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ROBUST DYNAMICAL DECOUPLING: FEEDBACK-FREE ERROR CORRECTION*

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Dynamical decoupling is a feed-back free scheme for quantum error correction against noise and decoherence errors. An efficiency analysis of dynamical decoupling is performed. Furthermore we provide the basic concepts of dynamical decoupling and quantum error correcting codes, and give an example of a hybrid protection scheme. Some interesting extensions of dynamical decoupling are discussed at the end.

1. Introduction

Recent advances in control and measurements of quantum systems have given rise to the exciting field of quantum information processing. In quantum information processing different aspects and ideas from quantum physics, computer science, and mathematics have promised futuristic technologies such as quantum computing (QC) and secure quantum communications[11]. The starting point for these application is usually a collection of separate, yet jointly controllable quantum systems, such as qubits. A universal set of the possible dynamics on the system allows for producing arbitrary unitary operators on the full Hilbert space of these qubits. These control operations are sometimes referred to as quantum gates in a discrete sense, or one may alternatively use a set of control Hamiltonians to achieve universality. Quantum information processing has other important

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existential requirements such as efficient state measurement and state preparation.

Isolation of the qubits of a quantum computer is a mathematical simplification. In reality every quantum system may interact with others. This interaction and lack of information about the state of those other systems is a major obstacle in quantum information processing. In other words, realistic and arbitrary control of quantum systems is always limited by the problem of decoherence and noise. These problems are associated with the undesired interactions of an ideally isolated quantum system, over which some means of manipulation and control already exists. More precisely, decoherence refers to the decay of *quantum* superpositions of pure states into mixed states of the possible measurement outcomes[11]. The term "error" often takes a more general meaning as it applies to classical circumstances also.

In the context of quantum information processing, "error correction" theory discusses problems (and solutions) due to the undesired interactions of quantum systems. Sometimes some knowledge of the error process is available and can be used to protect quantum systems against errors. For example, the collective decoherence models where certain global symmetries exist in the error process can easily be dealt with encoding of quantum information in the available decoherence free subspaces[10]. Universal error correction, in contrast, focuses on more general error models where apart from certain error rates and error correlations, not much is assumed about the nature of the errors affecting the system. Different regimes of errors naturally demand different error correction schemes. Practically speaking, these schemes can be operationally active or passive, use feedback, or use a larger operational Hilbert space (encoding). The domain of applicability and practicality of various schemes are different and so far no single scheme is practically capable of protecting against arbitrary error types/models. Despite this lack of generality, it is widely believed that hybrid methods incorporating various schemes can be efficiently used within a given physical implementation framework of a quantum information processor, for the purpose of quantum computing, and quantum state preservation [8,9].

In this summary we shall refer to the system's Hilbert space as $\mathcal{H}_S$, and to that of an external environment as $\mathcal{H}_B$, which we shall refer to as "the bath" or "the environment". We often assume $\mathcal{H}_S$ to be composed of one or more qubits. Generically ψ refers to a pure quantum state, while ρ refers to a density matrix. The starting point of the analysis in quantum error correction theory is the system-

bath Hamitonian ansatz:

$$H = I_B \otimes H_S + H_B \otimes I_S + \underbrace{\sum_\alpha B_\alpha \otimes S_\alpha}_{H_{SB}} \tag{1}$$

Operators I_B and I_S are the identity operators on $\mathcal{H}_S$ and $\mathcal{H}_B$. Typically no reliable operation or observation can be made on the environment. H_S refers to the system Hamiltonian, over which certain control is assumed. H_B is the Hamiltonian for the environment which might be unknown. H_{SB} is responsible for entangling the states of the system and the environment, which leads to decoherence, once the system density matrix ρ_S is reproduced after tracing out the environment: $\rho_S = \mathrm{Tr}_B(\rho_{SB})$. The isolation and the assumption of control over the system is a basic requirement of quantum information processing, but Eq. (1) is an approximation that can always be improved by incorporating further entities within the environment.

