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Encoded Universality in Physical Implementations of a Quantum Computer

D. Bacon,[1,2] J. Kempe,[1,3] D. P. DiVincenzo,[4] D. A. Lidar,[1*] and K. B. Whaley[1†]
Departments of Chemistry[1], Physics[2] and Mathematics[3], University of California, Berkeley 94270
[4]*IBM Research Division, T.J. Watson Research Center, Yorktown Heights, New York 10598*

We revisit the question of universality in quantum computing and propose a new paradigm. Instead of forcing a physical system to enact a predetermined set of universal gates (e.g., single-qubit operations and CNOT), we focus on the intrinsic ability of a system to act as a universal quantum computer using only its naturally available interactions. A key element of this approach is the realization that the fungible nature of quantum information allows for universal manipulations using quantum information encoded in a subspace of the full system Hilbert space, as an alternative to using physical qubits directly. Starting with the interactions intrinsic to the physical system, we show how to determine the possible universality resulting from these interactions over an encoded subspace. We outline a general Lie-algebraic framework that can be used to find the encoding for universality, and give examples relevant to solid-state quantum computing.

I. INTRODUCTION

Our generous universe comes equipped with the ability to compute. In physics, determination of the *allowable* manipulations of a physical system is of central importance. Computer science, on the other hand, has arisen in order to *quantify* what resources are needed in order to perform a certain algorithmic function. For computer science to be applicable to the real world, this quantification should be limited by what physics has determined to be allowable manipulations. Thus we arrive at the realization that because information is physical, computer science should be built on primitives which respect the laws of physics.

An important property of classical information which carries over to the quantum world is the fungible nature of information. A resource is fungible if interchanging it with another resource does not destroy the value of the resource. Whether we represent a classical bit by the presence or absence of a chad on a punch-card [1] or in the average orientation of a million electron spins, the intrinsic value of the information (the value of the bit) is untouched: it does not depend upon the medium in which it is represented. The fungible nature of information has been key to the exponential growth of the computer revolution: it does not matter that the information is being confined to ever smaller components on silicon chips. So, too, for quantum information: the plethora of experimentally proposed systems from which a quantum computer could be built is made possible by the fungibility of quantum information.

Another example of the fungible nature of quantum information is the idea that one can *encode* quantum information. The theory of quantum error correcting codes, for instance, makes use of this to allow computation in the presence of noise. [2] In this paper we argue that we can harness the fungible nature of quantum information in another important aspect, namely in universality constructions. In particular, we propose that instead of taking a physical system and adding external Hamiltonians and interactions to it so that a certain universal set of gates may be performed, one examines instead the intrinsic system interactions for their potential universality. The key to this approach is the recognition that the ability to encode quantum information allows for certain interactions to be universal over an encoded Hilbert subspace, even though they are not universal over the entire Hilbert space. Our task is then to search for the encodings providing such universality, given the intrinsic system interactions. Such encoded universality is common in quantum error correction but appears to have not been recognized in the discussions concerning the physical construction of a quantum computer.

*Current address: Dept. of Chemistry, Univ. of Toronto, Ontario, CANADA
†Corresponding author

II. UNIVERSALITY

The action of gates $\mathbf{G}$ on a circuit corresponds to unitary evolution of the system, with the gates given by the evolution operators derived from the system Hamiltonian $\mathbf{H}$: $\mathbf{G(t)} = \mathtt{T}\left\{\exp[-\mathtt{i}\int^{\mathbf{t}}\mathbf{H}(\tau)\mathbf{d}\tau]\right\}$. Here $\mathtt{T}$ denotes the time ordering operation. A finite set $\mathcal{G}$ of gates is defined to be *universal* if: i) each $\mathbf{U_i} \in \mathcal{G}$ acts on a constant number of qubits only, and ii) any quantum circuit (performing any desired unitary operation) can be simulated to arbitrary accuracy efficiently by a circuit with gates in $\mathcal{G}$.

