Topological Color Codes

“Topological Quantum Distillation”, Phys. Rev. Lett. 97 180501 (2006)

“Topological Computation without Braiding”, Phys. Rev. Lett. 98, 160502 (2007)

“Exact Topological Quantum Order in D=3 and Beyond”, Phys. Rev. B 75, 075103 (2007)

“Optimal Resources for Topological Stabilizer Codes”, Phys. Rev. A 76, 012305 (2007)

“Statistical Mechanical Models and Topological Color Codes”, arXiv:0711.0468

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Outline

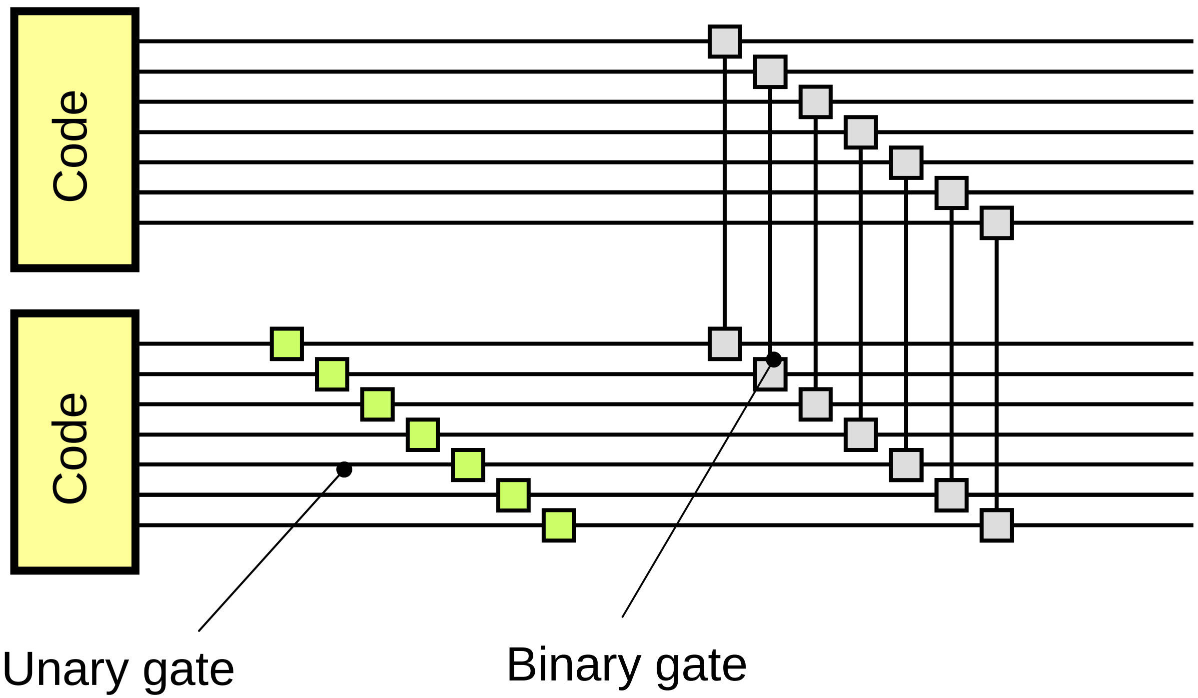
- Stabilizer codes and transversal gates
- Topological codes: Surface codes
- Color codes
- Triangular codes: Transversal Clifford gates.
- Conection with classical statistical mechanics.
- *D*-colexes
- 3D color codes
- Tetrahedral codes: universality.
- Topological Order

Stabilizer Codes

- A **stabilizer code**[1] C of length n is a subspace of the Hilbert space of a set of n qubits. It is defined by a stabilizer group S of Pauli operators, i.e., tensor products of Pauli matrices.

$$|\psi\rangle \in \mathcal{C} \qquad \Longleftrightarrow \qquad \forall\, s \in \mathcal{S} \quad s|\psi\rangle = |\psi\rangle$$

- Some stabilizer codes are specialy suitable for quantum computation. They allow to perform operations in a **transversal** and **uniform** way:



[1] D. Gottesman 95

Gate Sets

- Several codes allow the transversal implementation of the gates

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} \qquad K = \begin{pmatrix}1 & 0\\ 0 & i\end{pmatrix} \qquad \Lambda = \begin{pmatrix}I_2 & 0\\ 0 & X\end{pmatrix}$$

which generate the **Clifford group**. This is useful for quantum information tasks such as teleportation or **entanglement distillation**.

- Quantum **Reed-Muller** codes[1] are very special. They allow **universal computation** through transversal gates

$$K^{1/2} = \begin{pmatrix}1 & 0\\ 0 & i^{1/2}\end{pmatrix} \qquad \Lambda = \begin{pmatrix}I_2 & 0\\ 0 & X\end{pmatrix}$$

and transversal measurements of X and Z.

- We will see how both sets of operations can be transversally implemented in 2D and 3D topological color codes:

Color Codes = Transversality + Topology

[1] E. Knill *et al.*

Topological Stabilizer Codes

- For a TSC we mean a code in which:

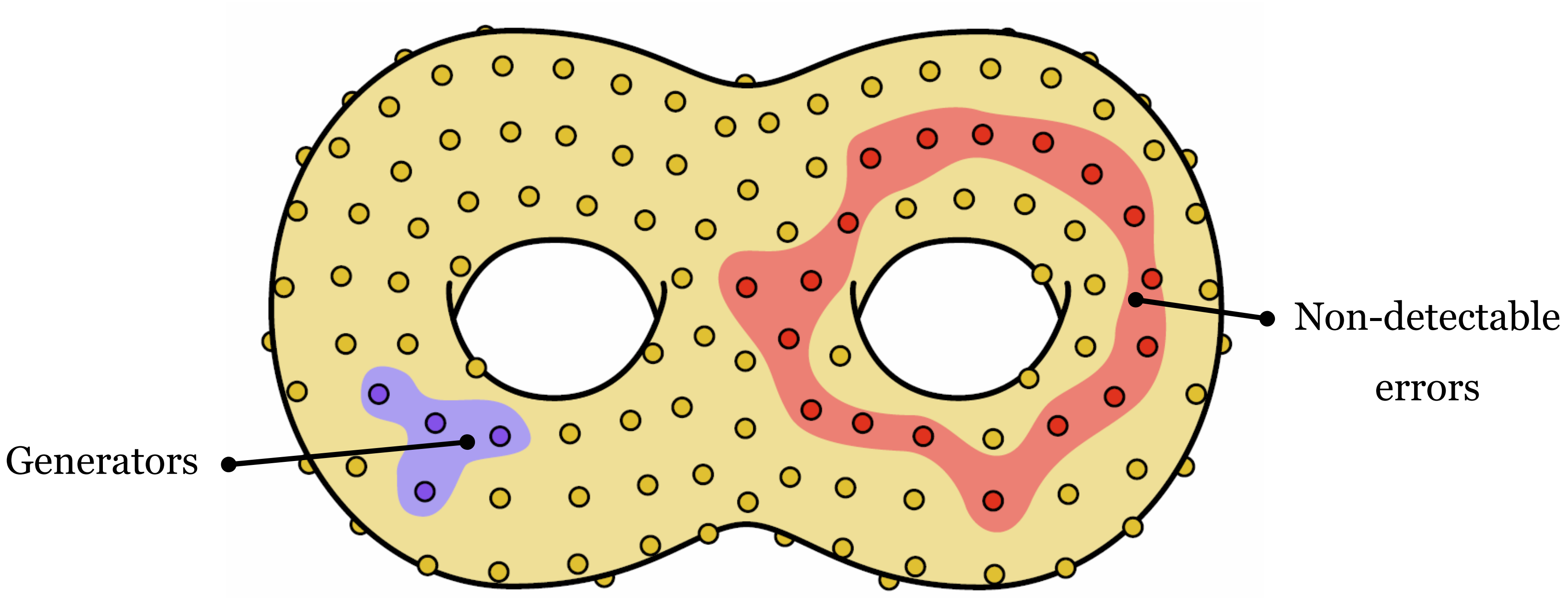
  a) the **generators** of the stabilizer are **local** and

