux cancelled ($\varphi = \pi$), but the junction wants $= 2\pi m$.

J, C $\Phi_0 / 2$ L

$$H = \frac{Q^2}{2C} + \frac{(\varphi - \pi)^2}{2L} - J\cos\varphi$$

The potential has two degenerate minima separated by a tunneling barrier, and if the capacitance C is large enough, the state can be well localized in one minimum or the other. These two states have persistent current circulating in opposite directions. Remarkably, since many Cooper pairs contribute to the current, the two states are "macroscopically" distinguishable.

ENERGY

$\phi_{\text{ext}} = \pi$

$|1\rangle$
$|0\rangle$

Δ

PHASE ϕ

This particular qubit design is not so good, because it is not well protected against fluctuations in the external flux, which drive dephasing. Other more clever designs, with more Josephson junctions, perform better.

Charge qubit

C

J

φ, Q
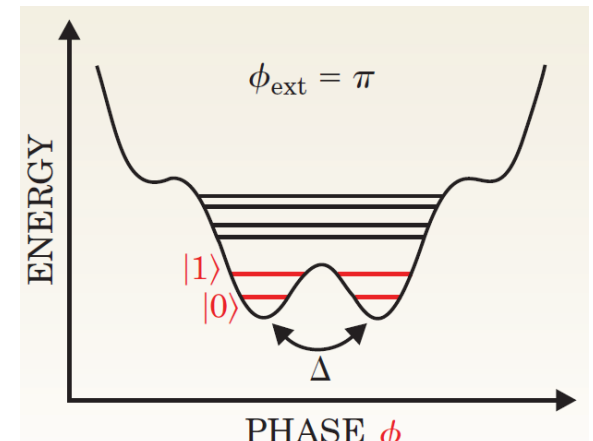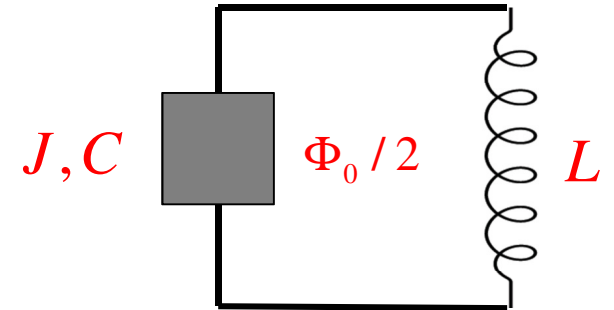
Small capacitance suppresses charge fluctuations; there are two low-lying states, with charge just above and below an offset charge (which can be adjusted with a gate voltage).

$$H = \frac{(Q - Q_g)^2}{2C} - J\cos\varphi \approx \frac{\left(\frac{1}{2} - Q_g\right)}{2C}\sigma_z - \frac{J}{2}\sigma_x$$

The operator $e^{-i\varphi}$ increases the charge by 1 Cooper pair, so the Josephson term flips the qubit defined by the two charge states.

ENERGY

$n_g = 0.5$

E_J

$|0\rangle$ $|1\rangle$

CHARGE n

This particular qubit design is not well protected against fluctuations in the offset charge Q_g. There are more clever designs that perform better.
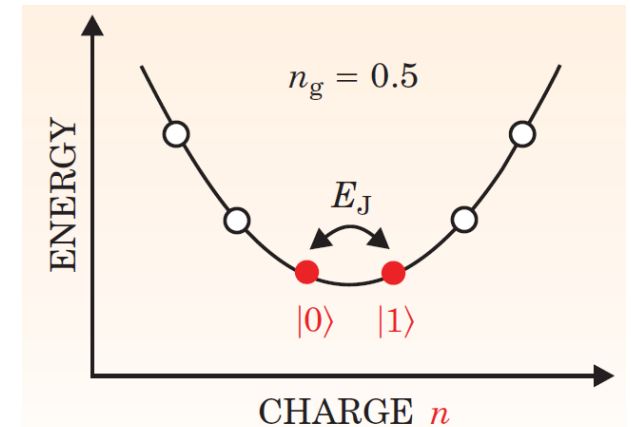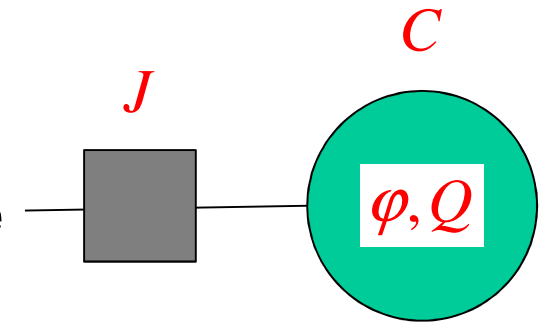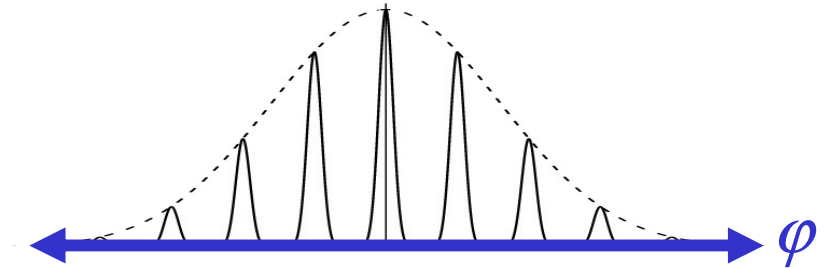
A qubit encoded in an oscillator