An alternative description involving only the system is also used: A quantum channel describes the linear transformation of the system density matrix ρ_S. This description in the Markovian regime is further simplified in the sense that the quantum channel description for short times describes the evolution of the system for all times and the whole evolution can be generated by exponentiating a "Lindblad super-operator" acting on the density matrices[1]. It is worth mentioning that while these pictures are mathematically interchangeable, the physical constraints often limit the way these pictures are used. Quantum error correcting codes[8] and decoherence free subspaces as error correction protocols can be described in the channel picture. Quantum error correcting codes in particular are the most predominant error correction strategy as they offer extensive universality within the Markovian regime and allow for fault tolerant quantum computation which technically refers to a robust implementation of quantum computing (in contrast with quantum state preservation or quantum memory). Quantum error correction carries the overhead of extra quantum computing qubits, entangling gates, measurements and ancillary qubits.

Dynamical Decoupling[17] is another error correction strategy which is implemented by application of a series of fast and strong/narrow pulses acting on the system that effectively renormalize the interaction Hamiltonian to remove undesired terms such as H_{SB}. Dynamical decoupling techniques have been traditionally used in NMR to remove unwanted intra-nuclear couplings and obtain high resolution spectra[4] . It can be shown that dynamical decoupling can approximately remove an arbitrary H_{SB} Hamiltonian, a technique which is referred to as universal dynamical decoupling. While this universality is a remarkable aspect of the dynamical decoupling theory, comparable to the universality of quantum error cor-

recting codes, physical constraints have obscured the prospect of effectively using dynamical decoupling in quantum information processing. These constraints are the requirements for perfect pulses and the ability to run closely packed sequence of pulses.

In this summary we analyze and discuss dynamical decoupling in the Hamiltonian setting, and sketch some interesting extensions of dynamical decoupling. We also cover the basics of quantum error correcting codes and present an example of a hybrid error correction scheme where both dynamical decoupling and quantum error correcting codes are used.

2. Propagators, Pulses, and Idealizations

In this summary we shall only focus on qubit systems. Some of the results and ideas that apply to the qubit case are extendible to other systems and settings. An isolated qubit is always driven by an $su(2)$ Hamiltonian given by $H_c = h_x X + h_y Y + h_z Z$. Operators X, Y, and Z refer to the corresponding Pauli operators. Without loss of generality, all Hamiltonian components which we shall consider are either traceless or a multiple of identity.

The unitary operations on this system can be generated by the Schrödinger propagator between the times t_0 and t_1:

$$U = T_+ \left[\exp(-i \int_{t_0}^{t_1} H_c(t') dt') \right] \tag{2}$$

For example the unitary operator X can be generated by turning on $H_c = \pi/2 h_x X$ for a duration $\delta = t_1 - t_0 = \frac{1}{h_x}$. We shall refer to $\delta \to 0$ as an "ideal pulse" in this summary. If a unitary operator is given by $e^{-i\Phi}$, we will simply refer to the Hermitian operator Φ as the *phase*.

As discussed in the previous section, a general Hamiltonian describing the qubit S plus an environment B is given by Eq. (1), and in the qubit case can be generally written as

$$H = H_c + H_e \tag{3}$$

$$= I_B \otimes (h_x X + h_y Y + h_z Z) + B_x \otimes X + B_y \otimes Y + B_z \otimes Z + B_0 \otimes I_S \tag{4}$$

where H_e loosely refers to the undesired parts of the qubit (and the environment) Hamiltonian. In this picture the propagator for a given navigation of the control H_c is

$$U_e = T_+ \left[\exp(-i \int_{t_0}^{t_1} (H_c(t') + H_e) dt') \right] \tag{5}$$

By scaling down $t_1 - t_0$ to 0 while scaling up $H_c(t)$ to keep the product constant, we get the ideal propagator: $U_e = I_B \otimes U$. In most of what follows we assume either ideal pulses or rectangular pulses.

