Protecting quantum information encoded in decoherence-free states against exchange errors

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The exchange interaction between identical qubits in a quantum-information processor gives rise to unitary two-qubit errors. It is shown here that decoherence-free subspaces (DFSs) for collective decoherence undergo Pauli errors under exchange, which, however, do not take the decoherence-free states outside of the DFS. In order to protect DFSs against these errors it is sufficient to employ a recently proposed concatenated DFS quantum-error-correcting code scheme [D. A. Lidar, D. Bacon, and K.B. Whaley, Phys. Rev. Lett. **82**, 4556 (1999)].

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I. INTRODUCTION

Preserving the coherence of quantum states and controlling their unitary evolution is one of the fundamental goals of quantum-information processing [1]. When the system Hamiltonian is invariant under particle permutations, the exchange operator E_{ij} interchanging particles i and j is a constant of the motion, and definite symmetry of a state will be conserved. Models of quantum computers based on identical bosons or fermions must of course respect this elementary requirement. It was pointed out in a recent paper [2] that active quantum-error-correcting codes (QECCs) [3] designed to correct independent single-qubit errors, will fail for *identical* particles in the presence of exchange errors. The reason is that exchange acts as a *two*-qubit error that has the same effect as a simultaneous bit flip on two different qubits. Of course, QECCs dealing explicitly with multiple-qubit errors are also available, so that exchange errors can readily be dealt with, provided one accepts longer code words than are needed to deal with single-qubit errors [4]. For example, in Ref. [2] a nine-qubit code is presented that can correct all single-qubit errors and all Pauli exchange errors. This is to be compared with the five-qubit ''perfect'' code that protects (only) against all single-qubit errors [5]. While the nine-qubit code is longer than the ''perfect'' code, it is shorter than a code required to protect against *all* two-qubit errors.

A different error model that has been considered by several authors is that in which qubits undergo *collective* rather than independent errors [6–9]. The underlying physics of this model has a rich history: it dates back at least to Dicke's quantum optics work on superradiance of atoms coupled to a radiation field, where it arose in the consideration of systems confined to a region whose linear dimensions are small compared to the shortest wavelength of the field [10]. The model was later treated extensively by Agarwal in the context of spontaneous emission [11]. It was only recently realized, however, that in the collective decoherence model there exist large decoherence-free subspaces (DFSs), which are ''quiet'' Hilbert subspaces in which no environmentally induced errors occur at all [7,8]. Such subspaces offer a passive protection against decoherence. Collective decoherence is an assumption about the manner in which the environment couples to the system: instead of independent errors, as assumed in the active QECC approach, one assumes that errors are strongly correlated, in the sense that all qubits can be permuted without affecting the coupling between system and bath. This is clearly a very strong assumption, and it may not hold exactly in a realistic system-bath coupling scenario. To deal with this limitation, we have shown recently how DFSs can be stabilized in the presence of errors that perturb the exact permutation symmetry, by concatenating DFSs with QECCs [9]. Concatenation is a general technique that is useful for achieving fault-tolerant quantum computation [12,13], and trades stability of quantum information for the price of longer code words. It is our purpose here to analyze the effect of exchange errors on DFSs for collective decoherence. These errors are fundamentally different from those induced by the system-bath coupling, since they originate *entirely* from the internal system Hamiltonian. We will show that by use of the very same concatenation scheme as introduced in Ref. [9] (which was designed originally to deal with system-bath induced errors), a DFS can be stabilized in the presence of exchange errors as well.

The structure of the paper is as follows. We begin by briefly recalling the origin of the exchange interaction in Sec. II and present some Hamiltonians modeling this interaction. We then present, in Sec. III, a short review of the Hamiltonian theory of DFSs. Next we discuss in Sec. IV the simplest model, of constant exchange matrix elements, and show that DFSs are immune to exchange errors in this case. Our main result is then presented in Sec. V, when we analyze the effect of exchange errors in the case of arbitrary exchange matrix elements. We show that a DFS is invariant under such errors, and conclude that concatenation with a QECC can generally stabilize DFSs against exchange.

II. MODELING EXCHANGE IN QUBIT ARRAYS

The exchange interaction arises by virtue of permutation symmetry between identical particles, in addition to some interaction potential. Exchange is caused by the system Hamiltonian and is unrelated to the coupling to an external environment. Exchange thus induces an extraneous *unitary* evolution on the system, but does not lead to decoherence.

