# Adiabatic Quantum Computation in Open Systems

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(Received 1 February 2005; published 16 December 2005)

We analyze the performance of adiabatic quantum computation (AQC) subject to decoherence. To this end, we introduce an inherently open-systems approach, based on a recent generalization of the adiabatic approximation. In contrast to closed systems, we show that a system may initially be in an adiabatic regime, but then undergo a transition to a regime where adiabaticity breaks down. As a consequence, the success of AQC depends sensitively on the competition between various pertinent rates, giving rise to optimality criteria.

DOI: 10.1103/PhysRevLett.95.250503 PACS numbers: 03.67.Lx, 03.65.Yz, 03.65.Ta

Adiabatic quantum computation (AQC) is a promising paradigm for quantum information processing, which appears particularly well suited to physical implementations [1]. In AQC, an algorithm is implemented via the slow evolution of a time-dependent Hamiltonian $H(t)$. AQC schemes have recently been proposed based on superconducting flux qubits [2]. An experimental implementation of an adiabatic optimization algorithm using nuclear magnetic resonance (NMR) techniques has already been reported [3]. Moreover, it has recently been shown that AQC and the standard circuit model of quantum computation are equivalent up to polynomial resource overhead [4,5].

The robustness of AQC against errors has recently been analyzed in several contexts [6,7]. An important consideration is that, if decoherence occurs in the instantaneous eigenstate basis, then AQC can be intrinsically robust against environmental noise provided one runs the algorithm at a temperature that is small compared to the minimum gap [6]. However, despite the importance of this result for the robustness of AQC, the choice of the system eigenstate basis as a preferred basis may not always be a good approximation (especially for non-Markovian environments), since it implicitly assumes that *the environment keeps track of the Hamiltonian evolution*. Moreover, from a more general point of view, a methodology to systematically study AQC under decoherence has not yet been developed. In this work, we introduce such a methodology and analyze the performance of AQC under decoherence modeled by a rather general class of master equations. Our approach is based on a recently introduced adiabatic approximation genuinely conceived for open quantum systems [8]. We show that this framework can be used to provide the optimal runtime of adiabatic quantum algorithms. This allows for the understanding of the performance and robustness of open-system AQC. We illustrate our method by discussing the adiabatic implementation of the Deutsch-Jozsa algorithm under dephasing.

*Adiabaticity in open quantum systems.*—Let us consider a quantum system coupled to an environment, or bath, evolving under the convolutionless master equation

$$\dot{\rho}(t) = \mathcal{L}(t)\rho(t). \tag{1}$$

An example of this class of master equations is given by (we use $\hbar = 1$ units throughout)

$$\dot{\rho} = -i[H, \rho] + \frac{1}{2}\sum_i ([\Gamma_i, \rho\Gamma_i^\dagger] + [\Gamma_i\rho, \Gamma_i^\dagger]). \tag{2}$$

Here $H(t)$ is the time-dependent effective Hamiltonian of the open system and $\Gamma_i(t)$ are time-dependent operators describing the system-bath interaction. Equation (2) with time-*independent* operators $\Gamma_i$ is usually referred to as the Markovian master equation, or Lindblad equation [9]. In a slight abuse of nomenclature, we will refer to the time-dependent generator $\mathcal{L}(t)$ [Eq. (1)] as the Lindblad superoperator and the $\Gamma_i(t)$ [Eq. (2)] as Lindblad operators.

The key idea required to establish a natural adiabatic approximation for open systems is to replace the concept of adiabatic evolution of eigenspaces of the Hamiltonian by adiabatic evolution of the Jordan blocks of the Lindblad superoperator [8]. In the superoperator formalism, the density matrix for a quantum state in a $D$-dimensional Hilbert space is represented by a $D^2$-dimensional "coherence vector" $|\rho\rangle\rangle = (\rho_1, \rho_2, \ldots, \rho_{D^2})^t$ and the Lindblad superoperator $\mathcal{L}$ becomes a $(D^2 \times D^2)$-dimensional supermatrix [9]. The master equation (1) generates a nonunitary evolution, since $\mathcal{L}(t)$ is non-Hermitian, and therefore, generally, nondiagonalizable. However, one can always transform $\mathcal{L}(t)$ into the Jordan canonical form, where it has a block-diagonal structure. This is achieved via the similarity transformation $\mathcal{L}_J(t) = S^{-1}(t)\mathcal{L}(t)S(t)$, where $\mathcal{L}_J(t) = \mathrm{diag}\,(J_1, \ldots, J_m)$ is the Jordan form of $\mathcal{L}(t)$, with $J_\alpha$ denoting the Jordan blocks. Instantaneous right $\{|\mathcal{D}_\beta^{(j)}(t)\rangle\rangle\}$ and left $\{\langle\langle\mathcal{E}_\alpha^{(i)}(t)|\}$ bases in the state space of linear operators can always be systematically constructed

