Concatenating Decoherence-Free Subspaces with Quantum Error Correcting Codes

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An operator sum representation is derived for a decoherence-free subspace (DFS) and used to (i) show that DFS's are the class of quantum error correcting codes (QECC's) with fixed, *unitary* recovery operators and (ii) find explicit representations for the Kraus operators of collective decoherence. We demonstrate how this can be used to construct a concatenated DFS-QECC code which protects against collective decoherence perturbed by independent decoherence. The code yields an error threshold which depends only on the perturbing independent decoherence rate. [S0031-9007(99)09301-1]

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Decoherence-free subspaces (DFS's) have recently emerged [1–6] as an alternative way to protect fragile quantum states against decoherence, alongside "conventional" quantum error correcting codes (QECC's) [7,8] and the new "dynamical decoupling" schemes [9]. This is of particular importance in quantum computation, where the promise of a speedup compared to classical computers hinges crucially on the possibility to maintain quantum coherence throughout the computation [10]. So far, DFS's and QECC's have been considered as distinct methods, often characterized as "passive" and "active," respectively. However, as we will show here, in fact, DFS's can be considered as a special class of QECC's, characterized as having a particularly simple form of recovery operators. Conditions for the existence of nontrivial DFS's are stringent: the decoherence process should be "collective," meaning that the bath couples in a symmetric way to all qubits. So far conditions for collective decoherence have been formulated in a Hamiltonian form [3,4], and in the Lindblad semigroup form [5,6]. Here we will present an alternative formulation in terms of the operator sum representation (OSR) [11], which has the advantage of establishing a direct link to the theory of QECC's. This OSR formulation enables us to combine DFS's and QECC's, into a concatenated scheme which can error-correct the more general physical situation of "cluster decoherence." For the price of longer code words, this concatenated scheme operates with a substantially reduced error threshold.

Hamiltonian formulation of decoherence-free subspaces.—Conditions for DFS's within the general (non-Markovian) framework of semigroup dynamics were derived in Ref. [4]. We first briefly rederive these conditions in a simplified form. Consider a closed quantum system, composed of a system S of interest defined on a Hilbert space $\mathcal{H}$ (e.g., a quantum computer) and a bath B. The evolution of the closed system is given by $\rho_{SB}(t) = \mathbf{U}\rho_{SB}(0)\mathbf{U}^\dagger$, where the unitary evolution operator (we set $\hbar = 1$) is $\mathbf{U} = \exp(-i\mathbf{H}t)$. The full Hamiltonian is $\mathbf{H} = \mathbf{H}_S \otimes \mathbf{I}_B + \mathbf{I}_S \otimes \mathbf{H}_B + \mathbf{H}_I$, where $\mathbf{H}_S$, $\mathbf{H}_B$, and $\mathbf{H}_I$ are, respectively, the system, bath, and interaction Hamiltonians, and $\mathbf{I}$ is the identity operator. Assuming initial decoupling between system and bath, the evolution of the closed system is given by $\rho_{SB}(t) = \mathbf{U}[\rho_S(0) \otimes \rho_B(0)]\mathbf{U}^\dagger$. Quite generally, the interaction Hamiltonian can be written as $\mathbf{H}_I = \sum_\alpha \mathbf{F}_\alpha \otimes \mathbf{B}_\alpha$, where $\mathbf{F}_\alpha$ and $\mathbf{B}_\alpha$ are, respectively, system and bath operators. Suppose that there exists a degenerate subset $\{|\tilde{k}\rangle\}$ of eigenvectors of the $\mathbf{F}_\alpha$'s such that

$$\mathbf{F}_\alpha|\tilde{k}\rangle = a_\alpha|\tilde{k}\rangle \quad \forall\ \alpha, |\tilde{k}\rangle\,. \tag{1}$$