To model exchange it is sufficient to consider a Hamiltonian of the form

$$H_{\rm ex}=\frac{1}{2}\sum_{i\neq j}^{K} J_{ij}E_{ij}, \tag{1}$$

where the sum is over all qubit pairs, J_{ij} are appropriate matrix elements, and

$$E_{ij}|\epsilon_1,\ldots,\epsilon_i,\ldots,\epsilon_j,\ldots,\epsilon_K\rangle =|\epsilon_1,\ldots,\epsilon_j,\ldots,\epsilon_i,\ldots,\epsilon_K\rangle, \tag{2}$$

where $\epsilon_i=0$ or 1. E_{ij} thus written is a general exchange operator operating on qubits i and j of a K-qubit state.

Typical examples of Hamiltonians leading to exchange are [14]: (i) the Heisenberg interaction between spins

$$H_{\rm Heis}=\frac{1}{2}\sum_{i\neq j} J_{ij}^{\rm H}\mathbf{S}_i\cdot\mathbf{S}_j, \tag{3}$$

where $\mathbf{S}_i=(\sigma_i^x,\sigma_i^y,\sigma_i^z)$ is the Pauli matrix vector of spin i; (ii) the Coulomb interaction

$$H_{\rm Coul}=\frac{1}{2}\sum_{i\neq j}\sum_{\sigma,\sigma'} J_{ij}^{\rm C}a_{i\sigma}^{\dagger}a_{i\sigma'}a_{j\sigma'}^{\dagger}a_{j\sigma}, \tag{4}$$

where $a_{i\sigma}^{\dagger}(a_{i\sigma})$ is the creation (annihilation) operator of an electron of spin σ in Wannier orbital i. J_{ij} is the exchange matrix element and is given for electrons by

$$J_{ij}^{C}=-e^2\int d\mathbf{r}d\mathbf{r}' \times\frac{w^*(\mathbf{r}-\mathbf{R}_i)w(\mathbf{r}'-\mathbf{R}_i)w^*(\mathbf{r}'-\mathbf{R}_j)w(\mathbf{r}-\mathbf{R}_j)}{|\mathbf{r}-\mathbf{r}'|}, \tag{5}$$

where $\mathbf{R}_i$ is a lattice vector and w is a Wannier function [14]. This is a rather generic form for the exchange matrix element; in other cases w would be replaced by the appropriate wave function and the Coulomb interaction $e^2/|\mathbf{r}-\mathbf{r}'|$ by the appropriate potential. The important point to notice is that the exchange integral depends on the overlap between the wave functions at locations i and j. Thus exchange effects generally decay rapidly as the distance $|\mathbf{R}_i-\mathbf{R}_j|$ increases. An important simplification is possible when interactions beyond nearest neighbors can be neglected (i.e., $J_{ij}=0$ if i and j are not nearest neighbors), in which case the approximation $J_{ij}\equiv J$ is often made.

In the Coulomb case the interpretation of $a_{i\sigma}^{\dagger}a_{i\sigma'}a_{j\sigma'}^{\dagger}a_{j\sigma}$ as an exchange operator is quite clear: spin σ is destroyed at orbital j and is created at orbital i, while spin σ' is destroyed at orbital i and is created at orbital j. The net effect is that spins σ and σ' are swapped between the electrons in orbitals i and j. In the Heisenberg case one can verify that the operator $\mathbf{S}_i\cdot\mathbf{S}_j$ also implements an exchange. Let I denote the identity operator, X_i the Pauli matrix σ_i^x operating on qubit i, etc. A qubit state is written as usual as a superposition over σ_z eigenstates $|0\rangle$ and $|1\rangle$. Then, defining

$$E_{ij}\equiv\frac{1}{2}(I+\mathbf{S}_i\cdot\mathbf{S}_j)=\frac{1}{2}(I+X_i\otimes X_j+Y_i\otimes Y_j+Z_i\otimes Z_j), \tag{6}$$

it is easily checked that Eq. (2) is satisfied [15].

