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Quantum Error-Correcting Codes by Concatenation

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Why Bei isn't here

Jonathan, November 24, 2011

Overview

- Shor's nine-qubit code revisited
- The code $[\![25, 1, 9]\!]$
- Concatenated graph codes
- Generalized concatenated quantum codes
- Codes for the Amplitude Damping (AD) channel
- Conclusions

Shor's Nine-Qubit Code Revisited

Bit-flip code: $|0\rangle \mapsto |000\rangle, \qquad |1\rangle \mapsto |111\rangle.$

Phase-flip code: $|0\rangle \mapsto |+++\rangle, \quad |1\rangle \mapsto |---\rangle.$

Effect of single-qubit errors on the bit-flip code:

- X-errors change the basis states, but can be corrected
- Z-errors at any of the three positions:

$$\left.\begin{array}{rcr} Z|000\rangle & = & |000\rangle \\ Z|111\rangle & = & -|111\rangle \end{array}\right\} \text{“encoded” } Z\text{-operator}$$

$\Longrightarrow$ Bit-flip code & error correction convert the channel into a phase-error channel

$\Longrightarrow$ Concatenation of bit-flip code and phase-flip code yields $[\![9,1,3]\!]$

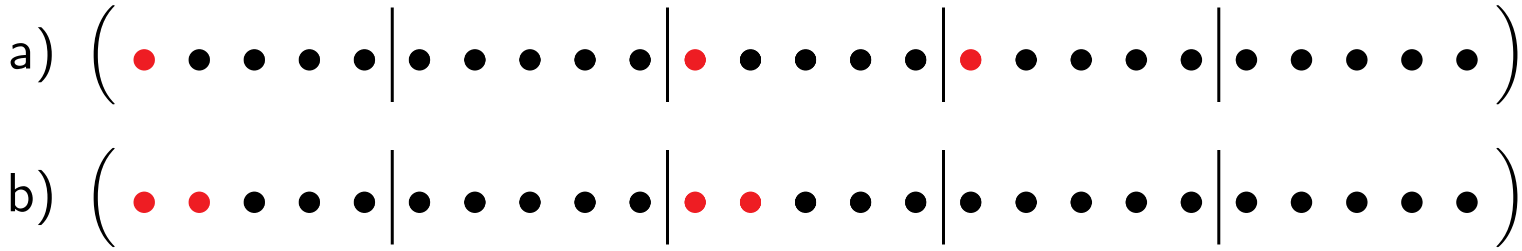
The Code $[\![25, 1, 9]\!]$

- The best single-error correcting code is $\mathcal{C}_0 = [\![5, 1, 3]\!]$
- Re-encoding each of the 5 qubits with $\mathcal{C}_0$ yields $\mathcal{C} = [\![5^2, 1, 3^2]\!] = [\![25, 1, 9]\!]$
- The code $\mathcal{C}$ is a subspace of five copies of $[\![5, 1, 3]\!]$
- The stabilizer of $\mathcal{C}$ is generated by five copies of the stabilizer of $\mathcal{C}_0$ and an encoded version of the stabilizer of $\mathcal{C}_0$
- The code $\mathcal{C}$ is degenerate
- m-fold self-concatenation of $[\![n, 1, d]\!]$ yields $[\![n^m, 1, d^m]\!]$

Level-decoding of $[\![25, 1, 9]\!]$

The code corrects up to $t = 4$ errors ($t < d/2$)

different error patterns:



- errror correction on both levels:
  corrects a), but fails for b)

- errror detection on lowest level, error correction on higer level:
  corrects b), but fails for a)

$\Longrightarrow$ optimal decoding must pass information between the levels

Overview

- Shor's nine-qubit code revisited
- The code $[\![25, 1, 9]\!]$

⇒ **Concatenated graph codes**
[Beigi, Chuang, Grassl, Shor & Zeng, Graph Concatenation for QECC, **JMP** 52 (2011), arXiv:0910.4129]

- Generalized concatenated quantum codes
- Codes for the Amplitude Damping (AD) channel
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Canonical Basis of a Stabilizer Code

- fix logical operators $\overline{X}_i$ and $\overline{Z}_\ell$
- the stabilizer $\mathcal{S}$ and the logical operators $\overline{Z}_\ell$ mutually commute
- the logical state $|\overline{00\ldots0}\rangle$ is a stabilizer state
- define the (logical) basis states as

$$|\overline{i_1 i_2 \ldots i_k}\rangle = \overline{X}_1^{i_1} \cdots \overline{X}_k^{i_k} |\overline{00\ldots0}\rangle$$

in terms of a classical code over a finite field:

- the logical state $|\overline{00\ldots0}\rangle$ corresponds to a self-dual code $\mathcal{C}_0$
- the basis states $|\overline{i_1 i_2 \ldots i_k}\rangle$ correspond to *cosets* of $\mathcal{C}_0$
- for a stabilizer code, the union of the cosets is an additive code $\mathcal{C}^*$

Graphical Quantum Codes

[D. Schlingemann & R. F. Werner: QECC associated with graphs, PRA **65** (2002), quant-ph/0012111]

[Grassl, Klappenecker & Rötteler: Graphs, Quadratic Forms, & QECC, ISIT 2002, quant-ph/0703112]

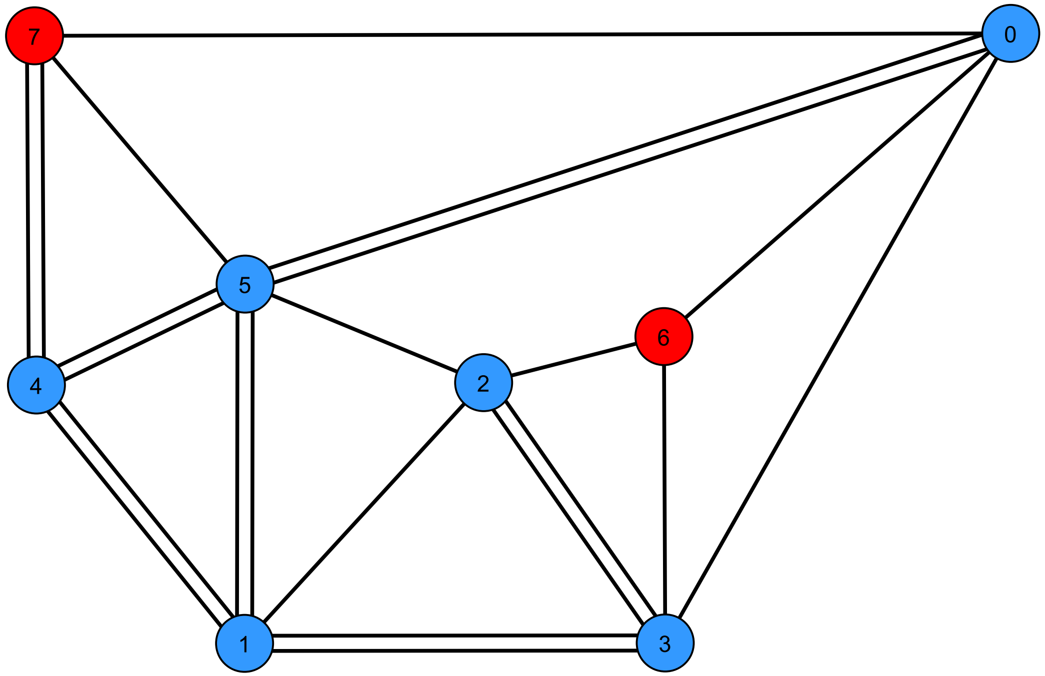
Basic idea

- a classical symplectic self-dual code defines a single quantum state $\mathcal{C}_0 = [\![n, 0, d]\!]_q$
- the standard form of the stabilizer matrix is $(I|A)$
- the generators have exactly one tensor factor X
- self-duality implies that A is symmetric
- A can be considered as adjacency matrix of a graph with n vertices
- logical X-operators give rise to more quantum states in the code $\mathcal{C} = [\![n, k, d']\!]_q$
- use additionally k *input* vectices