Somewhat surprisingly, most discussions of universality in quantum computing are cast in the language of gates, rather than Hamiltonians. As physicists we find, however, that in order to understand encoded universality, it is often more useful to think about a gate in terms of the generating Hamiltonian instead of the resulting gate. Given the notion of a universal set of gates, various physical implementations of such a quantum computer have been proposed. Naturally, in order to qualify as a valid quantum computer, each physical implementation must be shown to possess a universal set of gates arising from controllable interactions of the given physical system. To date, with some exceptions, [3–5] all physical proposals for a quantum computer have implemented a set of universal gates via a route in which: (i) the basic qubit is identified in the physical system, (ii) single-qubit unitary gates are shown to be possible on these qubits, and (iii) some two-qubit gate acting between qubits is shown to be possible. The latter is usually a CNOT or a controlled phase. For a few systems this standard model does provide a natural path towards building a quantum computer, but for other implementations severe device engineering must be performed in order to force the system to achieve this universal set of gates.

For instance, in many solid state implementations a fast intrinsic interaction is the exchange interaction (Heisenberg coupling)

$$\mathbf{E_{ij}} = \vec{\sigma}^{\mathbf{i}} \cdot \vec{\sigma}^{\mathbf{j}}, \tag{1}$$

(here $\vec{\sigma}^i$ is the vector of Pauli spin matrices $(\sigma_x, \sigma_y, \sigma_z)$ on the i-th qubit). We call $\mathbf{E_{ij}}$ the exchange interaction because addition of a global phase and rescaling gives a Hamiltonian which exchanges the qubits ($|x\rangle|y\rangle \to |y\rangle|x\rangle$). [6] The gates which can be obtained by tuning the exchange interaction strength J are the unitary gates $\mathbf{U_{ij}(t)} = \exp(\mathbf{itJE_{ij}})$. Given n qubits coupled via exchange interactions it can be shown [4] that these gates alone cannot be used to implement an arbitrary quantum circuit on n qubits. Thus designers of solid state quantum computers have had to supplement the exchange interaction by single-qubit rotations in order to make their set of interactions universal over all n qubits. However, adding this single-qubit capability generally leads to considerable device complexity and to disadvantages such as slow single-qubit gate times and possible additional sources of decoherence resulting from addition of auxiliary, non-intrinsic interactions to the system.

It was recently realized [3–5] that the exchange interaction by itself can be used to simulate a quantum circuit if a particular encoded space of three or more qubits is used to represent the quantum information. Thus while the exchange interaction on n qubits cannot be used to implement an arbitrary quantum circuit on n qubits, it can be used to simulate a quantum circuit on cn qubits, where c is a constant representing the encoding efficiency of such a code. The fact that the exchange interaction *alone* can be used to simulate a quantum circuit by using encoded quantum information allows one to avoid implementing the generally difficult single-qubit gates.

This example illustrates a new approach to universality very different from the conventional paradigm of using single qubit-gates supplemented by a two-qubit gate. The latter has consituted the primary guideline in recent years for deciding whether a physical system might be used as a quantum computer, without regard to the fact that this is just one of many possible paths to universality. Nevertheless, other approaches have also been proposed. Thus, it was shown [7,8] that *almost any* two-qubit unitary gate along with the exchange gate acting between n qubits is universal over all n qubits. We propose here a radically different route to achieve universality, namely, that starting from the natural interactions given by the physics of the proposed qubit system (e.g., $\mathbf{E_{ij}}$), the underlying potential of that interaction for encoded universality be investigated. For a given physical system, the natural interactions of choice should be determined by various factors including their speed, the ease with which they can be implemented, and their robustness towards decoherence processes. The path we take, then, is to proceed from the intrinsic interactions to encoded universality, rather than to artificially modify or build the interactions to fit a standard model.

III. GENERAL FORMALISM

The general formalism will be illustrated here with the example of the exchange interaction. We omit most of the proofs and details, which can be found in Ref. 4.