  b) **non-detectable** errors have a **global** (topological) nature.
- Usually we consider TSCs in which

  a) qubits are placed on a surface,

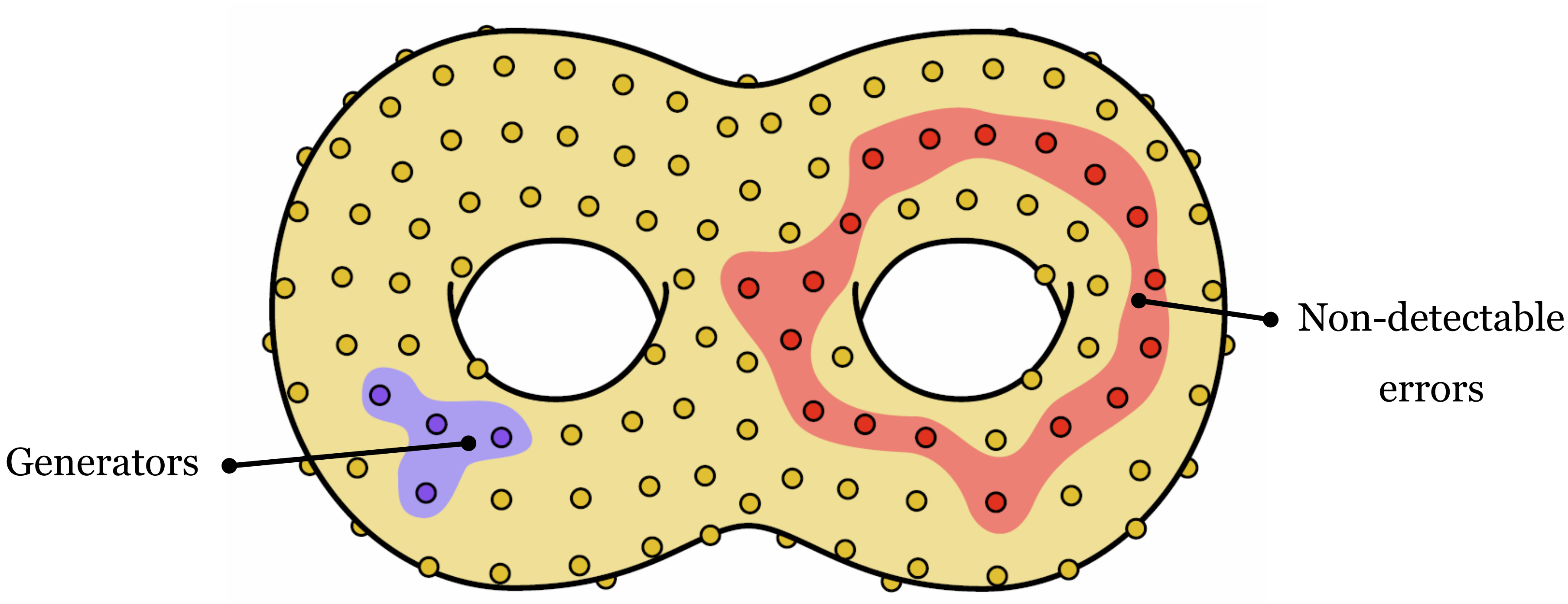
  b) the stabilizer $\mathcal{S}$ is composed of **boundaries** and its normalizer $N_{\mathcal{S}}$ of **cycles**,

  c) non-detectable errors are related to cycles which are not boundaries (homology…).



Topological Stabilizer Codes

- When working with stabilizer codes, it is enough to measure a set of generators of the stabilizer in order to perform error correction.
- The nice property of TSCs is their locality: one can construct arbitrarily robust codes while the generators of the stabilizer remain local and with a fixed support.
- It turns out that the best strategy to perform error correction within TSCs is to continuously measure local generators (Dennis *et al.* '02).

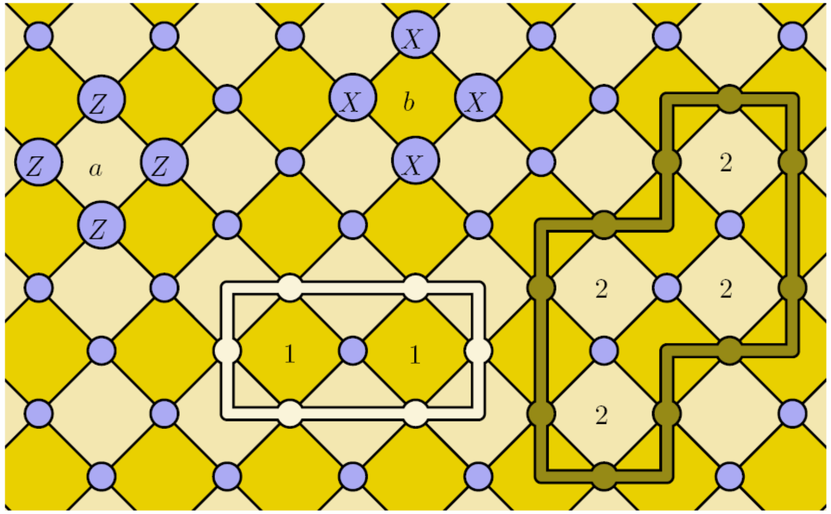


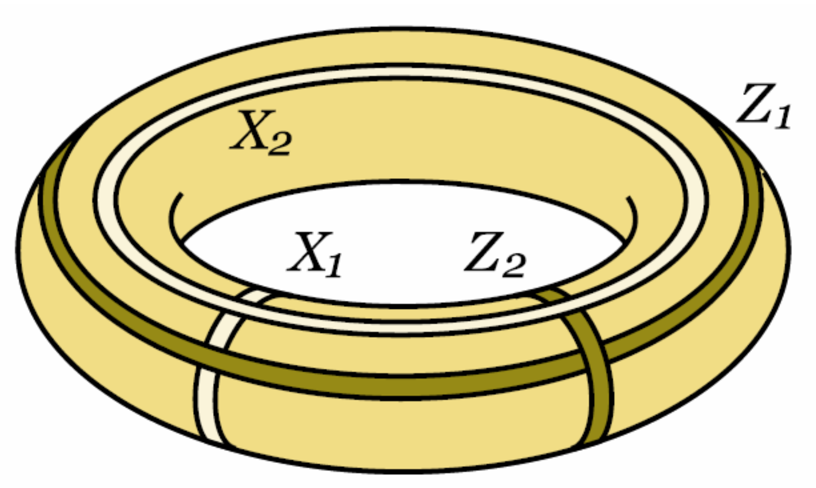
Surface Codes

- To construct a surface code (Kitaev '07, aka toric code), one starts from a **4-valent** lattice with **2-colorable** faces.
- Each **vertex** corresponds to a **qubit**.
- The generators of the stabilizer are light and dark **plaquette operators**:

$$B_a^Z := Z_1 Z_2 Z_3 Z_4$$

$$B_b^Z := X_5 X_6 X_7 X_8$$

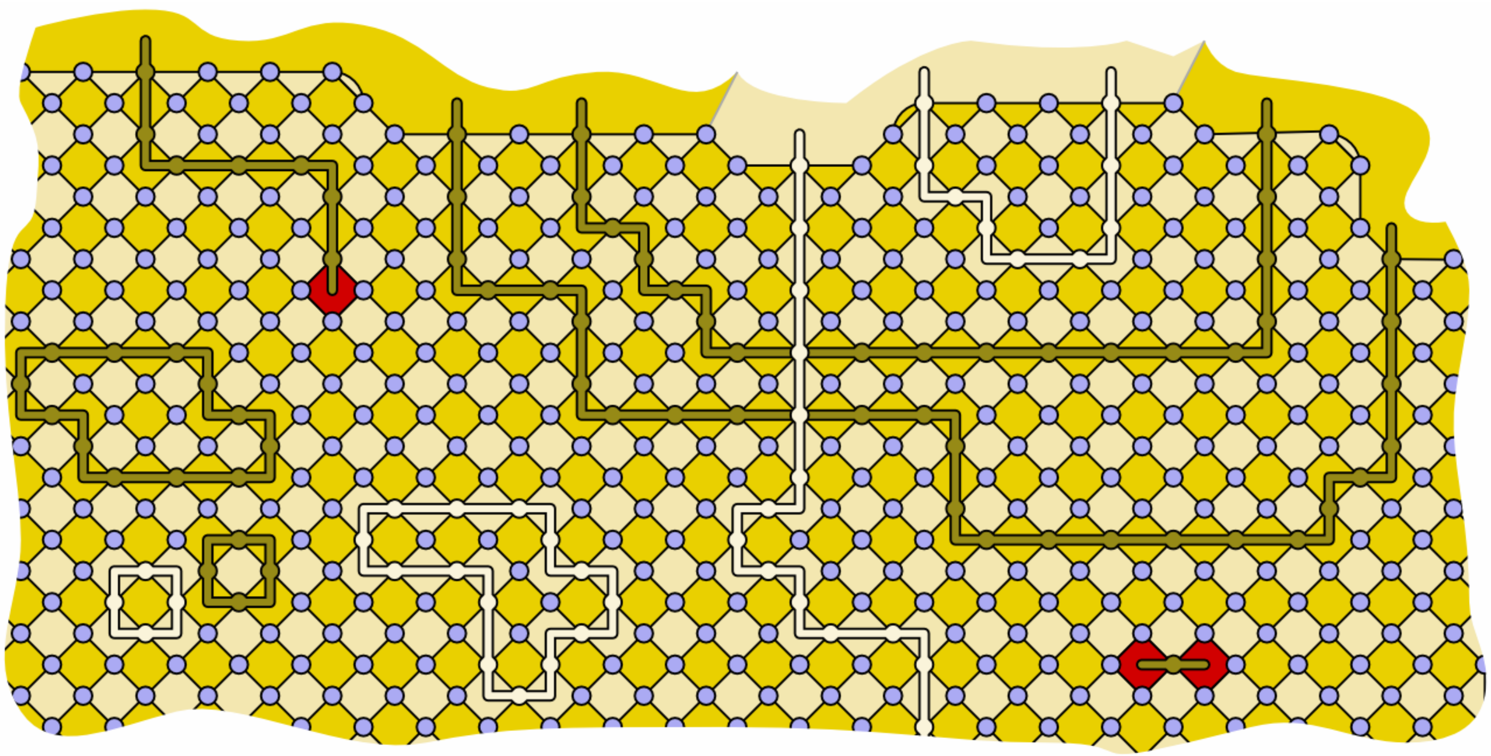




- Dark (light) **string operators** are products of Z-s (X-s).
- Plaquette operators generate the stabilizer: boundary string operators.
- Closed strings compose its normalizer.
- **Crossing** dark and light strings operators **anticommute**.
- **Encoded X-s** and **Z-s** can be chosen from those closed strings which are not boundaries.

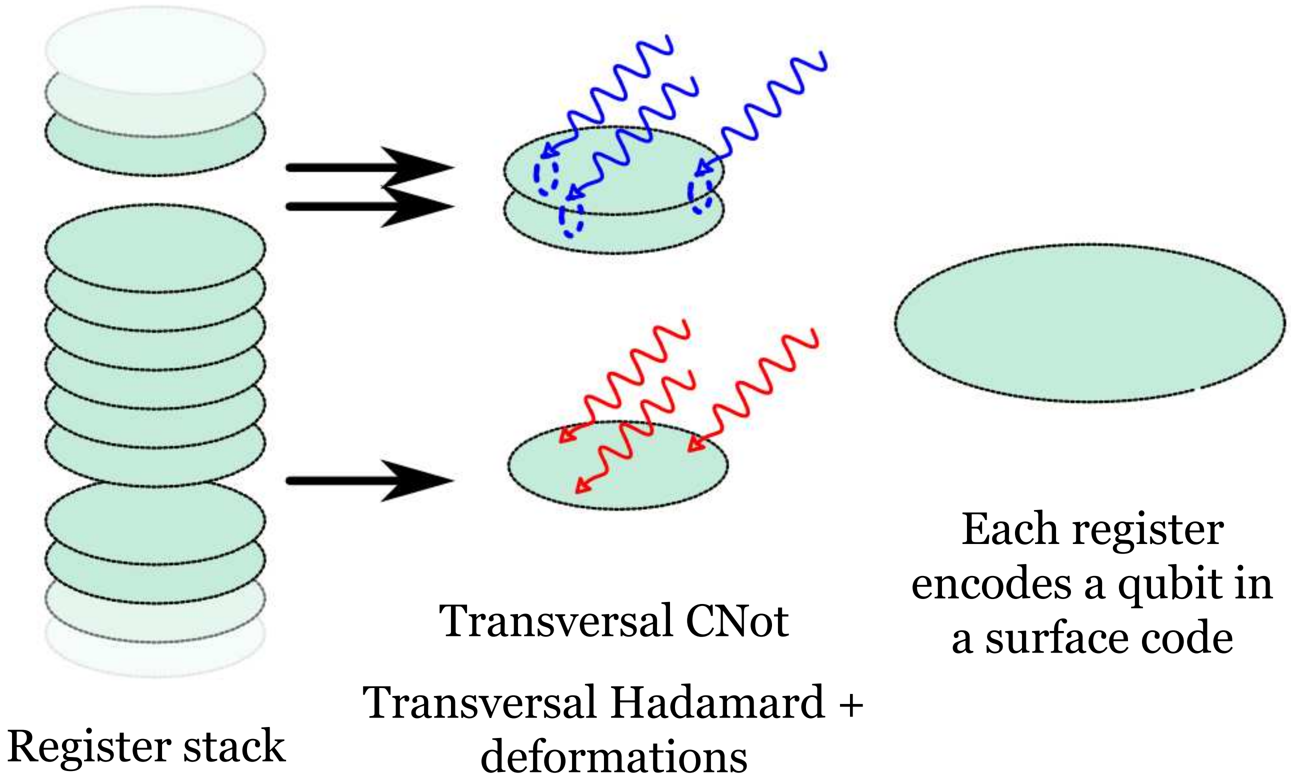
Borders

- To obtain **planar** codes, we need to introduce the notion of border.
- An open strings has **endpoints** at plaquettes of its color. The string operator generates **violations** of the corresponding plaquette stabilizers.
- Then, if a plaquette operator is **missing**, strings can end at it and still be ‘closed’.
- A dark (light) border is a big missing dark (light) plaquette, where dark (light) strings can end.
- Strings that start and end in the same border are boundaries.