Graphical Representation of $[\![6,2,3]\!]_3$

$$\left(\begin{array}{cccccc|cccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 2 & 2 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 2 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 1 & 2 & 2 & 1 & 0 & 2 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & 1
\end{array}\right)$$

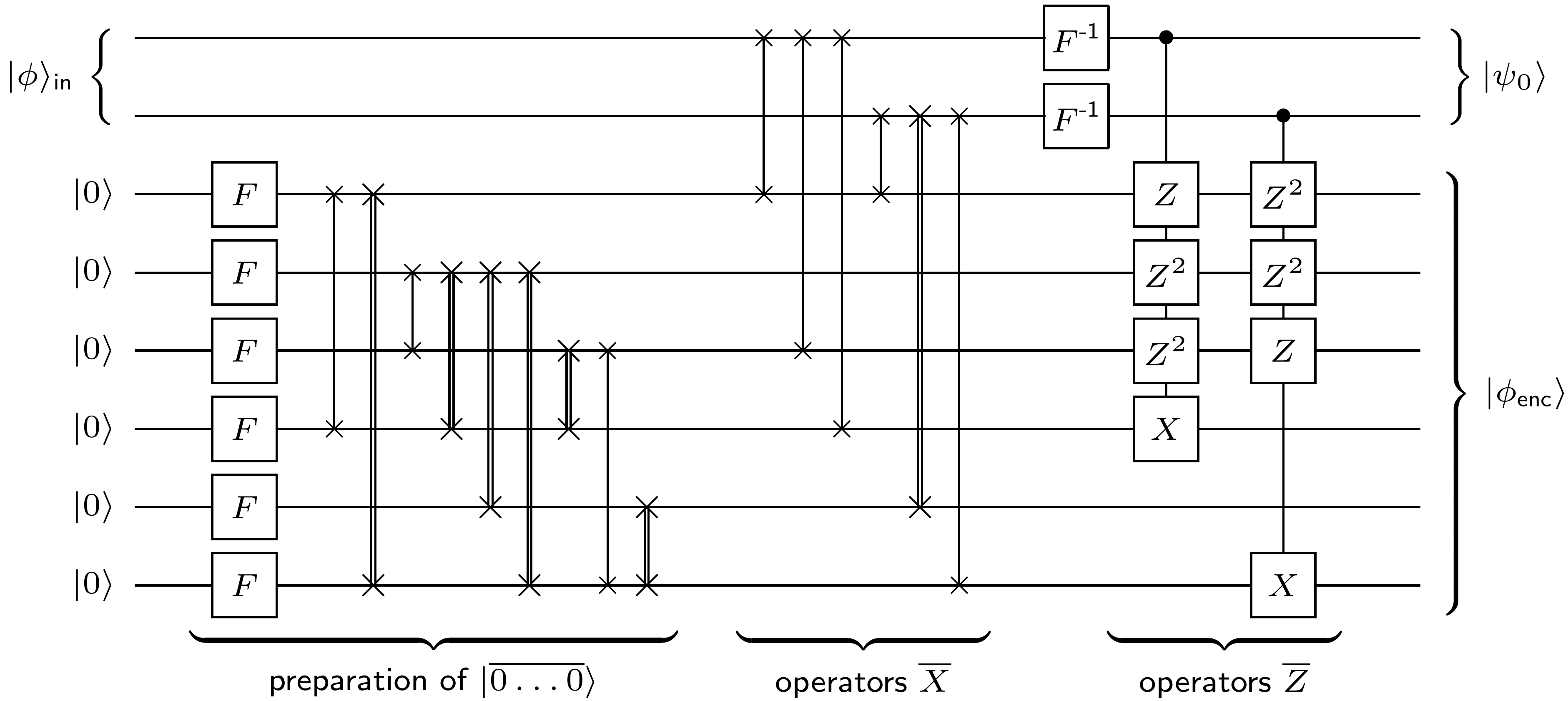
stabilizer & logical X-operators



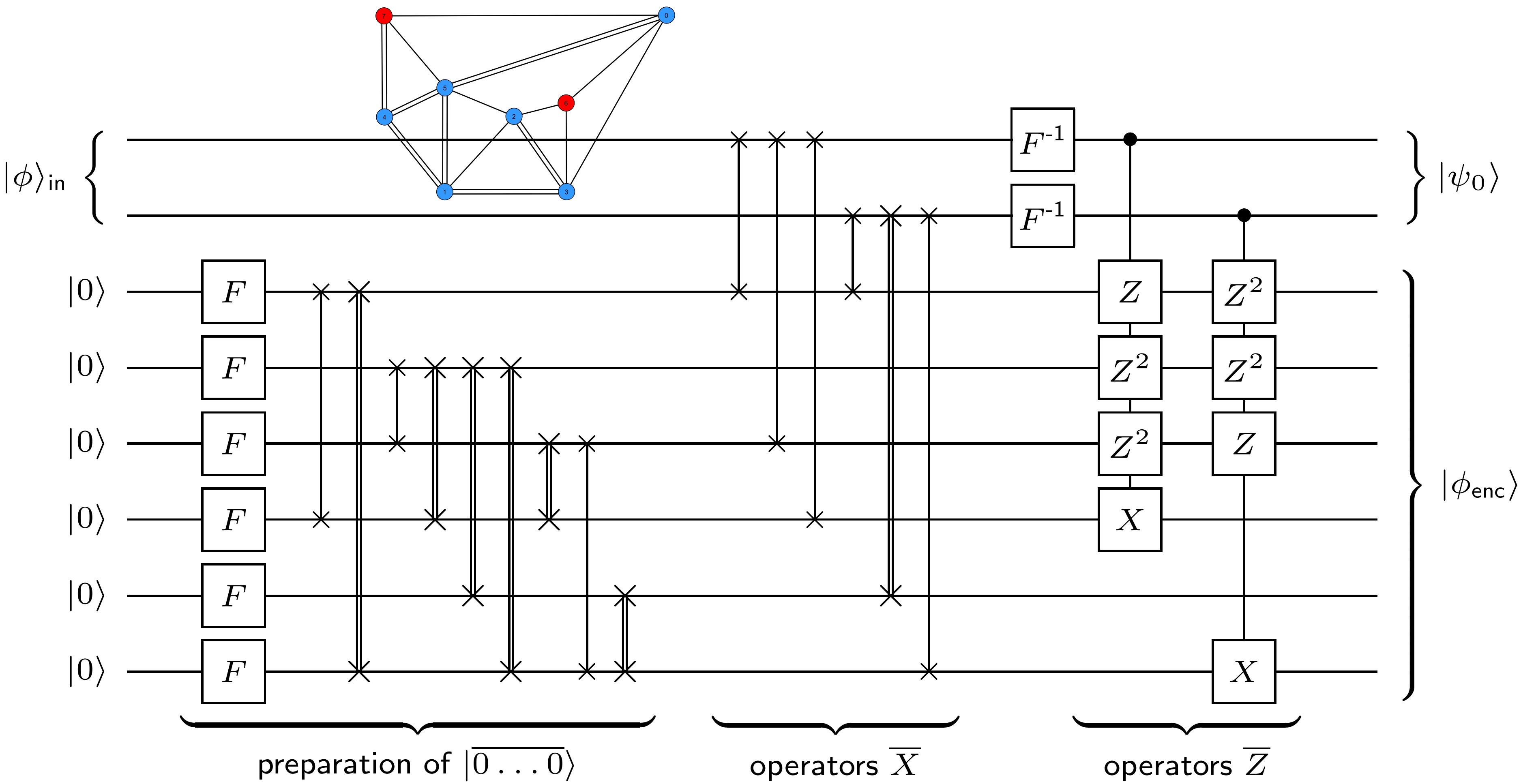
graphical representation

Encoder based on Graphical Representation

[M. Grassl, Variations on Encoding Circuits for Stabilizer Quantum Codes, LNCS 6639, pp. 142–158, 2011]