Assume that we are given a set of interactions on some d-dimensional Hilbert space – Hermitian Hamiltonians $h_i(t)\mathbf{H_i}$ where $h_i(t)$ are controllable coupling constants – that we want to use for our universal quantum computer. Which unitary gates can be approximated by turning these interactions on and off (perhaps in parallel)? The answer to this [7–9] is that every unitary gate in the Lie group corresponding to the Lie algebra *generated* by the Hamiltonians $\mathbf{H_i}$ can be approximated to arbitrary accuracy by a sequence of gates obtained through the temporal evolution of $h_i(t)\mathbf{H_i}$. This is true because of the following properties:

$$e^{i(\alpha\mathbf{A}+\beta\mathbf{B})} = \lim_{n\to\infty}(e^{i\alpha\mathbf{A/n}}e^{i\beta\mathbf{B/n}})^n \tag{2}$$

$$e^{[\mathbf{A},\mathbf{B}]} = \lim_{n\to\infty}(e^{-i\mathbf{A}\sqrt{\mathbf{n}}}e^{i\mathbf{B}/\sqrt{\mathbf{n}}}e^{i\mathbf{A}/\sqrt{\mathbf{n}}}e^{-i\mathbf{B}\sqrt{\mathbf{n}}})^n. \tag{3}$$

To approximate to a given accuracy, say $\exp(i(\alpha\mathbf{A}+\beta\mathbf{B}))$, by a sequence of gates with generators $\mathbf{A}$ and $\mathbf{B}$ one just truncates Eq. (2) for large enough n. Every element in the Lie algebra, that is the algebra formed via linear combination $(\alpha\mathbf{A}+\beta\mathbf{B})$ and via the Lie commutator $(i[\mathbf{A},\mathbf{B}])$, can be approximated to given accuracy using these results. Thus all gates in the Lie group corresponding to this Lie algebra generated by the $\mathbf{H_i}$ can be approximated to arbitrary accuracy by using the controllable coupling constants $h_i(t)$. The inverse of this statement is also true.

Let $\mathcal{A}$ be the Lie algebra generated by the $\mathbf{H_i}$. The Lie group generated by the Lie algebra $\mathcal{A}$ is a subgroup of the d-dimensional unitary group $U(d)$ and thus is a compact Lie group. An important property of compact Lie groups is that they, along with the Lie algebra which generates the Lie group, are completely reducible. A representation of a Lie algebra is irreducible if the action of the representation on its vector space does not possess an invariant subspace. An algebra which is completely reducible can be written as a direct sum of irreducible representations (irreps):

$$\mathcal{A} \cong \bigoplus_J \bigoplus_{\lambda=1}^{n_J} \mathcal{L}_J, \tag{4}$$

where $\mathcal{L}_J$ is the Jth irrep which appears with degeneracy n_J and is d_J-dimensional. For the operators $\mathbf{H_i}$ this implies that there is a basis for the Hilbert space on which $\mathbf{H_i}$ acts as

$$\mathbf{H_i} = \bigoplus_{\mathbf{J}} \mathbf{I_{n_J}} \otimes (\mathbf{L_J})_\mathbf{i} \tag{5}$$

where $(\mathbf{L_J})_\mathbf{i}$ are elements of the Jth irrep $\mathcal{L}_J$ and $\mathbf{I_d}$ is the d-dimensional identity operator. Clearly, understanding the irreps of the Lie algebra generated by the Hamiltonians $\mathbf{H_i}$ will allow us to understand the suitability of this Lie algebra for simulating a quantum circuit. In particular, Eq. (5) implies that with a *suitable encoding* the $\mathbf{H_i}$ can produce the action of a specific irrep $\mathcal{L}_J$. This action may or may not be useful for quantum computation, an issue we address below.

It is often helpful in understanding the universality of an interaction to find the *commutant* $\mathcal{A}' = \{\mathbf{X} : [\mathbf{X},\mathbf{A}] = \mathbf{0}, \forall\mathbf{A} \in \mathcal{A}\}$ of the Lie algebra $\mathcal{A}$. It should be clear from Eq. (5) that every element $\mathbf{K} \in \mathcal{A}'$ is reducible to:

$$\mathbf{K} = \bigoplus_{\mathbf{J}} \mathcal{M}_{\mathbf{n_J}} \otimes \mathbf{I_{d_J}} \tag{6}$$

where $\mathcal{M}_{n_J}$ is an n_J-dimensional (complex) square matrix, and d_J is the dimension of the Jth irrep. This structure, dual to $\mathcal{A}$, offers a useful way for seeing how the encoding arises: $\mathcal{A}'$ splits into a sum of degenerate irreps. The degeneracy of a particular representation gives the dimension of the space over which the encoding is made.