If $\mathbf{H}_S$ leaves the Hilbert subspace $\tilde{\mathcal{H}} = \mathrm{Span}[\{|\tilde{k}\rangle\}]$ invariant, and if we start within $\tilde{\mathcal{H}}$, then the evolution of the system will be *decoherence-free* (DF). To show this, expand the initial density matrices of the system and the bath in their respective bases: $\rho_S(0) = \sum_{ij} s_{ij}|\tilde{i}\rangle\langle\tilde{j}|$ and $\rho_B(0) = \sum_{\mu\nu} b_{\mu\nu}|\mu\rangle\langle\nu|$. Using Eq. (1), one can write the combined operation of the bath and interaction Hamiltonians over $\tilde{\mathcal{H}}$ as

$$\mathbf{I}_S \otimes \mathbf{H}_B + \mathbf{H}_I = \mathbf{I}_S \otimes \mathbf{H}_c \equiv \mathbf{I}_S \otimes \left[\mathbf{H}_B + \sum_\alpha a_\alpha \mathbf{B}_\alpha\right].$$

This clearly commutes with $\mathbf{H}_S$ over $\tilde{\mathcal{H}}$. Thus since neither $\mathbf{H}_S$ (by our own stipulation) nor the combined Hamiltonian $\mathbf{H}_c$ takes states out of the subspace,

$$\mathbf{U}[|\tilde{i}\rangle \otimes |\mu\rangle] = \mathbf{U}_S|\tilde{i}\rangle \otimes \mathbf{U}_c|\mu\rangle\,, \tag{2}$$

where $\mathbf{U}_X = \exp(-i\mathbf{H}_X t)$, $X = S, c$. Hence it is clear, given the initially decoupled state of the density matrix, that the evolution of the closed system will be $\rho_{SB}(t) = \sum_{ij} s_{ij}\mathbf{U}_S|\tilde{i}\rangle\langle\tilde{j}|\mathbf{U}_S^\dagger \otimes \sum_{\mu\nu} b_{\mu\nu}\mathbf{U}_c|\mu\rangle\langle\nu|\mathbf{U}_c^\dagger$. It follows using simple algebra that after tracing over the bath, $\rho_S(t) = \mathrm{Tr}_B[\rho_{SB}(t)] = \mathbf{U}_S\rho_S(0)\mathbf{U}_S^\dagger$, i.e., that the system

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evolves in a completely unitary fashion on $\tilde{\mathcal{H}}$: under the condition of Eq. (1) the subspace is DF. As shown in Ref. [4], Eq. (2) is also a necessary condition for a DFS.

Operator sum representation on a decoherence-free subspace.—In the OSR, the evolution of the density matrix is written as $\rho_S(t) = \mathrm{Tr}_B[\mathbf{U}(\rho_S \otimes \rho_B)\mathbf{U}^\dagger] = \sum_a \mathbf{A}_a \rho_S(0)\mathbf{A}_a^\dagger$, where the "*Kraus operators*" are given by

$$\mathbf{A}_a = \sqrt{\nu}\,\langle\mu|\mathbf{U}|\nu\rangle; \qquad a = (\mu,\nu), \tag{3}$$

($|\mu\rangle$, $|\nu\rangle$ are bath states) and satisfy the normalization constraint $\sum_a \mathbf{A}_a^\dagger \mathbf{A}_a = \mathbf{I}_S$.

Let $\tilde{\mathcal{H}}$ be an $\tilde{N}$-dimensional DFS. In this case it follows immediately from Eqs. (2) and (3) that the Kraus operators all have the following representation (in the basis where the first $\tilde{N}$ states span $\mathcal{H}$):

$$\mathbf{A}_a = \begin{pmatrix} g_a\tilde{\mathbf{U}}_S & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{A}}_a \end{pmatrix}; \qquad g_a = \sqrt{\nu}\,\langle\mu|\mathbf{U}_c|\nu\rangle. \tag{4}$$

Here $\bar{\mathbf{A}}_a$ is an arbitrary matrix that acts on $\tilde{\mathcal{H}}^\perp$ ($\mathcal{H} = \tilde{\mathcal{H}} \oplus \tilde{\mathcal{H}}^\perp$) and may cause decoherence there; $\tilde{\mathbf{U}}_S$ is $\mathbf{U}_S$ restricted to $\tilde{\mathcal{H}}$. This simple condition can be summarized as follows.