Formally, the basis states for a 2-dimensional code space are coherent superpositions of equally spaced states, infinitely squeezed in "position" φ and "momentum" Q:
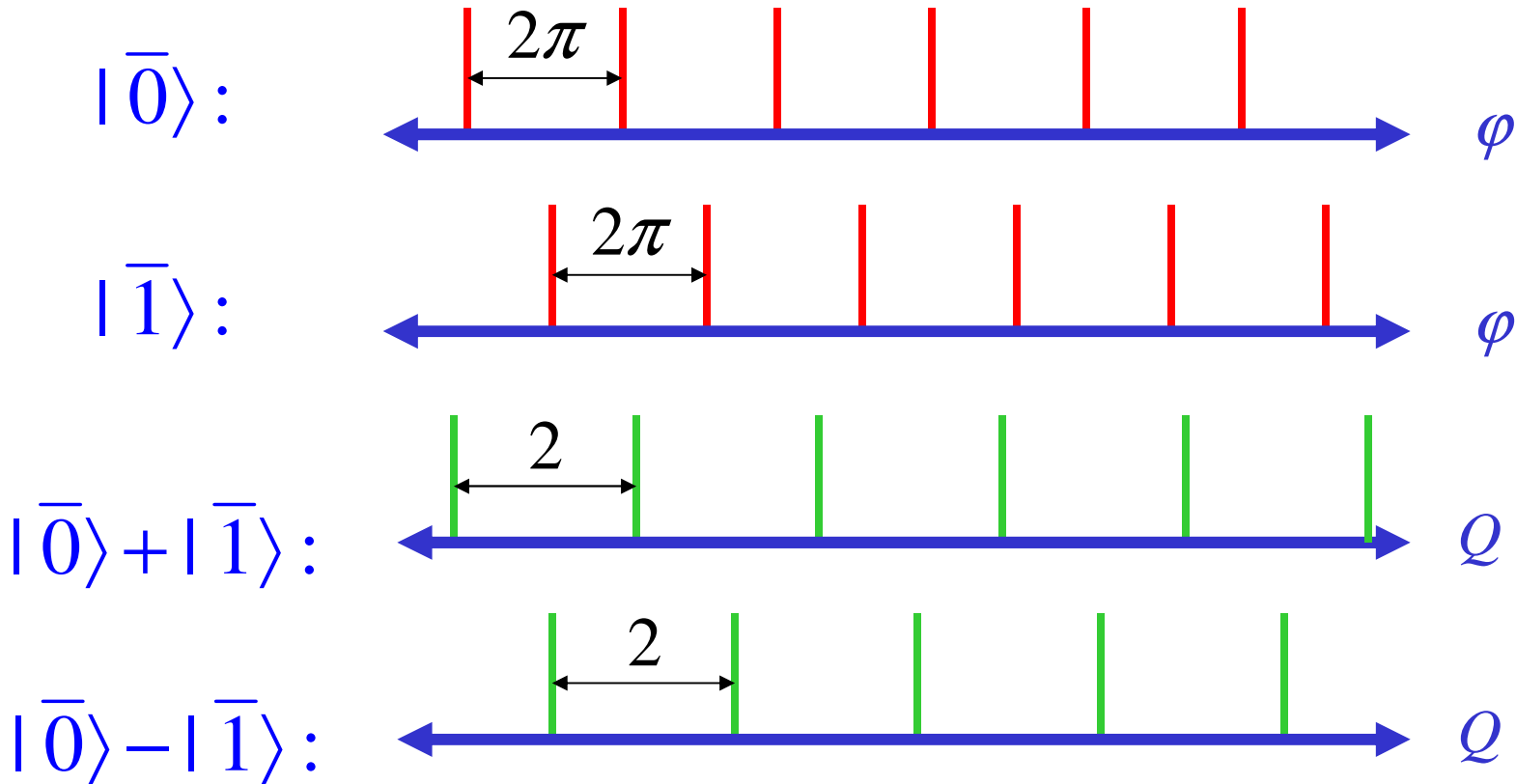
This code can correct all shifts that satisfy: $|\Delta\varphi| < \dfrac{\pi}{2}, \quad |\Delta Q| < \dfrac{1}{2}.$