III. REVIEW OF DECOHERENCE-FREE SUBSPACES

We briefly recall the Hamiltonian theory of DFSs [9,16]. Given is a system-bath interaction Hamiltonian

$$H_{\rm SB}=\sum_{\lambda} F_{\lambda}\otimes B_{\lambda}, \tag{7}$$

where F_λ and B_λ are, respectively, the system and bath operators. The decoherence-free states are those and only those states $\{|\psi\rangle\}$ that are simultaneous degenerate eigenvectors of all system operators appearing in $H_{\rm SB}$:

$$F_\lambda|\psi\rangle=c_\lambda|\psi\rangle. \tag{8}$$

The eigenvalues $\{c_\lambda\}$ do not depend on $|\psi\rangle$. The subspace spanned by these states is a DFS, meaning that under $H_{\rm SB}$ the evolution in this subspace is unitary, and there is no decoherence. This results in a passive protection against errors, to be contrasted with the active QECC approach. Of particular interest is the case where the $\{F_\lambda\}$ are collective operators, such as the total spin operators

$$S_\alpha=\sum_{i=1}^{K}\sigma_i^\alpha, \quad \alpha=x,y,z. \tag{9}$$

These operators satisfy su(2) commutation relations, just like the local σ_i^α Pauli operators:

$$[S_\alpha,S_\beta]=2i\varepsilon_{\alpha\beta\gamma}S_\gamma. \tag{10}$$

This situation, referred to above as collective decoherence, arises when the bath couples in a permutation-invariant fashion to all qubits. In this paper we shall confine our attention to collective decoherence and employ the term DFS exclusively in this context [17]. With a system-bath interaction of the form $H_{\rm SB}=\Sigma_\alpha S_\alpha\otimes B_\alpha$ (as, e.g., in the Lamb-Dicke limit of the spin-boson model), a combinatorial calculation shows (see Appendix) that the number of encoded qubits is $\lim_{K\to\infty}\log_2 K!/[(K/2+1)!(K/2)!]\approx K-\frac{3}{2}\log_2 K$. The resulting decoherence-free code thus asymptotically approaches unit efficiency (number of encoded qubits per physical qubits), and is therefore of significant interest. In the collective decoherence case, since the S_α are the generators of the semisimple Lie algebra su(2), the DFS condition Eq. (8) is satisfied with $c_\alpha=0$ [8]. This means that the decoherence-free states $\{|j\rangle\}$ are $su(2)$ *singlets*: they are states of zero total spin, and belong to the one-dimensional irreducible representation of su(2). For example, for $K=2$ qubits undergoing collective decoherence, there is just one decoherence-free state: $(|01\rangle-|10\rangle)/\sqrt{2}$, i.e., the familiar singlet state of two spin-1/2 particles. For as few as $K=4$ there are already two singlet states, spanning a full encoded decoherence-free qubit [7].

IV. DECOHERENCE-FREE STATES AND EXCHANGE WITH CONSTANT MATRIX ELEMENTS

A simple situation arises when we can assume that $J_{ij} \equiv J/K$ for *all* i,j, i.e., without the restriction to nearest-neighbor interactions. This long-range Ising model is thermodynamically equivalent to the mean-field theory of metallic ferromagnets, and there exist some examples of metals (e.g., $HoRh_4B_4$) that are well described by it [18]. At present the relevance of such materials to quantum computer architectures is not clear. We also stress that in the vast majority of physical examples exchange correlations decay exponentially fast with the distance between particles. The case of arbitrary exchange matrix elements is dealt with in the next section. We consider the long-range model here mainly for its simplicity and for the remarkable result that DFSs are completely immune to exchange errors in this case.

We have for $\mathbf{S}=(S_x,S_y,S_z)$

$$S^2=\mathbf{S}\cdot\mathbf{S}=3KI+2\sum_{i\neq j} X_i\otimes X_j+Y_i\otimes Y_j+Z_i\otimes Z_j, \quad (11)$$

so that the exchange Hamiltonian can be rewritten as

$$H_{\rm ex}=\frac{J}{4K}\sum_{i\neq j}^{K}(I+X_i\otimes X_j+Y_i\otimes Y_j+Z_i\otimes Z_j)$$

$$=\frac{J}{8K}[(K^2-4K)I+S^2]. \quad (12)$$

Whereas the DFS condition guarantees that no decoherence is caused by the coupling to the bath, uncontrolled unitary evolution due to the system Hamiltonian may still pose a significant problem. This is exactly the case in the presence of exchange errors, as described above. However, using Eq. (12) and recalling that the DFS states have zero total spin, we see that in the collective decoherence case the DFS is in fact automatically protected against exchange errors:

$$H_{\rm ex}|\psi\rangle=\left[\nu I+\frac{J}{8K}S^2\right]|\psi\rangle=\nu\,|\psi\rangle, \quad (13)$$

where $|\psi\rangle$ is a DFS state and $\nu\equiv(J/K)(K^2-4K)/8$. Since the constant ν does not depend on ψ, this implies that under the unitary evolution generated by $H_{\rm ex}$, a DFS state accumulates an overall, global phase $e^{i\nu t}$. This phase is not measurable and does not affect the decoherence time. Thus in the $J_{ij}\equiv J$ model a DFS does not undergo exchange errors, and the smallest DFS ($K=4$ physical qubits) already suffices to encode a full logical qubit.