Theorem I: A subspace $\tilde{\mathcal{H}}$ is a DFS iff (if and only if) all Kraus operators have an identical unitary representation upon restriction to it, up to a multiplicative constant.

Thus, in the OSR, the task of identifying a DFS reduces to finding a subspace in which all the Kraus operators act as the system unitary evolution operator. We now give an example for the important case of collective decoherence (CD). CD is generally described within the following scenario: the system operators $\{\mathbf{F}_\alpha\}$ in the interaction Hamiltonian form the Lie algebra $su(2)$ [3,6]. This means that the interaction Hamiltonian can be rewritten as

$$\mathbf{H}_I = \mathbf{S}_+ \otimes \mathbf{V}_+ + \mathbf{S}_- \otimes \mathbf{V}_- + \mathbf{S}_z \otimes \mathbf{V}_z. \tag{5}$$

Here $\mathbf{S}_\alpha = \sum_{i=1}^K \sigma_i^\alpha$ are *global* Pauli spin operators (i is the qubit index) satisfying the $sl(2)$ commutation relations, and $\mathbf{V}_\alpha$ are the bath operators coupled to these degrees of freedom. A more restricted case of CD arises when only phase damping processes are allowed, so that $\mathbf{V}_+ = \mathbf{V}_- = 0$. We will concentrate on this case, as it is fully analytically solvable. For simplicity we will assume throughout that $\mathbf{H}_S = 0$.

Pure phase damping on a single qubit is described by the Pauli σ^z matrix. In the standard basis it is easy to verify that the matrix representation of the global phase damping operator $\mathbf{S}_z$ is $\mathbf{S}_z = \mathrm{diag}[f(j)]$, where $f(j) =$ (number of 0's) $-$ (number of 1's) in the binary representation of j ($j = 0 \ldots 2^{K-1}$). For example, for two qubits, $\mathbf{S}_z = \mathrm{diag}[2,0,0,-2]$. Since $\mathbf{S}_z$ is diagonal, the action of the interaction Hamiltonian $\mathbf{H}_I = \mathbf{S}_z \otimes \mathbf{V}_z$ can be written simply as $\mathbf{H}_I|j\rangle|\nu\rangle = f(j)\,|j\rangle \otimes \mathbf{V}_z|\nu\rangle$. Hence the action of the full Hamiltonian is $\mathbf{H}|j\rangle|\nu\rangle = |j\rangle \otimes \mathbf{\Omega}_j|\nu\rangle$, where $\mathbf{\Omega}_j \equiv f(j)\mathbf{V}_z + \mathbf{H}_B$. Similarly, $\mathbf{H}^n|j\rangle|\nu\rangle = |j\rangle \otimes \mathbf{\Omega}_j^n|\nu\rangle$, whence $\exp(-i\mathbf{H}t)\,|j\rangle|\nu\rangle = |j\rangle \otimes \exp(-i\mathbf{\Omega}_j t)\,|\nu\rangle$, so that the Kraus operators can be evaluated explicitly: $\langle j'|\mathbf{A}_a|j\rangle = \delta_{jj'}\sqrt{\nu} \times \langle\mu|\exp(-i\mathbf{\Omega}_j t)\,|\nu\rangle$. Thus, *the Kraus operators for pure phase damping CD have a diagonal matrix representation in the standard basis*, which can be written compactly as $\mathbf{A}_a = \mathrm{diag}[g_a^{(f_j)}]$, $g_a^{(f_j)} = \sqrt{\nu}\,\langle\mu|\exp(-i\mathbf{\Omega}_j t)\,|\nu\rangle$. For example, in the case of two qubits we obtain $\mathbf{A}_a = \mathrm{diag}[g_a^{(2)}, g_a^{(0)}, g_a^{(0)}, g_a^{(-2)}]$, with $g_a^{(0)} = \sqrt{\nu}\,\langle\mu|\exp(-i\mathbf{H}_B t)\,|\nu\rangle$ and $g_a^{(\pm 2)} = \sqrt{\nu}\,\langle\mu|\exp(-i[\mathbf{H}_B \pm 2\mathbf{V}_z]t)\,|\nu\rangle$. By Theorem I the three blocks in $\mathbf{A}_a$ correspond to three DFSs. Next, consider the effect of this OSR on a general density matrix (omitting standard algebra): $[\rho(0)]_{jk} \overset{t}{\mapsto} \sum_a [\mathbf{A}_a \rho(0)\mathbf{A}_a^\dagger]_{jk} = [\rho(0)]_{jk} \sum_{\mu\nu} \nu g_{\mu\nu}^{(f_j)} g_{\mu\nu}^{(f_k)^*}$. As expected, no mixing of density matrix elements occurs. The time dependence of each element is determined by the sum in the last expression, which, motivated by the understanding that decoherence is taking place, we write formally as a decaying exponential (although without a Markovian approximation Poincaré recurrences may occur). Thus, $\exp(-t/\tau_{jk}) \equiv \sum_{\mu\nu} \nu g_{\mu\nu}^{(f_j)} g_{\mu\nu}^{(f_k)^*} = \sum_\nu \nu\langle\nu|\exp\{it[f(j) - f(k)](\mathbf{V}_z + [\mathbf{V}_z, \mathbf{H}_B]) + \ldots\}|\nu\rangle$, with higher order terms, all depending on powers of $f(j) - f(k)$, given by the Campbell-Hausdorff formula. In agreement with the general theory of DFS's [4,6], the DF states are those for which $f(j) = f(k)$, in which case $1/\tau_{jk} = 0$. Normally (i.e., in the "pointer basis" [12]), all other states are expected to have $1/\tau_{jk} > 0$, although in order to verify this one must specify $\mathbf{V}_z$. This expectation is confirmed for a harmonic bath [1].