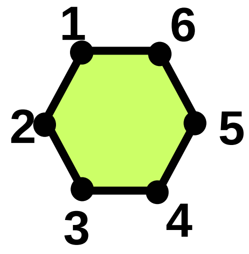
Pancake Quantum Computer

- Imagine a quantum computer in the form of a **stack of layers** (Dennis *et al.* '02)
- Each layer corresponds to a single-qubit encoded in a **surface code**.
- Measurements of the stabilizers are continuous to **keep track of errors**.
- CNot gates are performed in a purely **transversal** way, but others require **code deformations** and **distillation**.
- Can we find topological codes implementing other gates transversally? **Yes!**



Color Codes

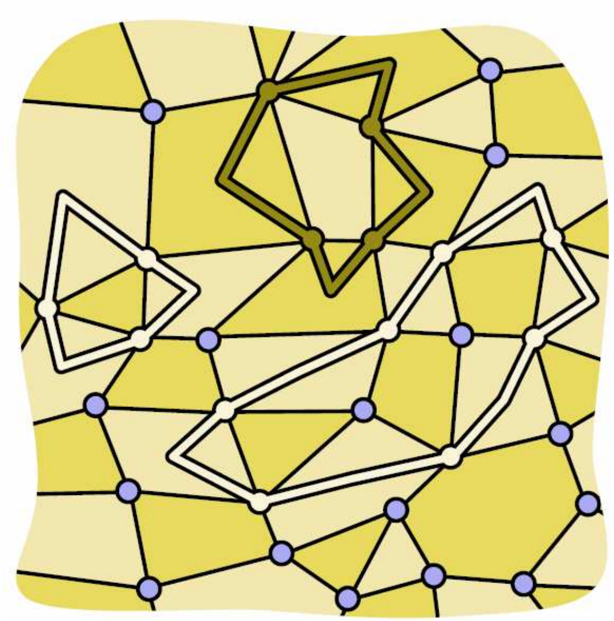
- Color codes are obtained from **trivalent** lattices with **3-colorable** faces.
- Faces are classified in red, green and blue.
- Each **vertex** corresponds to a **qubit**.



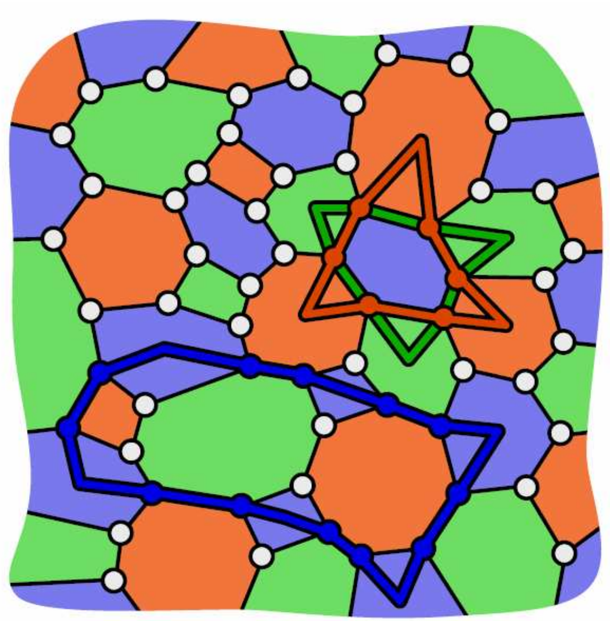
$$B_f^X = X_1 X_2 X_3 X_4 X_5 X_6$$

$$B_f^Z = Z_1 Z_2 Z_3 Z_4 Z_5 Z_6$$

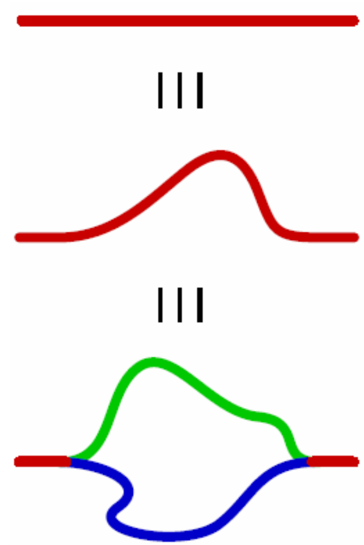
- The generators of the stabilizer are X and Z **plaquette operators**.
- As plaquettes, strings come in three colors.
- Strings not only can be deformed. A new feature appears: **branching** points.



Surface codes

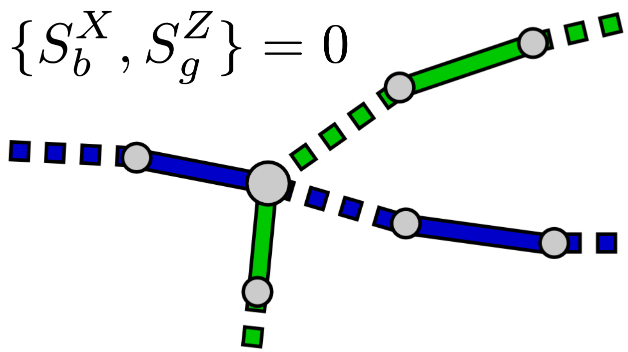


Color codes

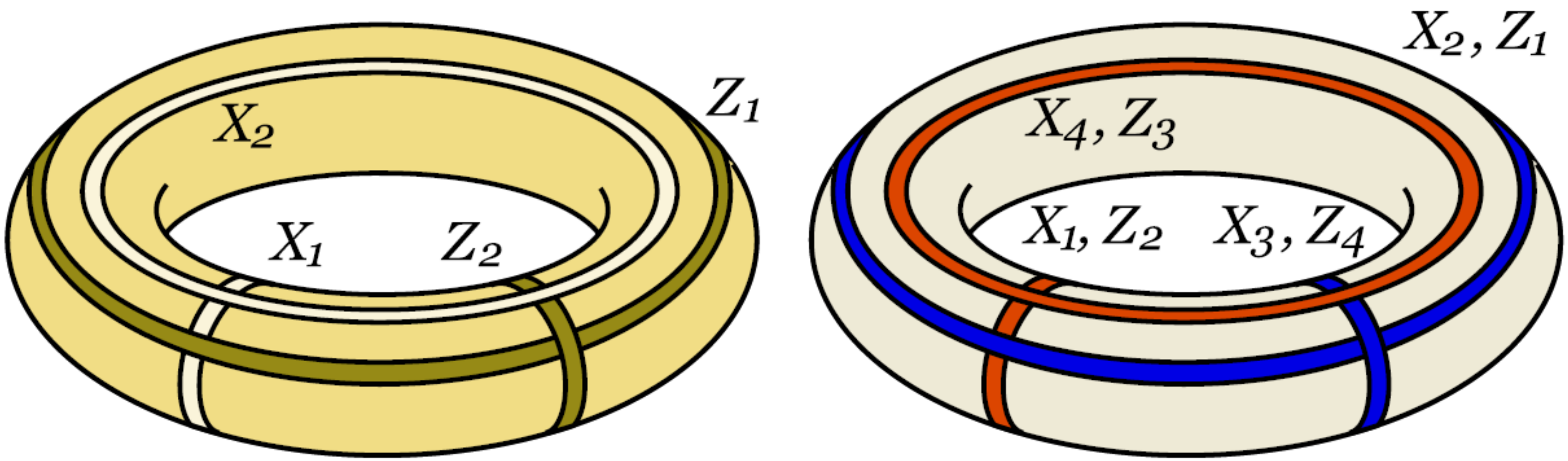


String Operators

- For each colored string S, there are a **pair** of **string operators**, S^X and S^Z, products of Xs or Zs along S.
- String operators either commute or anticommute.
- Two string operators **anticommute** when they have **different color and type** and **cross** an odd number of times.