V. DECOHERENCE-FREE STATES AND ARBITRARY EXCHANGE MATRIX ELEMENTS

We now analyze the effect of arbitrary exchange errors on DFS states for collective decoherence. We show that by concatenation with QECCs, DFSs can be stabilized against these errors.

A. Decoherence-free subspaces are invariant under exchange

The exchange operator commutes with the total-spin operators. To see this, use the definitions of these operators in Eqs. (2) and (9), and let $S_\alpha^{ij}\equiv(\sum_{k\neq i,j}^{K}\sigma_k^\alpha)$. Since they act on different qubits, S_α^{ij} clearly commutes with E_{ij}. Now, using $\sigma^\alpha\sigma^\beta=\delta_{\alpha\beta}I+i\varepsilon_{\alpha\beta\gamma}\sigma^\gamma$

$$S_\alpha E_{ij}=[S_\alpha-(\sigma_i^\alpha+\sigma_j^\alpha)]E_{ij}+(\sigma_i^\alpha+\sigma_j^\alpha)E_{ij}$$

$$=\left(\sum_{k\neq i,j}^{K}\sigma_k^\alpha\right)E_{ij}+\frac{1}{2}(\sigma_i^\alpha+\sigma_j^\alpha)\left(I+\sum_{\beta=x,y,z}\sigma_i^\beta\otimes\sigma_j^\beta\right)$$

$$=S_\alpha^{ij}E_{ij}+\sigma_i^\alpha+\sigma_j^\alpha+\frac{i}{2}\sum_{\beta,\gamma}\varepsilon_{\alpha\beta\gamma}(\sigma_i^\beta\otimes\sigma_j^\gamma+\sigma_i^\gamma\otimes\sigma_j^\beta). \quad (14)$$

The last term in this expression vanishes since $\varepsilon_{\alpha\beta\gamma}=-\varepsilon_{\alpha\gamma\beta}$ and we are summing over all β,γ values. Thus

$$S_\alpha E_{ij}=S_\alpha^{ij}E_{ij}+\sigma_i^\alpha+\sigma_j^\alpha=E_{ij}S_\alpha. \quad (15)$$

Now let $|\psi\rangle$ be a decoherence-free state (which it is for collective decoherence iff $S_\alpha|\psi\rangle=0$ [8]). Since $S_\alpha(E_{ij}|\psi\rangle)=E_{ij}S_\alpha|\psi\rangle=0$, it follows that $E_{ij}|\psi\rangle$ is also decoherence free. We have thus proved:

Theorem I. Let $\tilde{\mathcal{H}}$ be a decoherence-free subspace against collective decoherence errors, and E_{ij} an exchange operation on qubits i and j. Then $E_{ij}\tilde{\mathcal{H}}=\tilde{\mathcal{H}}$.

The significance of this result is that exchange errors act as errors on the *encoded* DFS qubits, i.e., they keep decoherence-free states inside the DFS. The exact way in which these errors are manifested is a difficult problem. Exchange operations are transpositions in the language of the permutation group S_K and are known to generate this group [19]. For a given number K of physical qubits the action of the exchange operators will realize a 2^K-dimensional reducible representation of S_K. The DFS for collective decoherence on these K qubits is the set of one-dimensional irreducible subspaces in the irreducible representations (irreps) of S_K, which appear with multiplicity $K!/[(K/2+1)!(K/2)!]$ (see Appendix). For $K=4$ the DFS is two-dimensional (equal to the multiplicity of the one-dimensional irreps), encoding one qubit. Therefore in this case exchange errors will act as the usual Pauli errors on a single (encoded) qubit. Correction of exchange errors for $K=4$ can then be done entirely within the DFS by using a quantum error correcting code for single-qubit errors. This observation naturally leads one to consider concatenating the DFS code words with such a code, as done in the concatenated code of Ref. [9]. That paper showed that the concatenated DFS-QECC code can in fact deal with the more general case of *both* errors inside the DFS (as is our case here), *and* errors that take states outside of the DFS. We investigate the correction of exchange errors in detail for the $K=4$ case in Sec. V B. For $K>4$ qubits, the dimension of the DFS is greater than 2 (e.g., for $K=6$ it is 5), and the action of exchange errors will correspondingly be represented by higher-dimensional irreps of S_K. To correct such unitary errors it will be necessary to resort to codes for ‘‘qu$k$its’’ $(k>2)$, such as stabilizer codes for higher-dimensional sytems [20], or polynomial codes [13]. We defer the discussion of this case to a future publication [21] and focus here on the $K=4$ case.