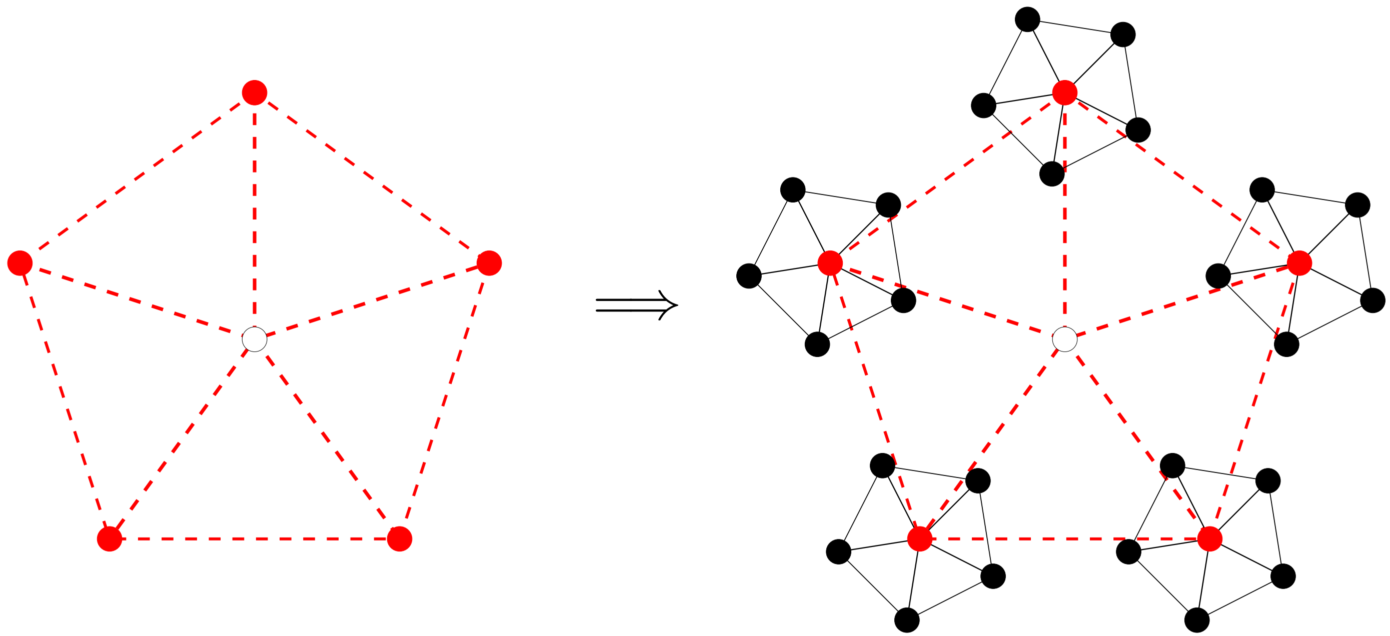
Encoder based on Graphical Representation



Concatenation of Graph Codes

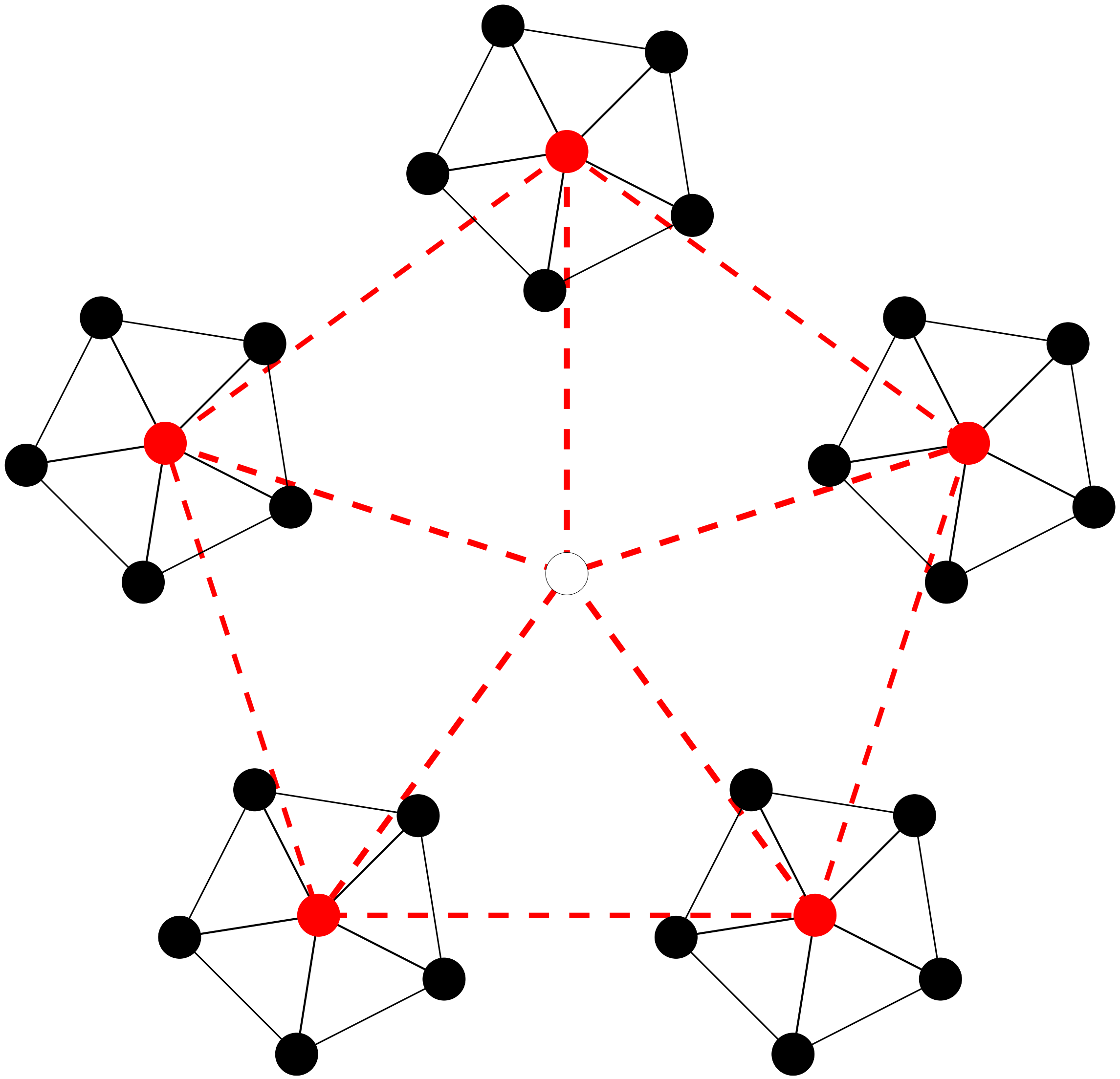
[Beigi, Chuang, Grassl, Shor & Zeng, Graph Concatenation for QECC, **JMP** 52 (2011), arXiv:0910.4129]