- As in surface codes, encoded X and Z operators can be chosen from closed string operators which are not boundaries.
- The number of **encoded qubits** is **twice** as in a surface code:

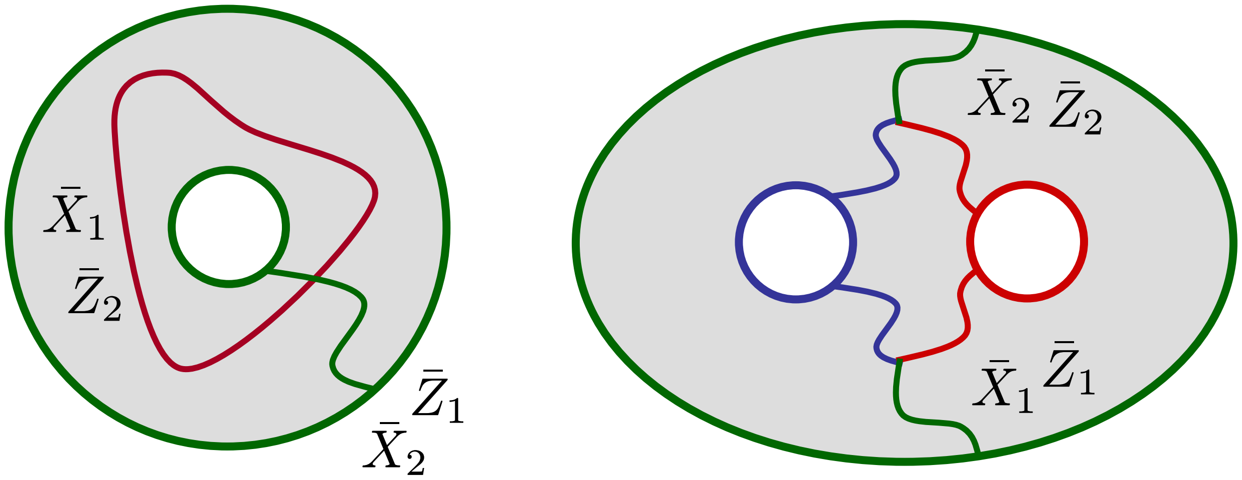


Surface code: 2 qubits

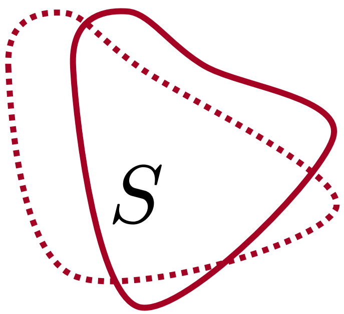
Color code: 4 qubits

Borders and String-Nets

- Borders are big missing plaquettes. Their color is that of the erased plaquette.

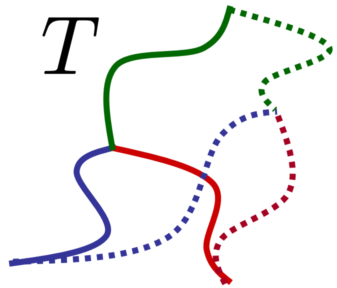


- Both examples encode 2 qubits, but the second requires **string-net operators**.
- These have a new feature, which turns out to be crucial in orther to be able to implement transversally the whole Clifford group:



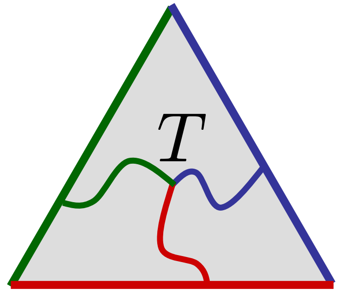
$$[S^X, S^Z] = 0$$

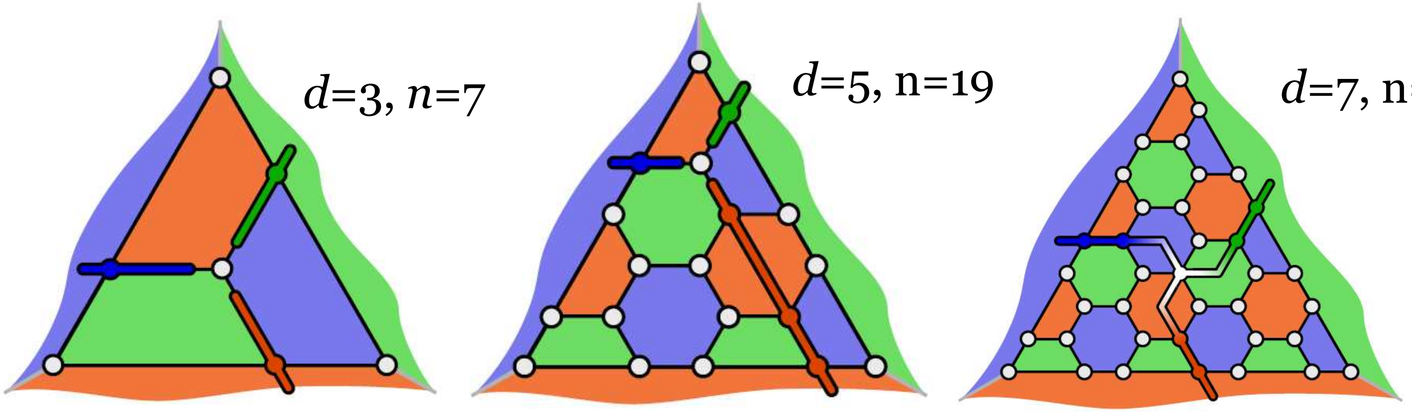
$$\{T^X, T^Z\} = 0$$



Triangular Codes

- These are color codes encoding a **single qubit**.
- All strings in such a code are boundaries (belong to the stabilizer).
- The encoded X and Z are given by the **string-net** operators T^X and T^Z.





- They require **less** qubits than their surface code counterparts.