B. Effect of exchange errors on the four qubit decoherence-free subspace

Suppose that the qubits undergo collective decoherence in clusters of four identical particles, but different clusters are independent (as they might be, e.g., in a polymer with an AAAABBBBAAAA... type of order). Each cluster would then support a two-dimensional DFS, accommodating a single encoded DFS qubit. The $K=4$ physical qubits DFS states can then be written as [7]

$$|\tilde{0}\rangle=\frac{|a\rangle-|b\rangle}{2},\quad |\tilde{1}\rangle=\frac{2|c\rangle-|a\rangle-|b\rangle}{2\sqrt{3}}, \tag{16}$$

where

$$|a\rangle\equiv|0110\rangle+|1001\rangle,\quad |b\rangle\equiv|1010\rangle+|0101\rangle,$$
$$|c\rangle\equiv|0011\rangle+|1100\rangle. \tag{17}$$

Note that the mutually orthogonal states $|a\rangle,|b\rangle$, and $|c\rangle$ are sums of complementary states. Moreover, the four qubits play a symmetrical role (i.e., 0 and 1 appear equally in all four positions in both $|\tilde{0}\rangle$ and $|\tilde{1}\rangle$). This dictates that exchange of qubits in symmetrical positions should have the same effect. In other words, we expect E_{12} to be indistinguishable from E_{34}, and similarly for $\{E_{13},E_{24}\}$ and $\{E_{23},E_{14}\}$ (although for a linear geometry most physical exchange mechanisms will yield $|J_{23}|>|J_{14}|$). This expectation is borne out; in the $\{|\tilde{0}\rangle,|\tilde{1}\rangle\}$ basis we find, using straightforward algebra, that the six exchange operators can be written as

$$E_{12}=E_{34}=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}=-\overline{Z},$$
$$E_{13}=E_{24}=\tilde{R}(\pi/3)=\frac{\sqrt{3}}{2}\overline{X}+\frac{1}{2}\overline{Z}, \tag{18}$$
$$E_{14}=E_{23}=\tilde{R}(-\pi/3)=-\frac{\sqrt{3}}{2}\overline{X}+\frac{1}{2}\overline{Z},$$

where $\tilde{R}(\theta)=R(\theta)\overline{Z}$, and

$$R(\theta)=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}.$$

Thus $\tilde{R}(\theta)$ is a reflection about the x axis followed by a counterclockwise rotation in the x,y plane. In writing these expressions, the matrices operate on column vectors such that $|\tilde{0}\rangle=\binom{1}{0}$ and $|\tilde{1}\rangle=\binom{0}{1}$, and $\overline{X},\overline{Z}$ are the *encoded* Pauli matrices, i.e., the Pauli matrices acting on the DFS states (and *not* on the physical qubits). Thus, exchange errors act as encoded Pauli errors on the DFS states.

Using this observation, it is possible to protect DFS states against such errors by concatenation with a QECC designed to correct single-qubit errors. The critical point is that this QECC will now correct single *encoded* qubit errors. This requires an additional encoding layer to be constructed. In particular, suppose we add such an encoding layer by using DFS qubits to build code words of the five-qubit ‘‘perfect’’ QECC [5]. These code words have the form $|\tilde{\epsilon}_1\rangle|\tilde{\epsilon}_2\rangle|\tilde{\epsilon}_3\rangle|\tilde{\epsilon}_4\rangle|\tilde{\epsilon}_5\rangle$, where $\epsilon=0,1$, and j in $\tilde{\epsilon}_j$ is now a *cluster* index. Since the five-qubit QECC can correct any single-qubit error, in particular it can correct the specific errors of Eq. (18) that the encoded DFS qubits would undergo under an exchange interaction on the physical qubits in a given cluster. However, the error detection and correction procedure must be carried out sufficiently fast so that exchange errors affecting multiple blocks at a time do not occur, or else concatenation with a code that can deal with $t>1$ independent errors is needed. The typical time scale for exchange errors to occur is $1/(2|J_{ij}|)$, where J_{ij} is the relevant exchange matrix element.