3. Dynamical Decoupling

For the sake of clarity we first focus on the simplest case which we refer to as a "canonical dynamical decoupling cycle". We start from Eq. (3) and restrict $H_e = B_z \otimes Z$. As in the previous section H_c is the controllable part of the Hamiltonian and we use it to produce unitary pulses acting on the system. The *canonical dynamical decoupling cycle with pulse X* is given by "$XF_{[\tau]}XF_{[\tau]}$", where the pulse sequence is applied from the right and $F_{[\tau]}$ refers to a free evolution of duration τ. The unitary propagator for this sequence is given by

$$\begin{aligned} U &= X\exp[-i\tau(B_z\otimes Z)]X\exp[-i\tau(B_z\otimes Z)] \\ &= \exp[-i\tau(B_z\otimes XZX)]\exp[-i\tau(B_z\otimes Z)] \\ &= \exp[-i\tau(-B_z\otimes Z)]\exp[-i\tau(B_z\otimes Z)] = I_S\otimes I_B \end{aligned} \tag{6}$$

The above sequence has removed H_e from the evolution of the system by *time reversal*. To generalize we note that any qubit Hamiltonian H_e can be decomposed as $H_e = H_e^{X,\|} + H_e^{X,\perp}$ such that $[X, H_e^{X,\|}] = 0$ and $\{X, H_e^{X,\perp}\} = 0$. For a general H_e we can rewrite the above sequence

$$\begin{aligned} U &= X\exp[-i\tau(H_e^{X,\|} + H_e^{X,\perp})]X\exp[-i\tau(H_e^{X,\|} + H_e^{X,\perp})] \\ &= \exp[-i\tau(H_e^{X,\|} - H_e^{X,\perp})]\exp[-i\tau(H_e^{X,\|} + H_e^{X,\perp})] \\ &=: \exp(-i2\tau\mathcal{D}_X[H_e]) = \exp[-i\tau(2H_e^{X,\|} + O(B_\alpha^2\tau)] \end{aligned} \tag{7}$$

In Eq. (7) the overall propagator is used to define an effective Hamiltonian $\mathcal{D}_X[H_e]$. We can look at the transformation of the propagator as a renormalization of the Hamiltonian:

$$B_x\otimes X + B_z\otimes Z + B_Y\otimes Y \overset{XF_{[\tau]}XF_{[\tau]}}{\longmapsto} B_x\otimes X + O(B_\alpha^2\tau) \tag{8}$$

Geometrically this process can be thought of a projection of H_e parallel to the "X axis".

The pulse used in the canonical dynamical decoupling sequence, removes all terms anti-commuting with it from the effective Hamiltonian. To remove every possible term it suffices (for example) to use the propagator of the above sequence as the "free evolution period" of a Y canonical dynamical decoupling sequence:

$$\underline{YF_{[2\tau]}}\,\underline{YF_{[2\tau]}} \curvearrowright \underline{YXF_{[\tau]}XF_{[\tau]}}\,\underline{YXF_{[\tau]}XF_{[\tau]}} \tag{9}$$

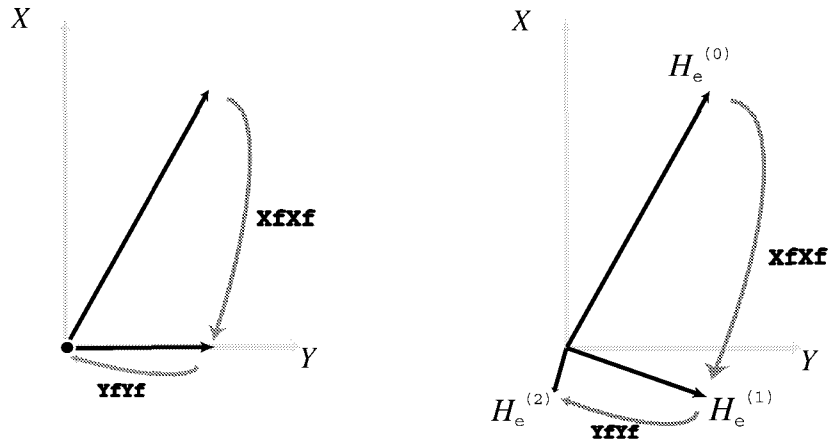

Figure 1. Projections and rotated projections in dynamical decoupling. The aim is to produce a combination of two orthogonal projectors (left), but in practice due to non-commutative terms, we obtain a combination of two tilted projectors.