We should point out a non-trivial aspect of the question of "universality given a set of interactions", which makes our task more difficult. This is the tensor product nature of quantum computation. We recall that a set of interactions is universal if it can be used to efficiently simulate a quantum circuit. An important property of the quantum circuit model is that at its most basic level it possess a tensor product of single qubit systems. Hidden within this seemingly trivial fact is the notion of locality: we believe that physics is local and hence we require our model of computation based on physics to preserve this locality. For the purposes of building a universal quantum computer this implies that we must, at some point, introduce a suitable tensor product structure in order to *efficiently* simulate a quantum circuit. The above discussion of representation theory for Lie algebras has been solely in terms of some abstract d-dimensional Hilbert space. We now admit two tensor product structures: first, on our physical system, and second, on the encoded qubits from which encoded universality will be constructed. The first tensor product structure is simply that implied by our physical system and is forced on us by the locality of physics. For example, if we are using the spins of single electrons on a quantum dot, the tensor product is just the natural one of these spin-qubits. The nature of this tensor product structure is also manifest in the set of interactions which will be present in the real world, and is represented in the Hamiltonian as operators acting on separate degrees of freedom.

The second tensor product is required by the fact that we are going to simulate a quantum circuit. Given a set of interactions, we know that these can be used to produce the action of a given set of irreps $\mathcal{L}_J$. However, without a mapping from the subspace on which the irrep acts to the tensor product structure of a quantum circuit, this is not enough to quantify the suitability of the interactions for universality. The exact cutoff where this tensor product structure comes into play is somewhat arbitrary. We will take the position here that we are searching for *small encodings*. That is, we seek to minimize the number of physical qubits used to encode a logical qubit. This logical qubit will be formed by a block of nearest neighbor qubits with the defining property that single encoded-qubit operations are possible within each such block. Given a set of such encoded qubits we then form a tensor product between pairs of blocks. We refer to this process of forming a tensor product code as "conjoining". These considerations are illustrated below.

IV. EXAMPLE OF ISOTROPIC EXCHANGE

As an example of our general discussion consider the Heisenberg model for spin-spin interactions. The most general form of this is given by a sum of fully anisotropic pair-wise spin-spin couplings

$$\mathbf{H_{ij}} = \mathbf{J^X_{ij}}\sigma^{\mathbf{i}}_{\mathbf{x}}\sigma^{\mathbf{j}}_{\mathbf{x}} + \mathbf{J^Y_{ij}}\sigma^{\mathbf{i}}_{\mathbf{y}}\sigma^{\mathbf{j}}_{\mathbf{y}} + \mathbf{J^Z_{ij}}\sigma^{\mathbf{i}}_{\mathbf{z}}\sigma^{\mathbf{j}}_{\mathbf{z}}. \tag{7}$$

We consider in detail here the isotropic limit in which the exchange couplings between a given ij pair are all equal, *i.e.*, $J^X_{ij} = J^Y_{ij} = J^Z_{ij} \equiv J_{ij}$, so that we are dealing then with the Heisenberg Hamiltonian

$$\mathbf{H}_{\mathrm{Hei}} = \sum_{\mathbf{i}\neq\mathbf{j}} \mathbf{J_{ij}E_{ij}} \tag{8}$$

where we assume that the coupling parameters J_{ij} can be turned on/off at will. To show how this intrinsic coupling can lead to encoded universality we analyze its action on three qubits.

The Lie algebra $\mathcal{L}_E$ generated by $\mathbf{E_{ij}}$ on three qubits allows us to implement the Hamiltonians in the set $\{\mathbf{E_{12}}, \mathbf{E_{23}}, \mathbf{E_{13}}, \mathbf{T} \equiv \mathbf{i}([\mathbf{E_{12}}, \mathbf{E_{23}}]\}$. A better basis for the Lie algebra is given by the set of operators $\mathbf{H_0} = \mathbf{E_{12}} + \mathbf{E_{23}} + \mathbf{E_{13}}$, $\mathbf{H_1} = \frac{\mathbf{1}}{\mathbf{4\sqrt{3}}}(\mathbf{E_{13}} - \mathbf{E_{23}})$, $\mathbf{H_3} = \frac{\mathbf{1}}{\mathbf{12}}(-\mathbf{2E_{12}} + \mathbf{E_{23}} + \mathbf{E_{13}})$, $\mathbf{H_2} = \mathbf{i}[\mathbf{H_3}, \mathbf{H_1}]$. We then find that