This 20-qubit concatenated DFS-QECC code is precisely the one discussed in Ref. [9], where it was shown that it offers protection against general collective decoherence symmetry-breaking perturbations. Our present result shows that this concatenated code is stable against exchange errors as well.

We note that it is certainly possible to find a shorter QECC than the five-qubit one to protect against the restricted set of errors in Eq. (18). However, such a code would not offer the full protection against general errors that is offered by concatenation with the perfect five-qubit code, and thus would not be as useful.

VI. SUMMARY AND CONCLUSIONS

To conclude, in this paper we considered the effect of unitary exchange errors between identical qubits on the protection of quantum information by decoherence-free subspaces (DFSs) defined for a qubit array. We showed that in the important case of ideal collective decoherence (qubits are coupled symmetrically to the bath), for which a perfectly stable DFS is obtained, DFSs are additionally invariant to exchange errors. Thus such errors generate rotations inside the DFS, but do not take decoherence-free states outside of the DFS. Consequently it is possible to use, without any modification, the concatenated DFS-QECC scheme of Ref. [9] in order to protect DFSs against exchange errors, while at the same time relaxing the constraint of ideal collective decoherence, and allowing for symmetry-breaking perturbations. This is useful for quantum memory applications. Since exchange interactions preserve a DFS, an interesting further question is whether they can be used *constructively* in order to perform controlled logic operations inside a DFS. We have found the answer to be positive, and that it is actually possible to perform universal computation in a fault-tolerant manner inside a DFS for collective decoherence using only two-body exchange interactions [22].

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APPENDIX: DIMENSION OF DECOHERENCE-FREE SUBSPACES FOR COLLECTIVE DECOHERENCE

In view of the fact that the total spin operators S_α satisfy spin-1/2 commutation relations, it follows from the addition of angular momentum that the operators S^2 and S_z have simultaneous eigenstates given by

$$S^2|S,m\rangle = S(S+1)|S,m\rangle, \qquad S_z|S,m\rangle = m|S,m\rangle, \tag{A1}$$

where $m=-S,-S+1,\dots,S$ and $S=0,1,\dots,K/2$ (for K even), $S=1/2,3/2,\dots,K/2$ (for K odd). The $|S,m\rangle$ states are known as Dicke states [10,11]. The degeneracy of a state with given S is

$$\frac{K!(2S+1)}{(K/2+S+1)!(K/2-S)!}, \tag{A2}$$

which for $S=0$, i.e., the singlet states, coincides with the dimension of the DFS for K qubits undergoing collective decoherence cited in the text.

It is interesting to derive this formula from combinatorial arguments relating to the permutation group of K objects, which we will do for $S=0$. The result follows straightforwardly from the Young diagram technique. As is well known (see, e.g., [19]), the singlet states of su(2) belong to the rectangular Young tableaux of $K/2$ columns and 2 rows. The multiplicity λ of such states is the number of ''standard tableaux'' (tableaux containing an arrangement of numbers that increase from left to right in a row and from top to bottom in a column), which is also the dimension of the irreducible representation of the permutation group corresponding to the Young diagram $\eta_{K/2,2}$ (an empty tableau) of $K/2$ columns and two rows. This number is found using the ''hook recipe'' [19], where one writes the ''hook length'' g_i (the sum of the number of positions to the right of box i, plus the number of positions below it, plus 1) of each box i in the Young diagram:

$$\lambda(\eta) = \frac{K!}{\prod_{i=1}^{K} g_i}. \tag{A3}$$

For example, for $\eta_{c,2}$ the hook lengths are

$c+1$	c	$c-1$	$\cdots$	3	2
c	$c-1$	$c-2$	$\cdots$	2	1

(A4)

and one finds, with $c=K/2$,

$$\lambda(\eta_{K/2,2}) = \frac{K!}{(K/2+1)!(K/2)!}, \tag{A5}$$

which is indeed the $S=0$ case of the general degeneracy formula, Eq. (A2).

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