Gottesman, Kitaev, Preskill

Protected phase gate

$$H = \frac{Q^2}{2C} + \frac{\varphi^2}{2L} - J(t)\cos(\varphi - \theta)$$

Codewords evolve while switch is closed.

$$|\varphi\rangle \rightarrow e^{-i\tau\varphi^2/2L}|\varphi\rangle = e^{-it\varphi^2/2\pi}|\varphi\rangle, \text{ for } \tau = Lt/\pi.$$

Therefore,

$$M_X \rightarrow M_{X'} = e^{-it\varphi^2/2\pi}e^{-2\pi iQ}e^{it\varphi^2/2\pi} = e^{-2\pi i(Q+\varphi t/\pi)} = M_X e^{-2i\varphi t}e^{-2\pi it} = M_X M_Z^{-1}$$

$$\overline{X} \rightarrow e^{-it\varphi^2/2\pi}e^{-\pi iQ}e^{it\varphi^2/2\pi} = e^{-\pi i(Q+\varphi t/\pi)} = \overline{X}e^{-i\varphi t}e^{-\pi it/2} = -i\overline{X}\overline{Z}, \text{ for } t = 1$$

The codespace rotates, and when it returns to the original codespace, the codewords have undergone a nontrivial holonomy (Berry phase), resulting in a nontrivial encoded gate. (And if two qubits are connected in series to the oscillator, a two-qubit phase gate.)

This is the action of encoded operation $\exp(i\frac{\pi}{4}Z)$ or $\exp(i\frac{\pi}{4}Z \otimes Z)$

Protected phase gate

But what if the switch stays closed too long or not long enough? Then evolution is actually

$$|\varphi\rangle \rightarrow e^{-i\varphi^2(1+\varepsilon)/2\pi}|\varphi\rangle, \text{ for } \tau = (1+\varepsilon)L/\pi.$$

where ε is the timing error.

$$P_{\text{error}}^{|+\rangle_C}(\varepsilon) \approx \exp\left(\frac{\varepsilon^2}{4\pi^2\kappa^6}\right)P_{\text{error}}^{|+\rangle_C}(\varepsilon = 0), \quad \kappa^{-2} = \sqrt{L/C}$$

Crudely,

$$e^{-i\varepsilon\varphi^2/2\pi}Qe^{i\varepsilon\varphi^2/2\pi} = Q + \varphi\varepsilon/\pi \quad \Rightarrow$$

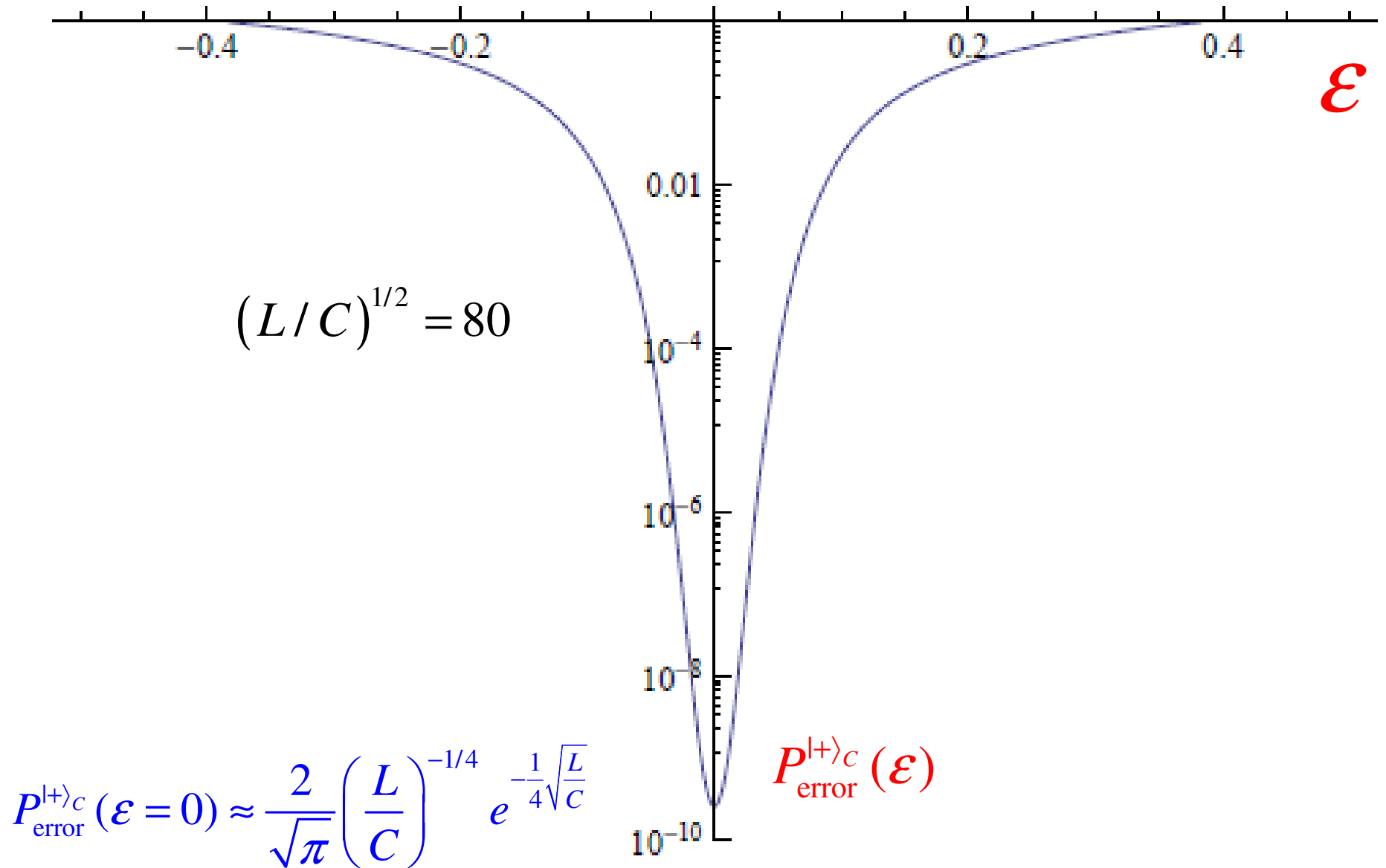$$\delta Q \approx (\varepsilon/\pi)\langle\varphi^2\rangle^{1/2} = \varepsilon/\left(\pi\sqrt{2\kappa^2}\right)$$