- self-concatenation of $[\![5,1,3]\!]$

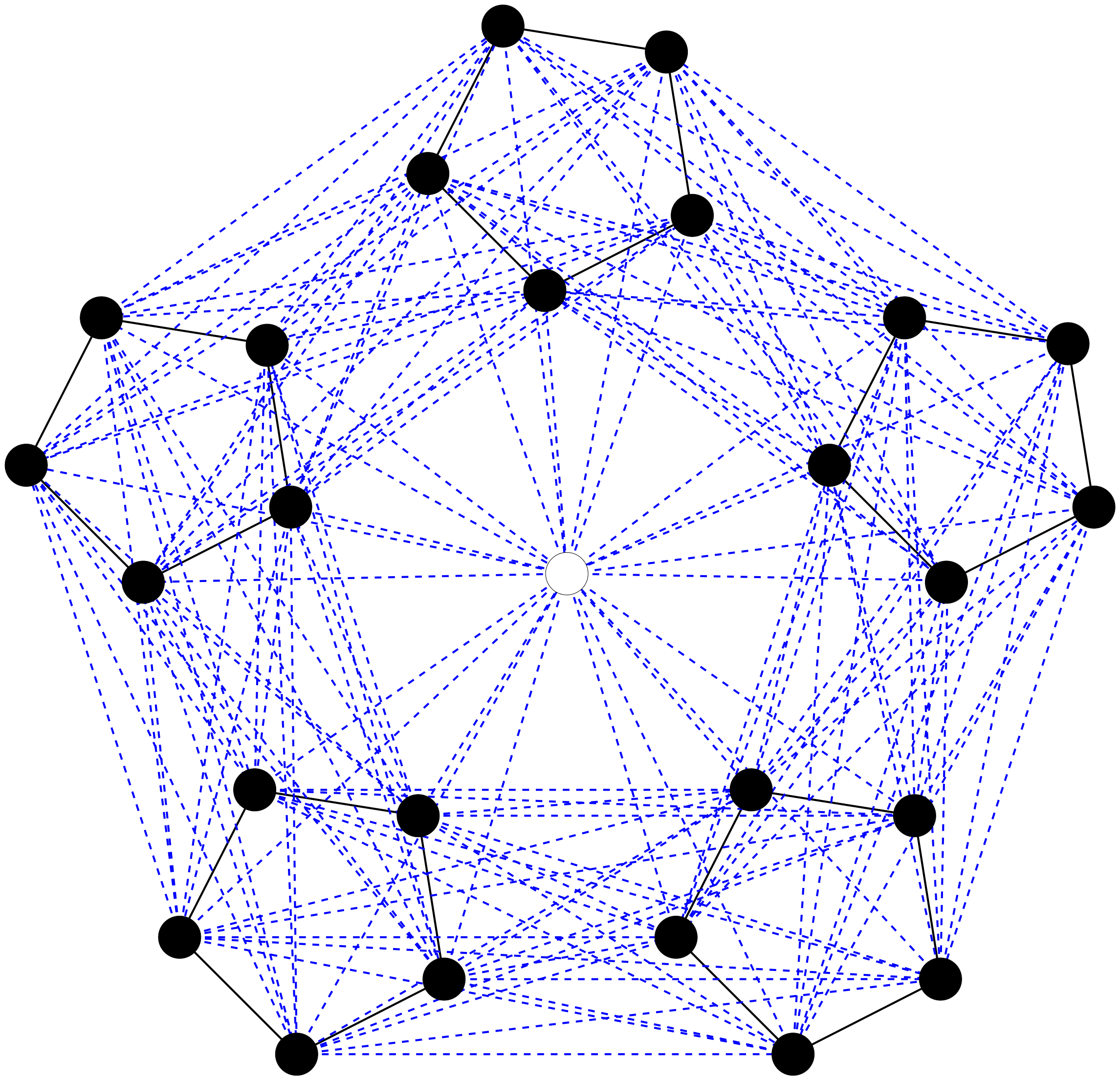


- measure the five auxillary nodes • in X-bases
- X-measurement corresponds to sequence of local complementations
  $\Longrightarrow$ many different choices

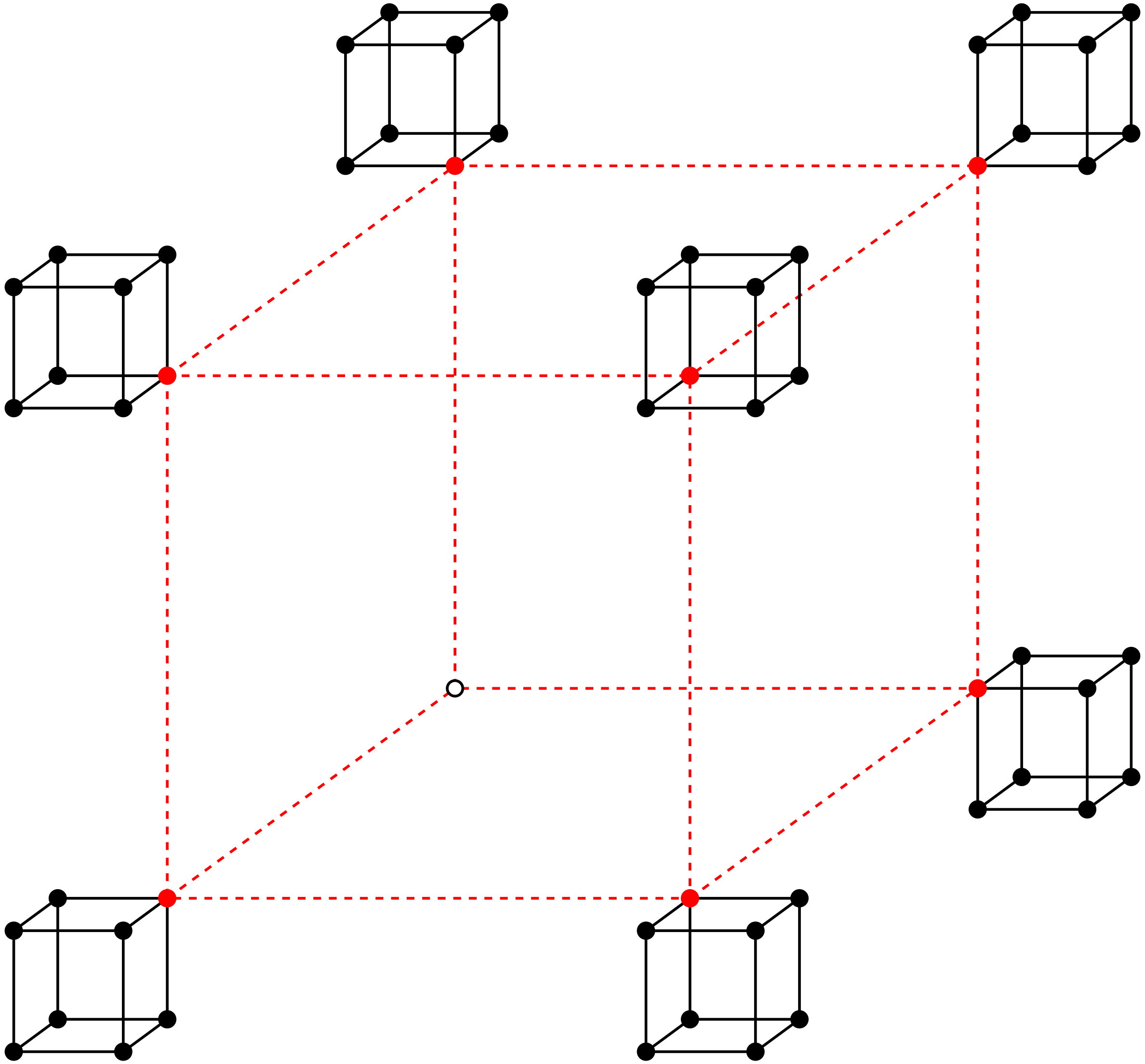
$$[\![5,1,3]\!] \Longrightarrow [\![25,1,9]\!]$$