Decoherence-free subspaces as quantum error correcting codes.—Quantum error correction can be regarded as the theory of reversal of quantum operations on a subspace [13]. This subspace, $C = \mathrm{Span}[\{|i_L\rangle\}]$, is interpreted as a "code" (with code words $\{|i_L\rangle\}$) which can be used to protect part of the system Hilbert space against decoherence (or "errors") caused by the interaction between system and bath. The errors are represented by the Kraus operators $\{\mathbf{A}_a\}$ [8]. To decode the quantum information after the action of the bath, one introduces "recovery" operators $\{\mathbf{R}_r\}$. A QECC is a subspace C and a set of recovery operators $\{\mathbf{R}_r\}$. Reference [8] gives two equivalent criteria for the general condition for QECC. It is possible to correct the errors induced by a given set of Kraus operators $\{\mathbf{A}_a\}$, (i) iff

$$\mathbf{R}_r\mathbf{A}_a = \begin{pmatrix} \lambda_{ra}\mathbf{I}_C & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_{ra} \end{pmatrix} \qquad \forall\ r,a, \tag{6}$$

or, equivalently, (ii) iff

$$\mathbf{A}_a^\dagger\mathbf{A}_b = \begin{pmatrix} \gamma_{ab}\mathbf{I}_C & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{A}}_a^\dagger\bar{\mathbf{A}}_b \end{pmatrix} \qquad \forall\ a,b. \tag{7}$$

In both conditions the first block acts on C; $\mathbf{B}_{ra}$ and $\bar{\mathbf{A}}_a$ are arbitrary matrices acting on $C^\perp$ ($\mathcal{H} = C \oplus C^\perp$). Let

us now explore the relation between DFS's and QECC's. First of all, it is immediate that DFS's are indeed a valid QECC. For, given the (DFS) representation of $\mathbf{A}_a$ as in Eq. (4), it follows that Eq. (7) is satisfied with $\gamma_{ab} = g_a^* g_b$. Note, however, that unlike the general QECC case which has a full-rank matrix γ_{ab}, in the DFS case this matrix has rank 1 (since the ath row equals row 1 upon multiplication by g_1^*/g_a^*), implying that a DFS is a highly *degenerate* QECC [7,8].