$$[\mathbf{H_0}, \mathbf{H}_\alpha] = \mathbf{0} \tag{9}$$

for all α and

$$[\mathbf{H}_\alpha, \mathbf{H}_\beta] = \mathbf{i}\epsilon_{\alpha\beta\gamma}\mathbf{H}_\gamma \tag{10}$$

with $\alpha, \beta, \gamma \in \{1, 2, 3\}$. $\mathbf{H_0}$ is an abelian invariant subalgebra of this Lie algebra and thus factors out as a global phase. The set $\{\mathbf{H_1}, \mathbf{H_2}, \mathbf{H_3}\}$, on the other hand, act as the generators of $su(2)$. Thus the exchange interaction between three qubits can be used to implement a single encoded-qubit $su(2)$. More precisely, we find that the $\mathbf{H}_\alpha$ for $\alpha \in \{1, 2, 3\}$ generates the algebra

$$\mathcal{L}^{(3)}_E = \left(\bigoplus_{i=1}^{4} \mathcal{S}_1\right) \oplus \left(\bigoplus_{i=1}^{2} \mathcal{S}_2\right) \tag{11}$$

where $\mathcal{S}_d$ is the d-dimensional irrep of $su(2)$. Note the degeneracy of the corresponding irreps. Corresponding to this decomposition the $\mathbf{H}_\alpha$ act as

$$\mathbf{H}_\alpha = \mathbf{0_4} \oplus \left(\frac{\mathbf{1}}{\mathbf{2}}\sigma_\alpha \otimes \mathbf{I_2}\right) \tag{12}$$

for $\alpha \in \{1,2,3\}$ where $\mathbf{0_4}$ is 4-dimensional zero operator (the 1 dimensional irreps all act as 0) and $\mathbf{I_2}$ is the two-dimensional identity operator. The action of the exchange is thus identical to that of an $su(2)$ operator on a single qubit when working over the encoded space defined by the above decomposition. If we encode our logical qubits as

$$|0_L\rangle = \frac{1}{\sqrt{2}}(|010\rangle - |100\rangle)$$
$$|1_L\rangle = \sqrt{\frac{2}{3}}|001\rangle - \sqrt{\frac{1}{6}}|010\rangle - \sqrt{\frac{1}{6}}|100\rangle \tag{13}$$

then we find that the action of $\mathbf{H}_\alpha$ is $\frac{1}{2}\sigma_\alpha$. Due to the degeneracy of the irrep, other encodings are also possible. Note that these states are nothing but $J = 1/2$ total angular momentum states of 3 spin-1/2 particles with a given projection along a certain axis, and can thus be found using elementary addition of angular momentum. [4]

Having shown that the action of the exchange interaction on three qubits can produce the effect of a single 2-dimensional representation of $su(2)$, it is natural to induce a tensor product structure between blocks of three qubits in order to simulate a quantum circuit. We thus *choose* an encoding scheme in which a single qubit is identified with 3 physical qubis. This is not the only choice of tensor product structure. In fact, any tensor product between sets of $k \geq 3$ qubits can be used to construct a universal gate set. We define the encoding efficiency of our code as the number of encoded bits of our simulated quantum computer ($e = \log_2(d)$ where d is the dimension of the encoded information) divided by the total number of qubits on which the encoding exists: $E = e/n = \log_2(d)/n$. Thus, for a scheme using the exchange interaction between three qubits, where we have a $d = 2$ dimensional encoding on $k = 3$ qubits, we find that $E = 1/3$. A peculiar property of the exchange interaction is that it has an asymptotic encoding efficiency of unity. In particular, if we take n qubits and use the exchange interaction to compute on these qubits then the dimension of the encoded space on which we can compute scales in such a way that the encoding efficiency is at least $E(n) = 1 - \frac{3}{2}\frac{\log_2 n}{n}$. [4] As $\lim_{n\to\infty} E(n) = 1$ we loose almost nothing in terms of encoding efficiency when we use the exchange interaction. We note, however, that one must still introduce a tensor product structure at some level in order to achieve universal quantum computation.