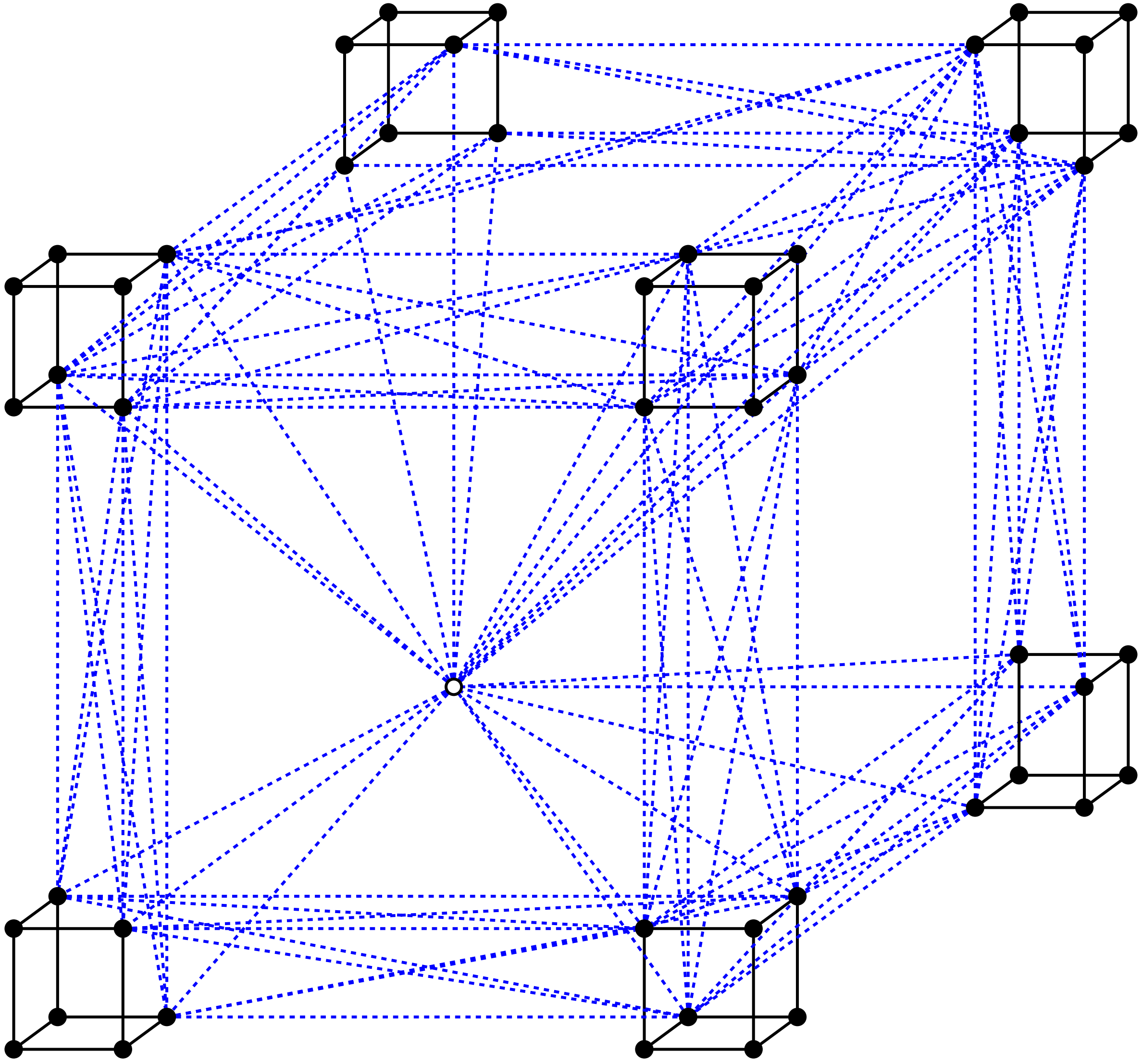
$[\![5,1,3]\!] \Longrightarrow [\![25,1,9]\!]$



$[\![7,1,3]\!] \Longrightarrow [\![49,1,9]\!]$



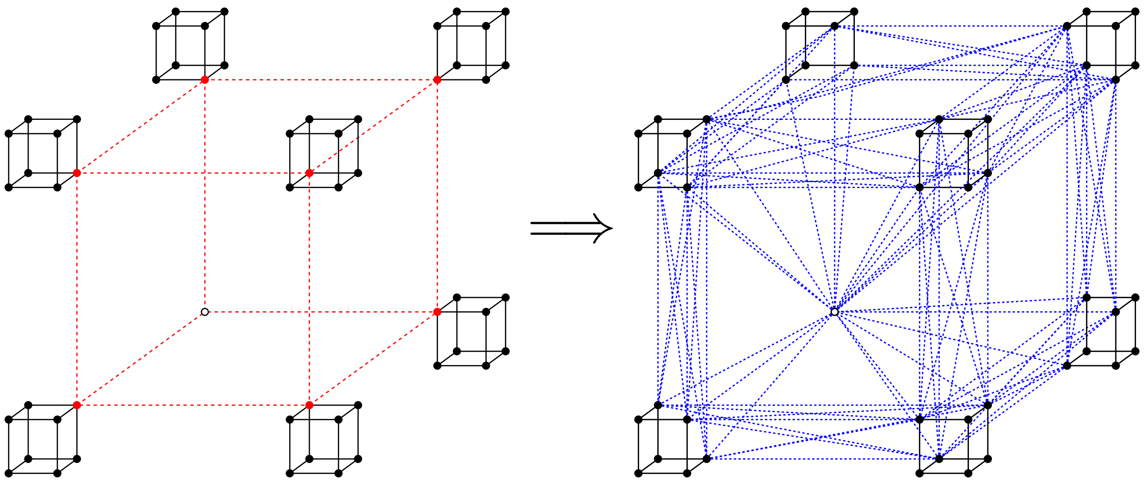
$[\![7,1,3]\!] \Longrightarrow [\![49,1,9]\!]$



General Concatenation Rule

(for qubit codes; see paper for qudit codes)

- Any edge connecting an input vertex with an auxiilary vertex is replaced by a set of edges connecting the input vertex with all neighbors of the auxillary vertex.
- Any edge between two auxiliary vertices A and B is replaced by a complete bipartite graph connecting any neighbor of A with all neighbors of B.