A DFS is an unusual QECC in another way: decoherence does not affect a perfect DFS *at all*. Since they are based on a perturbative treatment, other QECC's [e.g., stabilizer or GF(4) codes [7]] are specifically constructed to improve the fidelity to a given order in the error rate, which therefore always allows for some residual decoherence to take place. The absence of decoherence to any order for a perfect DFS is due to the existence of symmetries in the system-bath coupling which allow for an *exact* treatment. These symmetries are ignored by perturbative QECCs either for the sake of generality or because they simply do not exist, as in the case of independent couplings. Given a DFS, the only errors that can take place involve the unitary rotations of code words (basis states $\{|\tilde{i}\rangle\}$ of $\tilde{\mathcal{H}}$) inside the DFS, due to the system Hamiltonian $\mathbf{H}_S$ (this may actually be the desired evolution if one is implementing a computation inside $\tilde{\mathcal{H}}$ using $\mathbf{H}_S$). Thus, the complete characterization of DFSs as a QECC is given by the following.

Theorem II: Let C be a QECC for error operators $\{\mathbf{A}_a\}$, with recovery operators $\{\mathbf{R}_r\}$. Then C is a DFS iff upon restriction to C, $\mathbf{R}_r \propto \mathbf{U}_S^\dagger$ for all r.

Proof. First suppose C is a DFS. Then by Eqs. (4) and (6),

$$\mathbf{R}_r \begin{pmatrix} g_a \tilde{\mathbf{U}}_S & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{A}}_a \end{pmatrix} = \begin{pmatrix} \lambda_{ra} \mathbf{I}_C & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_{ra} \end{pmatrix}.$$

To satisfy this equation, it must be true that

$$\mathbf{R}_r = \begin{pmatrix} \frac{\lambda_{ra}}{g_a} \tilde{\mathbf{U}}_S^\dagger & \mathbf{C}_r \\ \mathbf{D}_r & \mathbf{E}_r \end{pmatrix}.$$

The condition $g_a \tilde{\mathbf{U}}_S^\dagger \mathbf{D}_r = \mathbf{0}$ implies $\mathbf{D}_r = \mathbf{0}$ by unitarity of $\tilde{\mathbf{U}}_S$. Also, since $\bar{\mathbf{A}}_a$ is arbitrary, generically the condition $\mathbf{C}_r \bar{\mathbf{A}}_a = \mathbf{0}$ implies $\mathbf{C}_r = \mathbf{0}$. Thus upon restriction to $C = \tilde{\mathcal{H}}$, indeed $\mathbf{R}_r \propto \tilde{\mathbf{U}}_S^\dagger$ (by unitarity of $\tilde{\mathbf{U}}_S$, $|\lambda_{ra}/g_a| = 1$). Now suppose $\mathbf{R}_r \propto \tilde{\mathbf{U}}_S^\dagger$. The very same argument applied to $\mathbf{A}_a$ in Eq. (6) yields $\mathbf{A}_a \propto \tilde{\mathbf{U}}_S$ upon restriction to C. Since this is exactly the condition defining a DFS in Eq. (4), the theorem is proved.

We conclude that DFS's are a particularly simple instance of general QECC's, where upon restriction to the code subspace, all recovery operators are proportional to the inverse of the system evolution operator.

Quantum error correction on a decoherence-free subspace.—A nonideal DFS will still be subject to some decoherence [6]. DFS's are efficient under conditions in which each qubit couples to the same environment (collective decoherence). Ordinary QECC's are designed to be efficient when each individual qubit couples to a different environment (independent decoherence). While neither code is efficient in the extreme limit when the other is, QECCs will still work for correlated errors [7,8], whereas DFS's will not work in the independent error case [6]. One would generally expect the likelihood of K qubits collectively coupling to the same environment to decrease with increasing K [1]. Thus an interesting situation (cluster decoherence) arises when small blocks of qubits undergo collective decoherence (e.g., groups of neighboring identical atoms on a polymer chain), while this symmetry is broken perturbatively by independent decoherence between blocks. Here we show how by adding an additional layer of QECC encoding, the DFS can be stabilized against such computational errors.