- A transversal H leaves the code invariant. For a transversal K, this is true only if the vertices per face are $v=4x$:

$$\hat{H} B_f^X \hat{H}^\dagger = B_f^Z \qquad \hat{K} B_f^X \hat{K}^\dagger = (-)^{\frac{v}{2}} B_f^X B_f^Z$$

$$\hat{H} B_f^Z \hat{H}^\dagger = B_f^X \qquad \hat{K} B_f^Z \hat{K}^\dagger = B_f^Z$$

- Under this condition, in triangular codes we can implement transversally H and K gates because:

$$\hat{H}\hat{X}\hat{H}^\dagger = \hat{Z} \qquad \hat{K}\hat{X}\hat{K}^\dagger = \pm i \hat{X}\hat{Z}$$

$$\hat{H}\hat{Z}\hat{H}^\dagger = \hat{X} \qquad \hat{K}\hat{Z}\hat{K}^\dagger = \hat{Z}$$

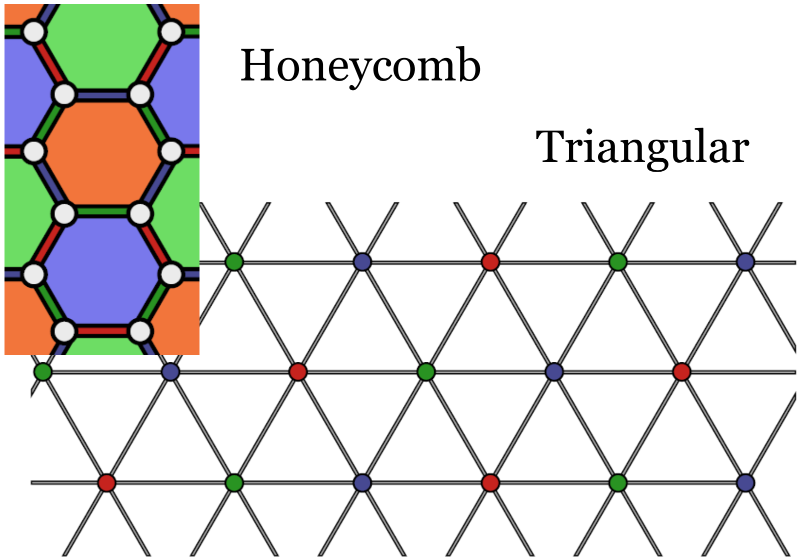
- The CNot is also transversal as in surface codes: both families of codes are CSS.
- Thus we can implement the whole **Clifford group** transversally.

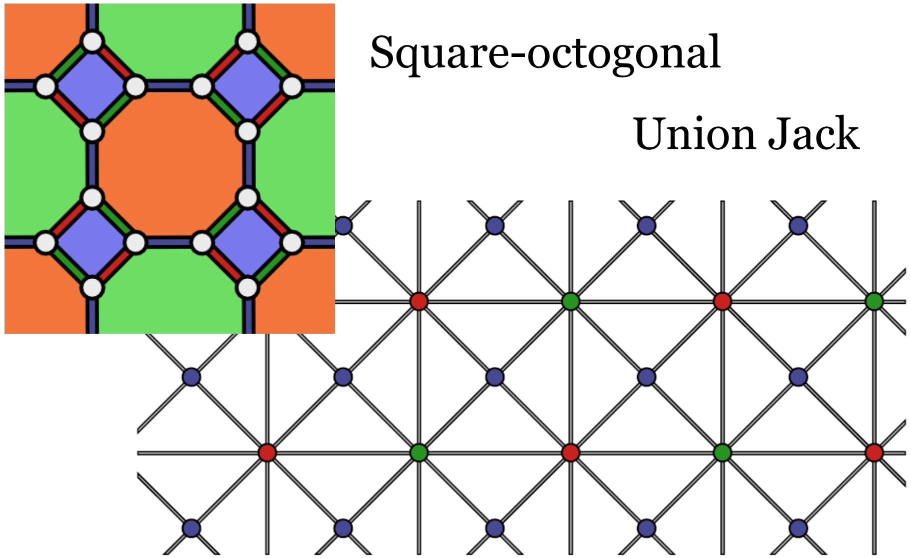
Classical Statistical Models

- Color codes can be connected with certain **classical 3-body Ising models**. Their partition function is the **overlapping** of a color code and certain product state:

$$\mathcal{H} := -J \sum_{\langle i,j,k \rangle} \sigma_i \sigma_j \sigma_k \qquad\qquad \mathcal{Z}(\beta J) \propto \langle \Psi_{\mathrm{c}} | \Phi_{\mathrm{P}} \rangle$$

- For honeycomb and 4-8 lattices, the model lives in triangular and union-jack lattices.
- Recall that transversal K gates are possible in 4-8 lattices but not in the honeycomb.
- At the same time, the **universality classes** of the classical models are **different**!

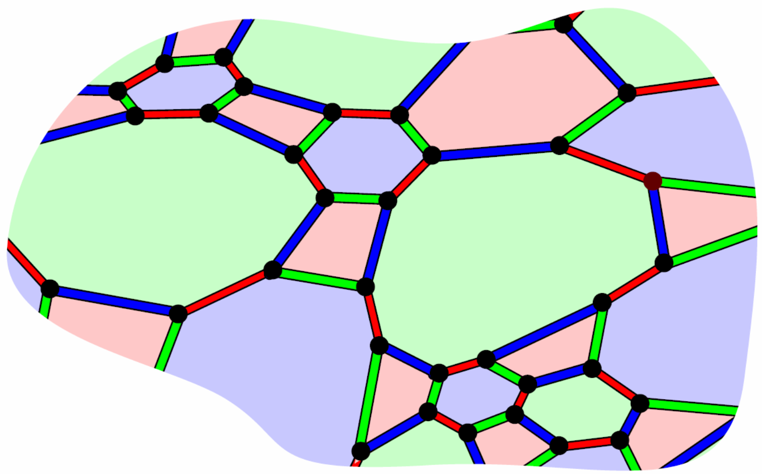


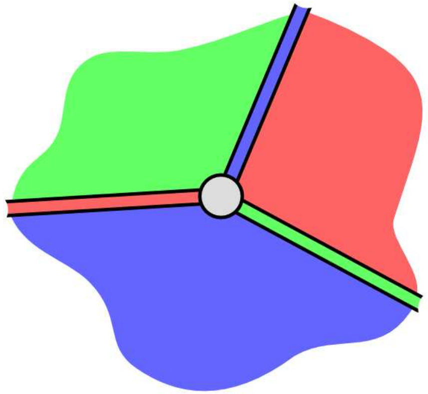


- **Random** versions of these classical models appear in the computation of the **threshold** of color codes (work in progress).

D-Colexes

- Color codes can be generalized to higher spatial dimensions *D*.
- First we have to generalize our 2D lattice. Note that **edges** can be **colored** in accordance with faces, so that at each vertex there are 3 links meeting, one of each color.

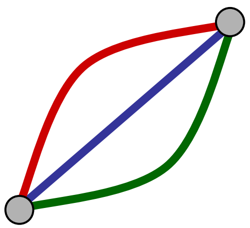




Local appearance of the lattice.

- In fact, the **whole structure** of the lattice is contained in its **colored graph**: faces can be reconstructed from edge coloring.

S



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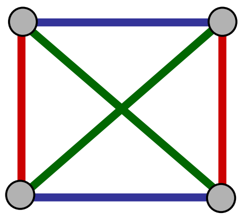