Overview

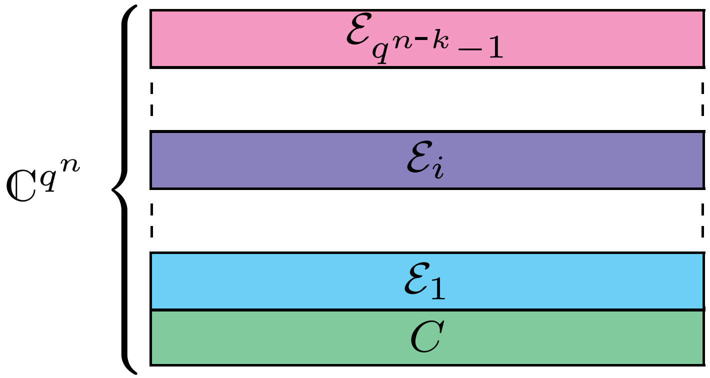
- Shor's nine-qubit code revisited
- The code $[\![25, 1, 9]\!]$
- Concatenated graph codes

⇒ **Generalized concatenated quantum codes**
[Grassl, Shor, Smith, Smolin & Zeng, PRA **79** (2009), arXiv:0901.1319]
[Grassl, Shor & Zeng, ISIT 2009, arXiv:0905.0428]

- Codes for the Amplitude Damping (AD) channel
- Conclusions

Stabilizer Codes

- stabilizer group $\mathcal{S} = \langle S_1, \ldots, S_{n-k} \rangle$ generated by $n-k$ mutually commuting tensor products of (generalized) Pauli matrices
- $C = [\![n, k, d]\!]$ is a common eigenspace of the S_i
- orthogonal decomposition of the vector space $\mathcal{H}^{\otimes n}$ into joint eigenspaces



- labelling of the spaces by the eigenvalues of the S_i
- errors that change the eigenvalues can be detected

Variations on $[\![5,1,3]\!]_2$

decomposition of $(\mathbb{C}^2)^{\otimes 5} = B^{(0)} = ((5, 2^5, 1))_2$ into 16 mutually orthogonal quantum codes $B_i^{(1)} = ((5, 2, 3))_2$

$\|0;1\rangle$	$\|1;1\rangle$	$\|2;1\rangle$	$\|3;1\rangle$	$\|4;1\rangle$	$\|5;1\rangle$	$\|6;1\rangle$	$\|7;1\rangle$	$\|8;1\rangle$	$\|9;1\rangle$	$\|10;1\rangle$	$\|11;1\rangle$	$\|12;1\rangle$	$\|13;1\rangle$	$\|14;1\rangle$	$\|15;1\rangle$	$\|1_L\rangle$
$\|0;0\rangle$	$\|1;0\rangle$	$\|2;0\rangle$	$\|3;0\rangle$	$\|4;0\rangle$	$\|5;0\rangle$	$\|6;0\rangle$	$\|7;0\rangle$	$\|8;0\rangle$	$\|9;0\rangle$	$\|10;0\rangle$	$\|11;0\rangle$	$\|12;0\rangle$	$\|13;0\rangle$	$\|14;0\rangle$	$\|15;0\rangle$	$\|0_L\rangle$
$B_0^{(1)}$	$B_1^{(1)}$	$B_2^{(1)}$	$B_3^{(1)}$	$B_4^{(1)}$	$B_5^{(1)}$	$B_6^{(1)}$	$B_7^{(1)}$	$B_8^{(1)}$	$B_9^{(1)}$	$B_{10}^{(1)}$	$B_{11}^{(1)}$	$B_{12}^{(1)}$	$B_{13}^{(1)}$	$B_{14}^{(1)}$	$B_{15}^{(1)}$	

new basis: $\{|i;j\rangle : i = 0,\ldots,15;\ j = 0,1\}$

Construction of $((15, 2^7, 3))_2$

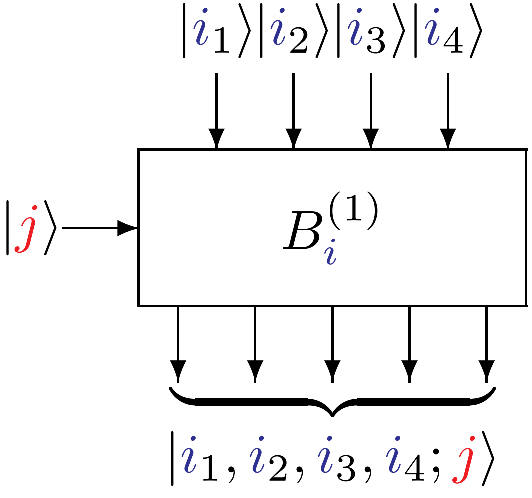
- basis $\{|i; j\rangle : i = 0, \ldots, 15;\ j = 0, 1\}$ of $B^{(0)} = ((5, 2^5, 1))_2$
  - states $|i; 0\rangle$ and $|i; 1\rangle$ are in the code $B_i^{(1)} = ((5, 2, 3))_2$
  - for $i \neq i'$, some states $|i; j\rangle$ and $|i'; j'\rangle$ have distance < 3
- protect the *quantum number* i
- a classical code of distance three suffices for this purpose
- generalized concatenated QECC $((3 \times 5, 16 \times 2^3, 3))$ with basis

$$\{|i; j_1\rangle|i; j_2\rangle|i; j_2\rangle : i = 0, \ldots, 15;\ j_1 = 0, 1; j_2 = 0, 1; j_3 = 0, 1\}$$

- normalizer code is a generalized concatenated code with
  - inner codes $\mathcal{B}^{(0)} = (5, 2^{10}, 1)_4$ and $\mathcal{B}^{(1)} = (5, 2^6, 3)_4$
  - outer codes $\mathcal{A}_1 = [3, 1, 3]_{16}$ and $\mathcal{A}_2 = [3, 3, 1]_{2^6}$