In the collective decoherence case [Eq. (5)], the smallest DFS which can encode one logical qubit is made up of four physical qubits [3]. Consider an operator basis which covers all possible errors which can occur on this 4-qubit DFS. From the discussion above, we know that a DFS has Kraus operators which all are direct sums of a fixed unitary transformation on $\tilde{\mathcal{H}}$ and variable transformations (whose exact form is irrelevant) on $\tilde{\mathcal{H}}^\perp$. Thus, perturbative errors (of size ϵ) on a DFS can conveniently be represented by Kraus operators with the following structure:

$$\mathbf{A}_a = \tilde{\mathbf{A}}_a + \epsilon \begin{pmatrix} \mathbf{Q}_1 & \mathbf{Q}_2 \\ \mathbf{Q}_3 & \mathbf{Q}_4 \end{pmatrix}. \tag{8}$$

Here $\tilde{\mathbf{A}}_a$ represents the (dominant) contribution due to ideal collective decoherence, and the second matrix represents the symmetry breaking perturbation. $\mathbf{Q}_1$ (a 2×2 matrix for the 4-qubit case) acts just on the DFS; $\mathbf{Q}_2$ (2×14) takes states from $\tilde{\mathcal{H}}^\perp$ into $\tilde{\mathcal{H}}$, but we need not worry about this as a separate process, since it can also be corrected using QECC inside $\tilde{\mathcal{H}}$; $\mathbf{Q}_3$ (14×2) takes states from $\tilde{\mathcal{H}}$ into $\tilde{\mathcal{H}}^\perp$; $\mathbf{Q}_4$ acts just on $\tilde{\mathcal{H}}^\perp$ and is irrelevant to our discussion. Thus, to first order in time, all of the relevant errors can be enumerated as (i) independent errors acting on the encoded DFS states. A basis for these errors are the Pauli operators: $\mathbf{X}$, $\mathbf{Z}$, $\mathbf{Y} = \mathbf{XZ}$, and $\mathbf{I}$ acting *only* on the DFS qubits $|0_L\rangle$ and $|1_L\rangle$ (e.g., $\mathbf{X}|0_L\rangle = |1_L\rangle$). (ii) Errors which take the system into $\tilde{\mathcal{H}}^\perp = \mathrm{Span}[\{|j_L\rangle\}_{j=2}^{15}]$. Define 14 operators $\mathbf{P}_j = |j_L\rangle(\langle 0_L| + \langle 1_L|)$, $\mathbf{P}_j : \tilde{\mathcal{H}} \mapsto \tilde{\mathcal{H}}^\perp$, out of a total of 28 in the $\mathbf{Q}_3$ block. In order to cover all possible errors which might take the DF states outside of $\tilde{\mathcal{H}}$, it suffices to consider the effect of $\mathbf{P}_j$ and $\mathbf{P}_j\mathbf{Z}$, where $\mathbf{Z}$ is the phase error operator acting on $\tilde{\mathcal{H}}$ as defined above. The possible errors which can occur on our 4-qubit DFS are thus given by $\mathcal{E} = \{\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathbf{P}_j, \mathbf{P}_j\mathbf{Z}\}$. The task is now to find an appropriate error correction scheme. To do so, we may use the DF states $|0_L\rangle$ and $|1_L\rangle$ to construct the well-known "perfect" 5-qubit QECC [14]. This concatenation yields new encoded states $|0_E\rangle$ and $|1_E\rangle$, composed of 20 physical qubits. In the standard approach to correcting errors