Optionally we can use $YX = Z$ to simplify[a] the above sequence: $ZF_{[\tau]}XF_{[\tau]}ZF_{[\tau]}XF_{[\tau]}$. This sequence is now *universal* in the sense that any H_e implied in the free evolution period $F_{[\tau]}$ is removed up to $O(B_\alpha^2\tau)$ since the only terms that commute with both X and Y are the "pure-bath" operators that have no effect on the system dynamics. Geometrically we can represent the combination of the two above transformations as two projections on the space of Hamiltonians. Ideally these two orthogonal projections will produce zero, but the higher order terms can be shown to produce extra rotations that suppress the cancellation and generally result in imperfect decoupling. This is illustrated in Fig. 1.

The unitary propagator of the sequence of sign-flipped Hamiltonians can be approximated by a Magnus expansion[17]: If the Hamiltonian for the Schrödinger evolution of a quantum system is given by $H(t)$, then the propagator $U(t)$ from 0 to t is given by $U = \exp(A_1 + A_2 + \ldots)$, where

$$A_1 = i\int_0^t dt_1 H(t_1) \tag{10}$$

$$A_2 = \frac{1}{2}\int_0^t dt_1 \int_0^{t_1} dt_2 [H(t_2), H(t_1)] \tag{11}$$

and A_i for $i > 2$ are more complicated i-dimensional integrals of commutators involving $H(t)$s at i times. A *k-th order universal dynamical decoupling sequence*

[a]We omit global phase factors of $i, -1$, *etc.*

produces a sequence of Hamiltonians and intervals such that A_1 to A_k contain only pure-bath terms $B \otimes I_S$. The sequence $ZF_{[\tau]}XF_{[\tau]}ZF_{[\tau]}XF_{[\tau]}$ is, for example, 1st order universal. If we can bound the operators B_α form above by some error rate j, then the second order Magnus terms for this sequence will be bounded by $c\tau_0^2 j^2$, where c is a numerical constant of the order of 10. The Magnus terms can be used to produce an effective "phase" or Hamiltonian for imperfections of dynamical decoupling.

In periodic dynamical decoupling, N periodic repetitions of a universal sequence for a total duration of $T = 4N\tau$ are used. The leading error associated with this decoupling can be easily estimated:

$$\|\Phi\| \lesssim Nc\tau_0^2 j^2 = c(Tj)^2/N. \tag{12}$$

The actual error in the fidelity is typically given by the square of this phase [15].

Eq. (12) dictates the use of shorter times between the pulses to obtain higher fidelities, however once more pulses are used, realistic "per pulse errors" take over and produce significant extra errors not covered by the above formula. It should be noted that even perfect control over H_c does not guarantee perfect pulses, as the presence of the H_e terms in the Hamiltonian, will result in systematic pulse errors proportional to the pulse width[b] and once the total duration for a physical realization of the sequence is fixed, one cannot make τ_0 arbitrary small by including increasingly many finite width pulses.

Despite these difficulties dynamical decoupling has been used with great success in the context of NMR. It is worth emphasizing that three main parameters enter the analysis of dynamical decoupling: model considerations, system-bath couplings, and pulse imperfections. These parameters will also appear in the analysis of quantum error correcting codes.

4. Basic Quantum Error Correcting Codes

Based on successful methods from "classical" error correcting codes, quantum error correcting codes are thought to be the most generic and best understood error protection schemes. The starting point is to obtain the error operators E_α, that describe the evolution of an ideally fixed (at $\rho(0)$) density matrix for the system via a Kraus operator expansion:

$$\rho(t) = \sum_\alpha E_\alpha(t)\rho E_\alpha(t)^\dagger \tag{13}$$

[b]It is possible to perform dynamical decoupling without the requirement of infinitely sharp pulses [16]. Nonetheless strong pulses and precise control are required to achieve high fidelities.