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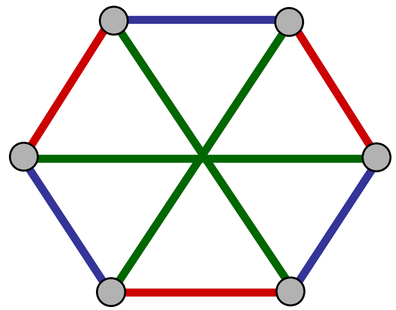
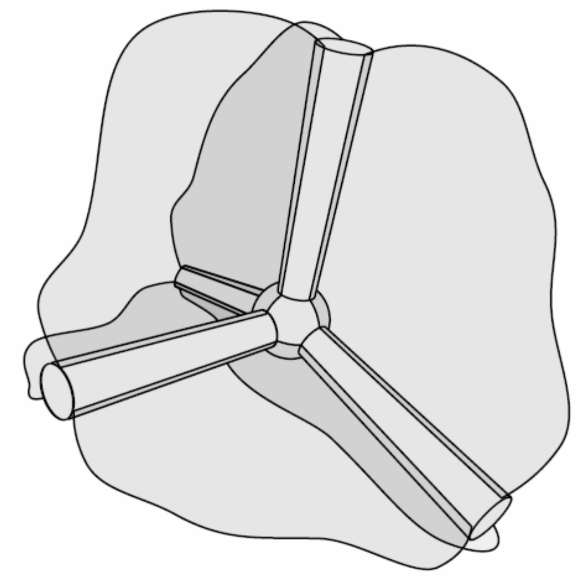
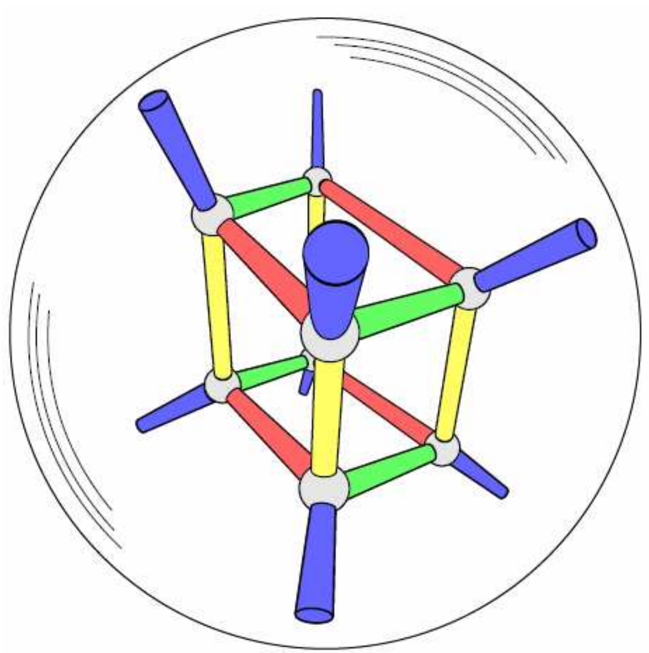


D-Colexes

- In dimension *D*, we consider graphs with *D+1* edges meeting at each vertex, of *D+1* different colors.
- Such graphs, with certain additional properties, give rise to *D*-manifolds. We call the resulting colored lattices *D*-colexes (for color complex).
- Of particular interest is the case D=3:



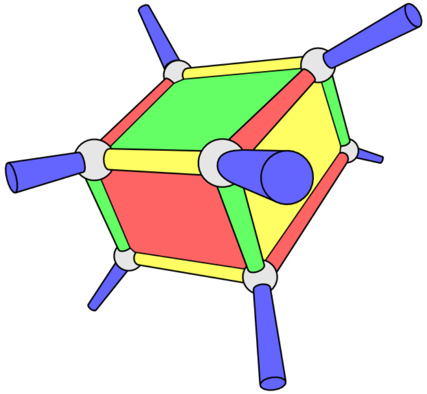
The neigborhood of a vertex.



The simplest 3-colex in projective space.

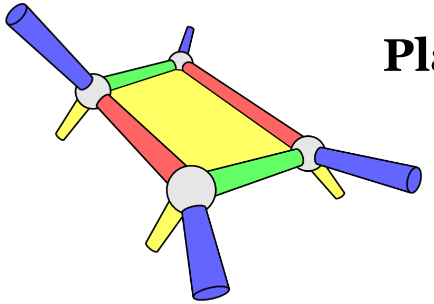
3D Color Codes

- Again one qubit per vertex, but now we have face and (3-) cell operators generating $\mathcal{S}$.



Cell operators

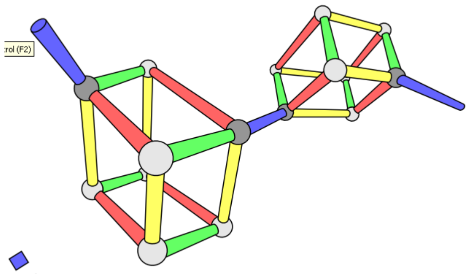
$$B_c^X = \bigotimes_{i=1}^{8} X_i$$

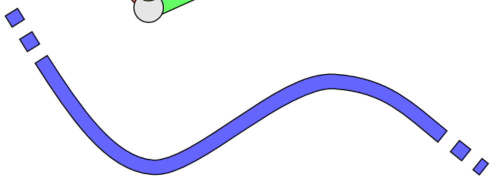


Plaquette operators

$$B_f^Z = \bigotimes_{i=1}^{4} Z_i$$

- **Strings** are constructed as in 2-D, but now come in **four colors**.
- The new feature are **membranes**. They come in **six color** combinations and, as strings, have **branching** properties.

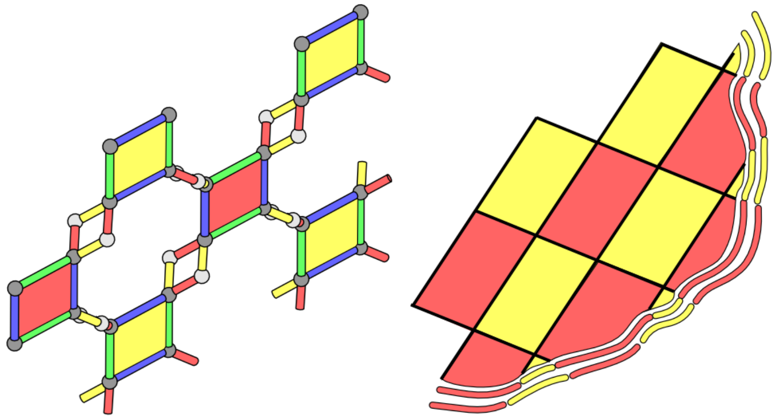




b-string

$$S^Z = \bigotimes_{\text{string}} Z_i$$

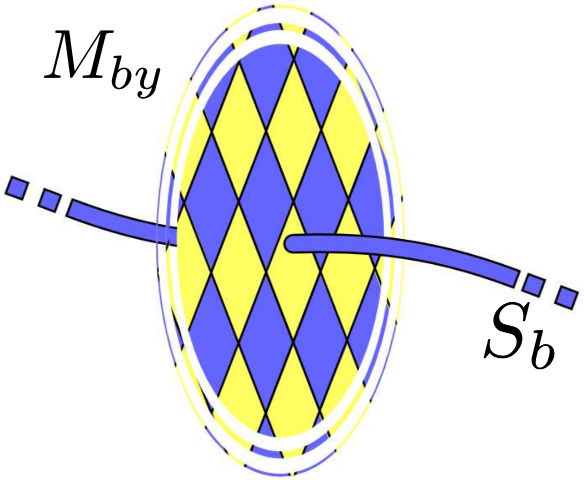
$$M^X = \bigotimes_{\text{membrane}} X_i$$