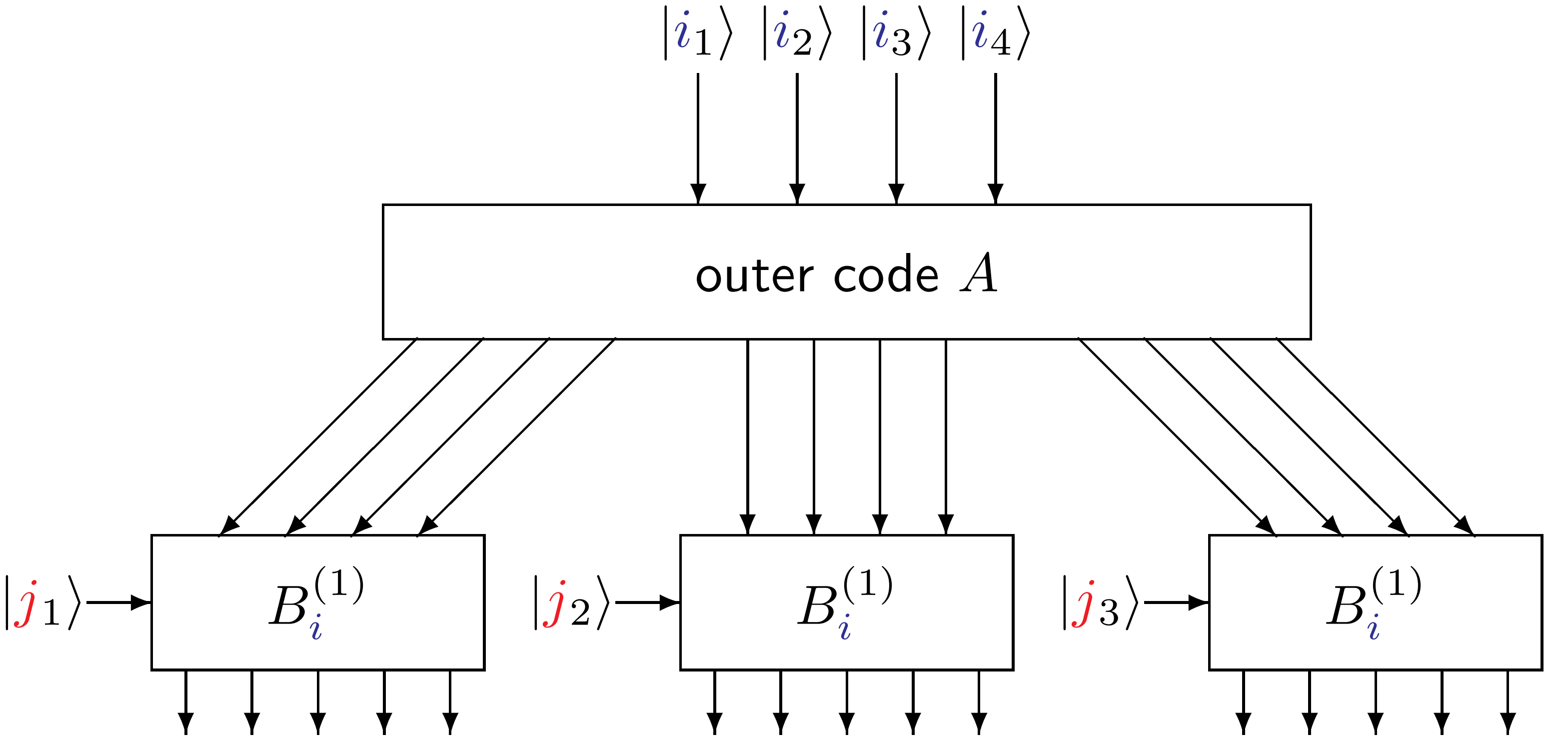
Encoding of $((15, 2^7, 3))_2$

encoder for the nested codes $((5, 2, 3))_2 \leq ((5, 2^5, 1))_2$



Encoding of $((15, 2^7, 3))_2$

generalized concatenated encoder



A New Qubit Non-Stabilizer Code

[Grassl, Shor, Smith, Smolin & Zeng, PRA **79** (2009), arXiv:0901.1319]

- the classical outer code can be any code, not only linear codes
- from the Hamming code $[18,16,3]_{17}$ over $GF(17)$ one can derive a code

$$\mathcal{A} = (18, \lceil 16^{18}/17^2 \rceil, 3)_{16}$$ [Dumer, Handbook CT]

- the resulting GCQC has parameters $((90, 2^{81.825}, 3))_2$
- the quantum Hamming bound reads $K(1+3n) \leq 2^n$, here $K < 2^{81.918}$
- the best stabilizer code has parameters $[\![90, 81, 3]\!]_2$
- the linear programming bound yields $K < 2^{81.879}$
- our code encodes 0.825 qubits more than any stabilizer code and at most 0.054 qubits less than the best possible code

A New Qutrit Non-Stabilizer Code

[Grassl, Shor, Smith, Smolin & Zeng, PRA **79** (2009), arXiv:0901.1319]

- inner code $B^{(0)} = \bigoplus_{i=0}^{80} B_i^{(1)}$ with each $B^{(1)} = ((10, 3^6, 3))_3$
- from the Hamming code $[84, 82, 3]_{83}$ over $GF(83)$ one can derive a code

$$\mathcal{A} = (84, \lceil 81^{84}/83^2 \rceil, 3)_{81}$$ [Dumer, Handbook CT]

- the resulting GCQC has parameters $((840, 3^{831.955}, 3))_2$
- the quantum Hamming bound reads $K(1 + 8n) \leq 3^n$, here $K < 3^{831.979}$
- the best stabilizer code has parameters $[\![840, 831, 3]\!]_3$
- the linear programming bound yields $K < 3^{831.976}$
- our code encodes 0.955 qutrits more than any stabilizer code and at most 0.021 qutrits less than the best possible code
- first non-stabilizer qutrit code better than any stabilizer code

A New Stabilizer Code $[\![36, 26, 4]\!]_2$

[Grassl, Shor & Zeng, ISIT 2009, arXiv:0905.0428]

inner codes: chain of nested stabilizer codes

$$B^{(0)} = [\![6, 6, 1]\!]_2 \supset B^{(1)} = [\![6, 4, 2]\!]_2 \supset B^{(2)} = [\![6, 0, 4]\!]_2.$$

classical outer codes

$$\mathcal{A}_1 = [6, 3, 4]_{2^{6-4}}, \qquad \mathcal{A}_2 = [6, 5, 2]_{2^{4-0}}, \qquad \mathcal{A}_3 = [6, 6, 1]_{2^6}$$

dimension

$$|\mathcal{A}_1| \times |\mathcal{A}_2| = (2^2)^3 (2^4)^5 = 2^6 2^{20} = 2^{26}$$

minimum distance

$$d \geq \min\{1 \times 4, 2 \times 2, 4 \times 1\} = 4$$

previously, only a code $[\![36, 26, 3]\!]_2$ was known [http://www.codetables.de]

Varying Inner Codes

[Dettmar et al., Modified Generalized Concatenated Codes . . . , IEEE-IT **41**:1499–1503 (1995)]

inner codes: chain of nested stabilizer codes

$$B_j^{(0)} = [\![n_j, n_j, 1]\!]_2 \supset B_j^{(1)} = [\![n_j, n_j - 6, 3]\!]_2$$

for $n_j \in \{7, \ldots, 17\} \cup \{21\}$

classical outer codes

$$\mathcal{A}_1 = [65, 63, 3]_{2^6}, \qquad \mathcal{A}_2 = [65, 65, 1]_{2^{2n_j-6}}$$

generalized concatenated quantum codes

$$[\![n, n-12, 3]\!]_2 \qquad \text{with } n = \sum_{j=1}^{65} n_j \in \{455, \ldots, 1361\} \cup \{1365\}$$

direct and simple construction of quantum codes with different length

A New Distance-Three Qubit Code

[Grassl, Shor & Zeng, ISIT 2009, arXiv:0905.0428]

inner codes: chain of nested stabilizer codes

$$B^{(0)} = [\![8, 8, 1]\!]_2 \supset B^{(1)} = [\![8, 6, 2]\!]_2 \supset B^{(2)} = [\![8, 3, 3]\!]_2.$$

classical outer codes

$$\mathcal{A}_1 = (6, 164, 3)_{2^{8-6}}, \qquad \mathcal{A}_2 = [6, 5, 2]_{2^{6-3}}, \qquad \mathcal{A}_3 = [6, 6, 1]_{2^{8+3}}$$

dimension

$$|\mathcal{A}_1| \times |\mathcal{A}_2| \times \dim(B^{(2)})^6 = 164 \times (2^3)^5 \times 2^{3\times 6} \approx 2^{40.358}$$

minimum distance

$$d \geq \min\{1 \times 3, 2 \times 2, 3 \times 1\} = 3$$

LP bound $K < 2^{40.791}$, hence the best stabilizer code is $[\![48, 40, 3]\!]_2$

Overview

- Shor's nine-qubit code revisited
- The code $[\![25, 1, 9]\!]$
- Concatenated graph codes
- Generalized concatenated quantum codes

$\Rightarrow$ **Codes for the Amplitude Damping (AD) channel**
[Duan, Grassl, Ji & Zeng, Multi-Error-Correcting Amplitude Damping Codes, ISIT 2010, arXiv:1001.2356]

- Conclusions

Amplitude Damping (AD) Channel

- with some probability, an excited quantums state $|1\rangle$ decays into the ground state $|0\rangle$, i. e., $|1\rangle \rightarrow |0\rangle$
- modeled by error operator $A_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}$
- at low temperature, spontaneous excitation $|0\rangle \rightarrow |1\rangle$ is negligible
- from $\sum_k A_k^\dagger A_k = I$ we get $A_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}$
- channel model

$$\mathcal{E}_{\text{AD}}(\rho) = A_0 \rho A_0^\dagger + A_1 \rho A_1^\dagger$$

notes:

- The channel operators do not contain identitiy I.
- Similar to the classical $\mathcal{Z}$-channel, but also error A_0.