which is small compared to

$$\langle Q^2\rangle^{1/2} = \sqrt{\kappa^2/2}$$

provided that $\varepsilon \ll \pi\kappa^2$

But … what happens when we "open the switch" – i.e. decouple qubit and oscillator?

# Overrotation and intrinsic error

$(L/C)^{1/2} = 80$

$P_{error}^{|+\rangle_C}(\mathcal{E})$

$P_{error}^{|+\rangle_C}(\mathcal{E}=0) \approx \dfrac{2}{\sqrt{\pi}}\left(\dfrac{L}{C}\right)^{-1/4} e^{-\frac{1}{4}\sqrt{\frac{L}{C}}}$

$\mathcal{E}$

Noisy measurements

This gadget uses repetition coding. Preparations and measurements are in the X basis, and measurements are decoded by majority voting. Because the gadget uses more gates than measurements, it resists measurement noise better than gate noise.

For highly biased noise in particular, we can crank up the measurement error rate without lowering very much the threshold gate error rate:
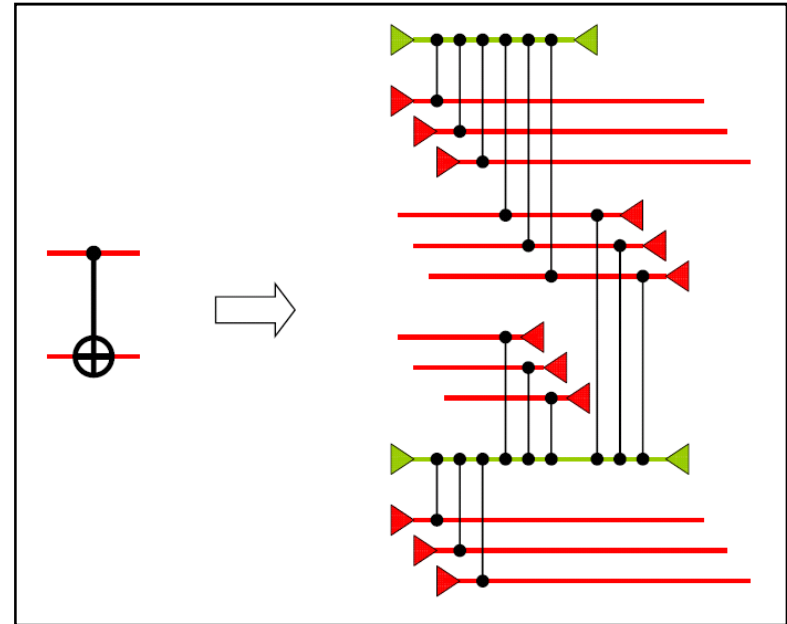
Bias	10^4	10^3	100	10	1
$\varepsilon_{meas} + \varepsilon_{prep}$	5%	5%	2%	2%	2%
$\varepsilon_{gate, threshold}$	1.4×10^{-3}	6×10^{-4}	1.8×10^{-4}	1.9×10^{-5}	2×10^{-6}