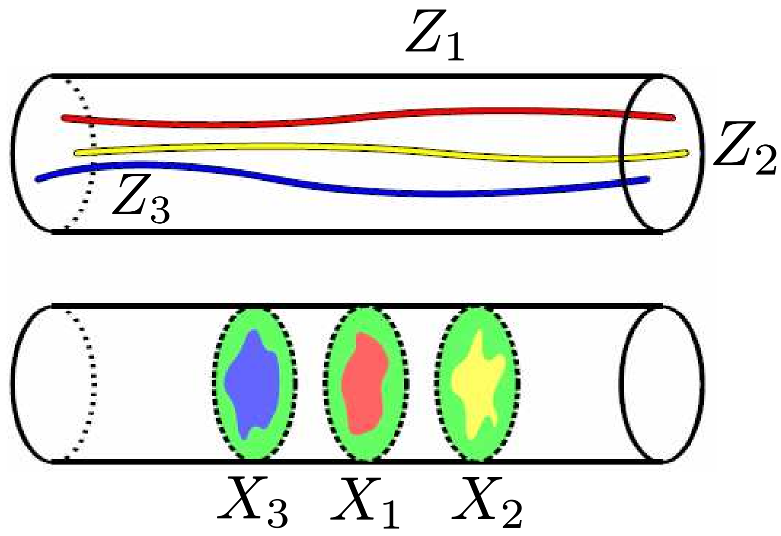
ry-membrane

3D Color Codes

- Now there are **3 independent** colors for strings (and 3 combinations for membranes).
- The number of **encoded qubits** is $3h_1 = 3h_2$, where h_i is the *i*-th Betty number.
- String and membrane operators **anticommute** only if they **share a color** and the string **crosses** an odd number of times the membrane.
- **Encoded X** and **Z** operators can be chosen from closed string and membrane operators which are not boundaries.



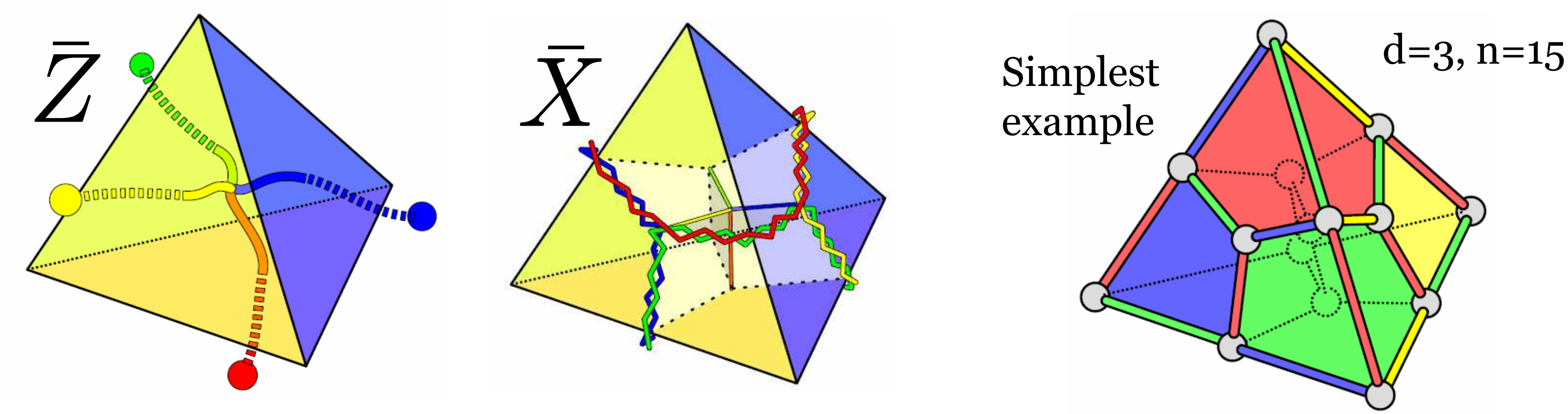
$$\{S_b^Z, M_{by}^X\} = 0$$



A pauli basis for the operators on the 3 qubits encoded in S2xS1.

Tetrahedral Codes

- 3-colexes cannot be constructed in our everyday 3D world keeping the locality structure unless we allow boundaries.
- As in *2D*, **borders** are big erased cells and they have the **color** of the **erased cell**.
- Given a border of color *c*, strings can end at it if they are *c*-strings and membranes can end at it if they are *xy*-strings with *x* and *y* different of *c*.
- The analogue of triangular codes are **tetrahedral** codes, which encode a **single** qubit.



- The desired **transversal** ***K***

$$|\hat{0}\rangle := \prod_c (1 + B_c^X)|\mathbf{0}\rangle = \sum_{\mathbf{v}\in V} |\mathbf{v}\rangle \qquad |\hat{1}\rangle := \hat{X}|\hat{0}\rangle \qquad \hat{K}^{1/2}|\hat{0}\rangle = |\hat{0}\rangle$$

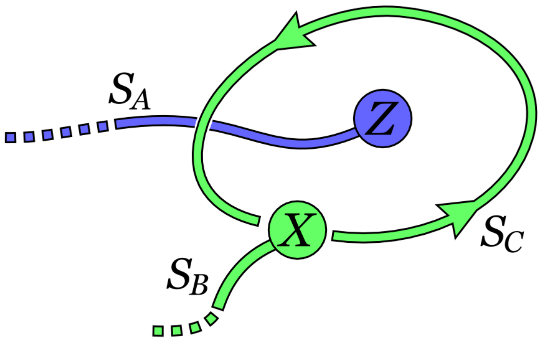
$$\forall\, \mathbf{v} \in V \quad \mathrm{wt}(\mathbf{v}) \equiv 0 \mod 8 \qquad l = 1,3,5,7 \qquad \hat{K}^{1/2}|\hat{1}\rangle = i^{l/2}|\hat{1}\rangle$$

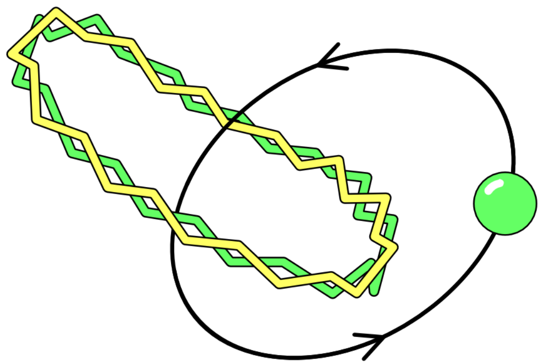
Topological Order

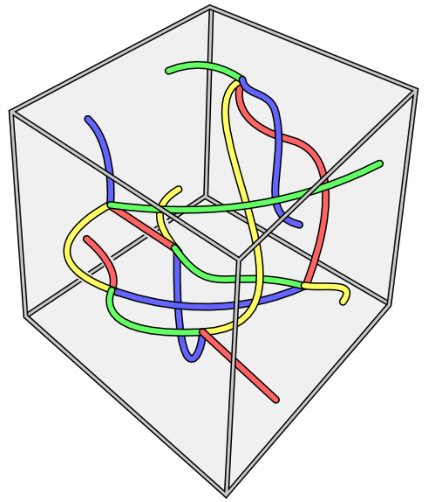
- A physical system showing **topological order** can be related to every TSC:

$$H = -\sum_{O \in \mathcal{S}'} O \qquad\qquad \mathcal{S}' = \text{Set of local generators of } \mathcal{S}$$

- For *2*-colexes, the excitations are abelian **anyons**, because monodromy operations can give global phases.
- For *3*-colexes, **charges and fluxes** exist. The topological content is related to the fact that charges can wind arround fluxes.
- For *D*-colexes, the resulting systems are **brane-net condensates**. The excitations are abelian **branyons**. For *D*>3, different topological orders are possible.







Conclusions

- *D*-colexes are *D*-valent complexes with *D*-colorable edges.
- Topological color codes are obtained from *D*-colexes.
- 2-colexes allow transversal Clifford gates.
- 3-colexes allow the same transversal gates as Reed-Muller codes.
- 2D color codes are related to classical 3-body Ising models.
- Brane-net condensate models arrise from color codes.