Approximate Error Correction

(see [Leung, Nielsen, Chuang & Yamamoto, *Physical Review A*, 56(4):2567–2573, 1997])

Perfect error correction

Knill-Laflamme conditions for code with basis $|c_i\rangle$ and for error operators A_k:

$$\langle c_i|A_k^\dagger A_l|c_j\rangle = \delta_{ij}\alpha_{kl}, \qquad \text{where } \alpha_{kl} \in \mathbb{C}$$

Approximate error correction

correction of errors up to some order t (t-code)

$$\langle c_i|A_k^\dagger A_l|c_j\rangle = \delta_{ij}\alpha_{kl} + O(\gamma^{t+1}) \qquad \text{where } \alpha_{kl} \in \mathbb{C}$$

Example: code from [Leung et al.] with $t = 1$

$$|0\rangle_L = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle) \qquad |1\rangle_L = \frac{1}{\sqrt{2}}(|0011\rangle + |1100\rangle)$$

Amplitude Damping (AD) Channel

Relation to Pauli errors

$$A_0 = \frac{1+\sqrt{1-\gamma}}{2} I + \frac{1-\sqrt{1-\gamma}}{2} Z$$

$$A_1 = \frac{\sqrt{\gamma}}{2}(X+iY) \quad \text{and} \quad A_1^\dagger = \frac{\sqrt{\gamma}}{2}(X-iY)$$

- quantum error-correction is linear in the error operators
- A_1 and $A_1^\dagger$ span the same space of operators as X and Y

$\Longrightarrow$ codes for an asymmetric quantum channel can be used for the AD channel

but: We don't need to correct for $A_1^\dagger$.

Expansion of the Errors

Relation to Pauli errors

$$A_0 = \frac{1+\sqrt{1-\gamma}}{2} I + \frac{1-\sqrt{1-\gamma}}{2} Z$$

$$A_1 = \frac{\sqrt{\gamma}}{2}(X+iY) \quad \text{and} \quad A_1^\dagger = \frac{\sqrt{\gamma}}{2}(X-iY)$$

note: $1-\sqrt{1-\gamma} = \frac{1}{2}\sqrt{\gamma}^2 + \frac{1}{8}\sqrt{\gamma}^4 + \frac{1}{16}\sqrt{\gamma}^6 + \frac{5}{128}\sqrt{\gamma}^8 + O(\sqrt{\gamma}^{10})$

$\Longrightarrow$ For a t-code, it is sufficient to independently correct $t+1$ errors Z and $2t+1$ errors X.

Proposition [Gottesman, PhD thesis]
An $[[n,k]]$ CSS code of X-distance $2t+1$ and Z-distance $t+1$ is an $[[n,k]]$ t-code.

Quantum Dual Rail Code

Lemma

Using the quantum *dual-rail code* $\mathcal{Q}_1$ which encodes a single qubit into two qubits, given by

$$|0\rangle_L = |01\rangle, \quad |1\rangle_L = |10\rangle,$$

two uses of the AD channel simulate a quantum erasure channel.

Proof

For the basis states $|i\rangle_L$ of $\mathcal{Q}_1$ we compute

$$\begin{aligned} (A_0 \otimes A_0)|i\rangle_L &= \sqrt{1-\gamma}|i\rangle_L \\ (A_0 \otimes A_1)|i\rangle_L = (A_1 \otimes A_0)|i\rangle_L &= \sqrt{\gamma}|00\rangle \\ (A_1 \otimes A_1)|i\rangle_L &= 0. \end{aligned} \tag{1}$$

Hence for any state ρ of the code $\mathcal{Q}_1$, we get

$$\mathcal{E}_{\mathsf{AD}}^{\otimes 2}(\rho) = (1-\gamma)\rho + \gamma(|00\rangle\langle 00|).$$

Quantum Dual Rail Code

Theorem

If there exists an $[[m, k, d]]$ quantum code $\mathcal{Q}_2$, then there is a $[[2m, k]]$ t-code $\mathcal{Q}$ correcting $t = d - 1$ amplitude damping errors.

Proof

- $\mathcal{Q}$ is the concatenation of $\mathcal{Q}_2$ with the quantum dual rail code $\mathcal{Q}_1$
- the effective channel for the outer code $\mathcal{Q}_2$ is

$$\mathcal{E}_{\mathsf{AD}}^{\otimes 2}(\rho) = (1-\gamma)\rho + \gamma(|00\rangle\langle 00|)$$

- $\mathcal{Q}_2$ corrects $d - 1$ erasure errors

Length Comparison with CSS and Stabilizer Codes

CSS code					stab. code	concatenation
n	k	t	$t+1$	$2t+1$	n'	$2m$
12–13	1	2	3	5	11	10
19–20	1	3	4	7	17	20
25–30	1	4	5	9	23–25	22
33–41	1	5	6	11	29	32
39–54	1	6	7	13	35–43	34
47–70	1	7	8	15	41–53	44–48
53–79	1	8	9	17	47–61	46–50
61–89	1	9	10	19	53–81	56
67–105	1	10	11	21	59–85	58

Length Comparison with CSS and Stabilizer Codes

CSS code					stab. code	concatenation
n	k	t	$t+1$	$2t+1$	n'	$2m$
14–17	2	2	3	5	14	16
20–27	2	3	4	7	20–23	20
27–37	2	4	5	9	26–27	28
34–45	2	5	6	11	32–41	32
41–62	2	6	7	13	38–51	40–46
48–71	2	7	8	15	44–59	44–52
55–87	2	8	9	17	50–78	52–54
62–102	2	9	10	19	56–83	56–56
69–110	2	10	11	21	62–104	64–82

Distance Comparison with Stabilizer Codes

n	k	t	$2t+1$	d	t'
8	1	1	3	3	1
10	1	2	5	4	1
20	1	3	7	7	3
22	1	4	9	7–8	3
32	1	5	11	11	5
34	1	6	13	11–12	5
48	1	7	15	13–17	6–8
50	1	8	17	13–17	6–8
56	1	9	19	15–19	7–9
58	1	10	21	15–20	7–9

n	k	t	$2t+1$	d	t'
8	2	1	3	3	1
16	2	2	5	6	2
20	2	3	7	6–7	2–3
28	2	4	9	10	4
32	2	5	11	10–11	4–5
46	2	6	13	12–16	5–7
52	2	7	15	14–18	6–8
54	2	8	17	14–18	6–8
56	2	9	19	14–19	6–9
82	2	10	21	18–28	8–13

Distance Comparison with Stabilizer Codes

n	k	t	$2t+1$	d	t'
16	5	1	3	4–5	1–2
22	5	2	5	6–7	2–3
28	5	3	7	7–9	3–4
36	5	4	9	8–11	3–5
42	5	5	11	9–13	4–6
50	5	6	13	11–16	5–7
60	5	7	15	13–19	6–9
78	5	8	17	15–25	7–12
86	5	9	19	18–28	8–13
98	5	10	21	19–32	9–15

n	k	t	$2t+1$	d	t'
16	6	1	3	4	1
24	6	2	5	6–7	2–3
28	6	3	7	6–8	2–3
36	6	4	9	8–11	3–5
48	6	5	11	10–15	4–7
58	6	6	13	12–19	5–9
64	6	7	15	14–21	6–10
84	6	8	17	17–27	8–13
92	6	9	19	18–29	8–14
104	6	10	21	19–33	9–16

Conclusions

- concatenation yields large codes from small components
- generalized concatenation for quantum codes allows the use of classical outer codes
- outer codes need not be linear
- construction of non-additive quantum codes with higher dimension than stabilizer codes
- simple construction of QECCs with varying length
- structured encoding circuits

- concatenation allows to transform channels