with the 5-qubit code, one uses a quantum network to calculate the syndrome on ancilla qubits and uses this syndrome to apply the appropriate correction procedure on the encoded qubits. Thus, in order to apply a similar procedure to correct for the standard bit flip errors which now act on our 4-qubit DF states, we simply convert the gates used in the standard error correction procedure to gates which perform the same operations on $|0_L\rangle$ and $|1_L\rangle$ and which do not disturb the states in $\tilde{\mathcal{H}}^{\perp}$. Under these conditions, it is obvious that $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$ can be corrected if the DF qubits decohere independently (the condition on standard QECC), as we stipulated. Furthermore, consider the following modified controlled-not gate: $\mathbf{C}|0_L, 0_L\rangle = |0_L, 0_L\rangle$, $\mathbf{C}|1_L, 0_L\rangle = |1_L, 0_L\rangle$, $\mathbf{C}|j_L, 0_L\rangle = |j_L, j_L\rangle$, plus unspecified operations (which do not have to concern us) which ensure that $\mathbf{C}$ is unitary. Using this gate, one can correct for the effect of $\mathbf{P}_j$ on the system as follows: attach an ancilla DF qubit in the $|0_L\rangle$ state, perform $\mathbf{C}$ on this state, and repeat the procedure on every DF qubit of the 5-qubit code word. This detects whether one of the $\mathbf{P}_j$ errors has occurred. Essential to this procedure is the fact that we do not disturb the original $|0_L\rangle$ and $|1_L\rangle$ states. Once an error of this type has been detected, one may recover the erred state to $|0_L\rangle$. The error has then been reduced to a standard Pauli one on a DFS state and can be fixed by QECC. For example (the first qubit, $x = 0$ or 1, belongs to the code word and the second is the ancilla): $|x_L\rangle|0_L\rangle \overset{\mathbf{P}_2}{\mapsto} (|x_L\rangle + |2_L\rangle)|0_L\rangle \overset{\mathbf{C}}{\mapsto} |x_L\rangle|0_L\rangle + |2_L\rangle|2_L\rangle$. Next the ancilla is measured: if the result is 0, no error has occured; if it is 2 then the state is recovered to $|0_L\rangle$ and the resulting standard Pauli error can be fixed by comparison of the erred DF qubit to the other four DF qubits in the code word. Finally, the extreme case of single independent physical qubit errors is also dealt with by the concatenated DFS-QECC code, because the above procedure automatically also corrects the errors which are the subject of the standard QECC procedures, representing them as linear combinations drawn from our natural DFS basis of errors $\mathcal{E}$. Of course, the 20-qubit code presented above is less efficient than the standard 5-qubit QECC code if these independent errors are the dominant ones.

The final question regarding the concatenated DFS-QECC scheme concerns the threshold for fault tolerant quantum computation [15]. The threshold probability of error has been estimated to be in the range of 10^{-6}–10^{-3} per operation [16]. In the present context it is simplest to discuss this issue in the language of the Markovian semigroup master equation. Following the notation of Ref. [6], we consider error generators $\{\mathbf{F}_\alpha\}$ yielding a decohering term $[\mathbf{F}_\alpha, \rho(t)\mathbf{F}_\beta^\dagger]$ in the master equation. Assume these generators produce errors with a rate λ. Then the fidelity $F(t) = \mathrm{Tr}[\rho_S(0)\rho_S(t)]$ is generally reduced by a term of $O(\lambda^2 t)$. Upon inclusion of a symmetry-breaking perturbation by error generators $\{\mathbf{G}_p\}$ of order ϵ, with $\lambda \gg \epsilon$, we find, following the arguments in Ref. [6], that this leads to a decrease in the fidelity, not of order $O(\lambda\epsilon t)$ as one might naively expect (since $[\mathbf{G}_p, \rho(t)\mathbf{F}_\beta^\dagger]$ terms have appeared), but, remarkably, only of order $O(\epsilon^2 t)$ (due to the $[\mathbf{G}_p, \rho(t)\mathbf{G}_q^\dagger]$ terms) [17].

Perturbing a DFS thus produces error rates which are solely proportional to the strength of the additional perturbing process, and which are not dependent on the rate of the error process which generates the DFS. Applying a QECC to this system, as discussed above, improves the fidelity to $1 - O(\epsilon^4 t^2)$. Now in quantum computation one generally envisions a realization of a quantum computer which has an *a priori* low decoherence rate λ. But in the ordinary QECC case, it is this rate λ which sets the error threshold. Hence, in the scenario envisioned here, we have effectively decoupled the rate λ from the error threshold, and a significant improvement may therefore result, if the additional perturbation ϵ is sufficiently small. Concatenation of a DFS with QECC thus has the potential to achieve the goal of truly fault tolerant quantum computation, not just quantum memory.

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