Error correcting codes are efficient in the Markovian regime where the correlations in time are minimal, and the short time behavior of the system dictates (E_α instead of $E_\alpha(t)$) that of the longer times. Quantum error correcting codes encode (embed) "logical qubit" states into a larger Hilbert space (of many more qubits). The embedding is such that the effect of all error operators E_α on all encoded states $|i\rangle_L$ are mutually orthogonal:

$$ {}_L\langle i| E_\alpha^\dagger E_\beta |j\rangle_L = \delta_{ij} c_{\alpha\beta} \tag{14}$$

where c_{ab} are the elements of a Hermitian matrix. This ensures that the effect of each error on any state in the logical Hilbert space can be measured via a "syndrome measurement" protocol and the proper state can be obtained via a "recovery" protocol. Structurally quantum error correcting codes rely on successive measurements, near parallelism in the applied operations, enlargement of the qubit Hilbert space[c] and availability of refreshable, pure ancilla qubits. Quantum error correcting codes are most efficient for the independent noise model, in which qubits are acted upon by errors individually and randomly such that there is no correlation in time (successive errors) and space (multiple qubit errors). For this case the theory can be applied fault-tolerantly together with actual encoded quantum operations. Within this theory one can show that the same embedding of the logical states in a bigger Hilbert space can be applied recursively so that errors, with rates lower than a certain threshold, can be efficiently removed. This is what is known as the threshold theorem for concatenated error correcting codes: For error rates below a certain threshold, using an exponential number of resources (qubits, pulses, and measurements) leads to super-exponential improvement in performance. Different assumptions on error model and modifying the actual error correction/computation schemes results in different thresholds [14,7].

It is possible to combine quantum error correcting codes with quantum operations. The stabilizer theory of the quantum error correcting codes[3] provides a relatively simple way of embedding the universal quantum operations with a given stabilizer based quantum error correcting code, and is the basis of fault-tolerant quantum computation.

The stabilizer formalism relates to dynamical decoupling also: It can be shown that the stabilizer elements of a given quantum error correcting code can be used as canonical decoupling cycles on the physical qubits of that code for removing the error operators that the code can correct, from the interaction terms. In other words the stabilizer generator elements become the canonical dynamical decoupling pulses; *e.g.*, for a single qubit the stabilizer generators can be taken to be X

[c]The smallest code that can correct arbitrary 1-qubit errors is given in term of 5-qubit states.

and Y.

5. Hybrid Error Correction Schemes

While both dynamical decoupling and quantum error correcting codes work for arbitrary error types, they each depend on certain resources and have their domain of practicality. However, certain modes of decoherence might allow for different techniques to be used together. One such example is the error correction of spontaneous emission on an electronic-level qubit. The coupling of a bound electron (system) with the electromagnetic field (environment) results in spontaneous emission, in which the electron that might be in a superposition of the excited ($|1\rangle$) and the ground ($|0\rangle$) state decays to $|0\rangle$, and the quantum information stored in the superposition is lost. Assume a collection of such electron-level qubits labeled by $i = 1,\ldots,n$. Evolution of the system can be described in the quantum trajectories picture. In this picture, each trajectory is separated into non-unitary evolution intervals, interrupted by sudden application of error operators (E_i) ($E_i = |0\rangle_i\langle 1|$ for spontaneous emission on the qubit i). The final density matrices of all trajectories can then be averaged to give the actual probabilistic density matrix as a function of time. The non-unitary evolution in this picture is generated by a non-Hermitian conditional Hamiltonian, $H_{\text{cond}} = H_{\text{system}} - \frac{i}{2}\sum_i E_i^\dagger E_i$. In the case of spontaneous emission we obtain $H_{\text{cond}} = i\sum_i |1\rangle_i\langle 1|$. We showed[5] that a simple quantum error correcting code involving only one extra qubit together with a dynamical decoupling pulses sequence with pulse type X can be used to correct spontaneous emission errors as long as the error rate is small and the pulse operations are faster than the average time between the errors. We further investigated the possibilities of fault-tolerant quantum computation within this setting.