Once one has introduced a tensor structure for the encoding, it is necessary to show that the natural couplings of the system can produce a non-trivial action between the tensor components of this encoded tensor product structure. In the above case, where we have shown that we can obtain full control over an encoded qubit, what we now need to show is that a non-trivial action between the encoded qubits can be enacted. This is nothing more than a map from the encoded universality to the fully universal set of physical gates consisting of single qubit gates supplemented by a non-trivial coupling between the qubits, i.e., to the standard model. This point of the universality proof is typically the most daunting, but really amounts to nothing more than understanding the Lie algebra generated by the interaction over two tensored encoded qubits. For the exchange interaction it can be shown that nearest neighbor exchange interactions can be used to produce such an action. [4] In particular, we find that the effect of the exchange interaction on 6 qubits becomes

$$\mathcal{L}_E^{(6)} = \left(\bigoplus_{i=1}^{7}\mathcal{S}_1\right) \oplus \left(\bigoplus_{i=1}^{5}\mathcal{S}_5\right) \oplus \left(\bigoplus_{i=1}^{3}\mathcal{S}_9\right) \oplus \mathcal{S}_5 \tag{14}$$

When we conjoin the two three-qubit codes, we find that these codes lie entirely within the last two irreps of this decomposition. This in turn implies that non-trivial interactions between the encoded irreps are possible. [4] In fact the original single encoded qubit $su(2)$ is also contained within this decomposition. The important point, however, is that on the tensor product between encoded qubits we can indeed couple these encoded qubits in such a way as to achieve a map to the unencoded, fully universal set of gates. Thus we have established that our encoding can efficiently (to within a factor of three in spatial resources) simulate the unencoded set of universal gates.

A similar analysis can be applied to the anisotropic Heisenberg model (XY model). There it turns out that the smallest encoding one can achieve is a logical qutrit (3-level system) into three physical qubits. [10] For both the isotropic and anisotropic Heisenberg models these universality results are general, i.e., they can be shown to hold for arbitrary numbers of qubits, n. [4,10] However, the approach of conjoining small blocks that encode qubits via a tensor product can be generally used to construct the required mapping of universality for arbitrary interactions, even when a general proof appears inaccessible.

V. OUTLOOK

We have shown that the commonly assumed paradigm of universal quantum computation requiring a set of both single-qubit and two-qubit gates can be replaced by a more general notion of encoded universality. This approach allows universality of quantum computation to be achieved via encodings specific to the natural or convenient physical interactions in a system. In the example given here we have discussed an instance where encoded universality can be achieved using only a *single* underlying physical interaction, namely the exchange interaction. This result is of particular significance for solid state implementations of quantum computation. The approach of coupling blocks containing small numbers of physical qubits (each forming one encoded qubit) via tensor products appears to be applicable to any interaction.

The results for encoded universality that we have described here are primarily existential, and do not address the very practical concerns of the overhead in temporal and spatial resources which may result from encoding. The practical question is then: what is the trade-off in resources of time (gates) and space (qubits) resulting from the universality encoding for a given set of physical interactions? Finding a universality encoding generally immediately reveals the spatial overhead, *i.e.*, the number of physical qubits required to perform the encoding, and may also provide information about the asymptotic overhead. Thus, for the exchange interaction, the maximum spatial overhead is a factor of three, and becomes arbitrarily small asymptotically. [5] The overhead in temporal resources, *i.e.*, how many physical gates are required to implement a given set of encoded universal gates, is somewhat more challenging to deal with. General issues of optimization of physical gates for circuits of encoded qubits have been addressed. [11] It has recently been shown explicitly that the encoded universal operations deriving from the exchange interaction can be realised within ~ 10 clock cycles. [5] However there does not exist to our knowledge any general methodology to find the optimal gate sequence for a specific two-qubit (or two-qutrit) gate.

The finding of encoded universality for the Heisenberg Hamiltonians suggest that this approach will be very useful for consideration of solid state quantum computation. Establishment of results concerning the existence of encodings for general classes of Hamiltonians is therefore now of interest. The new paradigm of encoded universality clearly opens up significant new opportunities for exploring the architectural realization of quantum computation.

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