6. Extensions of Dynamical Decoupling

To conclude, in the following subsections we briefly review some of the most recent results and ideas in dynamical decoupling.

6.1. *Inter-qubit couplings*

Consider a collection of n qubits in which the undesired interactions not only involve the couplings with the environment, but also inter-qubit couplings such as nearest neighbor couplings terms: $\sigma_x^i\sigma_x^j$. An interesting extension of dynamical decoupling is to remove these error terms, along with the system-bath interaction. The optimization of this sequence in terms of the number of pulses required becomes a combinatorial optimization problem[12].

6.2. *Zeno Effect*

On a different note an interesting connection is observed between dynamical decoupling and the quantum Zeno effect which traditionally is understood in terms of repeated measurements on a quantum system to inhibit or enhance couplings. This traditional view has been transformed to include arbitrary quantum operations and naturally includes dynamical decoupling as a special case [2]. Furthermore recently it has been shown[13] that quantum error correcting codes can be defined in terms of the Zeno effect, which hints to a possibly deeper physical connection between different error correction schemes.

6.3. *Concatenation*

The idea of concatenation, *i.e*, the recursive embedding of encoded qubits in the same error correcting code, has been used for threshold calculations. This threshold refers to an initial error rate below which, even after using an exponential number of concatenated qubits, the final error rate is inhibited "super-exponentially". We showed[6] that the same idea can be used with dynamical decoupling with a similar threshold. Given the number of pulses used N, the phase associated with concatenated dynamical decoupling scales as

$$\|\Phi\| \lesssim \frac{1}{c}(cj/N)^N. \tag{15}$$

This should be contrasted with the result obtained in the case of periodic decoupling, Eq. (12). The exponential improvement in terms of the number of pulses required thus strongly suggests the use of concatenated pulse sequences over the periodic schemes.

6.4. *Error per Gate*

One of the main difficulties with both dynamical decoupling and quantum error correcting codes is the requirement for near perfect quantum gates. For example, consider an error correcting code designed to reduce the rate e to some lower error rate e^2. In fault-tolerance theory, however, the error per gate is another important factor. Especially when a high number of quantum gates are executed, one needs to make sure that the gate errors are still corrected with the original error correcting design. Now, suppose the errors are based on a continuous model such as a Hamiltonian picture. In this picture we may associate linear error accumulation (well-defined error rate) to short times. Now one might ask "is the error probability fundamentally different when there are gates acting on the system in comparison to when no gate is being applied?" If the error rates for "gate-free"

and "gate" evolution are of the same order, then there is effectively only one important error rate in the system. Define the gate-effective error generator H_e' as $\exp(-i\delta H_e') = \exp(-i\delta(H_e + H_P))\exp(i\delta H_P)$, where $\exp(-i\delta H_P)$ is the propagator for a pulse P, and H_e is the free evolution error generator. For the case where all operators belong to $su(2)$, we have shown that $\|H_e'\| \leq \|H_e\|$. Thus for a simple case we confirm that the natural error parameter for the free evolution of the system is indeed larger than the parameters that are associated with gate errors, which simplifies fault tolerance.

6.5. *Dynamically Decoupled Quantum Computing*

While dynamical decoupling is an essential part of engineering interactions for quantum computing in many proposals, a *universal* decoupling process will naturally remove the "desired" evolution of the system along with the undesired parts. Due to this, the prospects of combining dynamical decoupling with quantum computation have been limited to a scenario where dynamical decoupling stops so that a quantum gate or a measurement is performed and is then resumed. This method is obviously prone to errors accumulating during the computation phase and the advantages of dynamical decoupling with state preservation are rendered useless. Nonetheless a protocol can be constructed in which an encoding of a few qubits are employed so that the quantum computing operations can be embedded along with the dynamical decoupling pulse sequence which corrects for single qubit errors. In this construction dynamical decoupling reflects and brings back all trajectories of the qubits that leave the encoded subspaces without modifying the desired encoded dynamics.

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