 

such that they obey the orthonormality condition $\langle\langle \mathcal{E}_\alpha^{(i)}(t)|\mathcal{D}_\beta^{(j)}(t)\rangle\rangle = \delta_{\alpha\beta}\delta^{ij}$, and such that the Jordan block structure is preserved under the action of the Lindblad superoperator, i.e., $\mathcal{L}(t)|\mathcal{D}_\alpha^{(j)}(t)\rangle\rangle = |\mathcal{D}_\alpha^{(j-1)}(t)\rangle\rangle + \gamma_\alpha(t)|\mathcal{D}_\alpha^{(j)}(t)\rangle\rangle$ and $\langle\langle \mathcal{E}_\alpha^{(i)}(t)|\mathcal{L}(t) = \langle\langle \mathcal{E}_\alpha^{(i+1)}(t)| + \langle\langle \mathcal{E}_\alpha^{(i)}(t)|\gamma_\alpha(t)$, with $|\mathcal{D}_\alpha^{(-1)}\rangle\rangle \equiv 0$ and $\langle\langle \mathcal{E}_\alpha^{(n_\alpha)}| \equiv 0$ [8]. Here subscripts enumerate Jordan blocks ($\alpha \in \{1, \ldots, m\}$), superscripts enumerate basis states inside a given Jordan block ($i, j \in \{0, \ldots, n_\alpha - 1\}$, $n_\alpha$ is the dimension of the Jordan block), and $\{\gamma_\alpha\}$ are the (generally complex-valued) Lindblad-Jordan (LJ) eigenvalues. Then, an open quantum system is said to undergo adiabatic dynamics when its Hilbert-Schmidt space can be decomposed into decoupled LJ eigenspaces with distinct, time-continuous, and noncrossing instantaneous eigenvalues of $\mathcal{L}(t)$ [8]. Just as in the closed-systems case, one can express the condition for adiabaticity in terms of the total time of evolution. To this end, we expand $|\rho(t)\rangle\rangle = \sum_{\beta=1}^{m}\sum_{j=0}^{n_\beta-1} p_\beta^{(j)}(t)e^{\int_0^t \gamma_\beta(t')dt'}|\mathcal{D}_\beta^{(j)}(t)\rangle\rangle$. It is convenient to express the variables in terms of the dimensionless time $s = t/T$, where $T$ denotes the total evolution time. Then, adiabatic dynamics in the interval $0 \le s \le 1$ occurs if and only if the following time condition is satisfied: $T \gg \max_\alpha\{T_\alpha^c\}$, where $T_\alpha^c$ denotes the *crossover time* for the Jordan block $J_\alpha$ [8]. For the particular case of one-dimensional blocks, which appears in our example below, we have [8]

$$T_\alpha^c = \max_{0\le s\le 1}\left|\sum_{\beta\neq\alpha}\left[Q_{\beta\alpha}(0) - Q_{\beta\alpha}(s)e^{T\Omega_{\beta\alpha}(s)} + \int_0^s ds' e^{T\Omega_{\beta\alpha}(s')}dQ_{\beta\alpha}(s')/ds'\right]\right|, \quad (3)$$

where $\Omega_{\beta\alpha}(s) = \int_0^s \omega_{\beta\alpha}(s')ds'$, $\omega_{\beta\alpha}(s) = \gamma_\beta(s) - \gamma_\alpha(s)$ (the gap between Jordan eigenvalues), $Q_{\beta\alpha}(s) \equiv V_{\beta\alpha}(s)/\omega_{\beta\alpha}^2(s)$, and $V_{\beta\alpha}(s) = p_\beta(s)\langle\langle \mathcal{E}_\alpha(s)|\frac{d\mathcal{L}(s)}{ds}|\mathcal{D}_\beta(s)\rangle\rangle$ (matrix elements of the time derivative of the Lindblad superoperator). Note that a quantity analogous to $Q_{\beta\alpha}$ appears in the standard condition for adiabaticity in closed systems [8]. In the expression for $V_{\beta\alpha}(s)$, upper indices in $p_\beta^{(j)}(s)$ and in the basis vectors $\{|\mathcal{D}_\beta^{(j)}(s)\rangle\rangle\}$ and $\{\langle\langle \mathcal{E}_\alpha^{(i)}(t)|\}$ were removed because the Jordan blocks are one dimensional. The crossover time $T_\alpha^c$ provides a decoupling time scale for each Jordan block: provided $T \gg T_\alpha^c$ the Jordan block $J_\alpha$ is adiabatically decoupled from all other blocks associated to a different eigenvalue.

*Performance of open-systems adiabatic quantum algorithms.*—The performance of AQC under decoherence can be analyzed consistently within the present picture of open-systems adiabaticity. In particular, the maximal crossover time $\max_\alpha\{T_\alpha^c\}$ determined by Eq. (3) provides the time scale over which the adiabatic approximation holds. Provided the evolution is as slow as is set by this time scale, the density operator evolves separately in sets of Jordan blocks related to distinct eigenvalues of $\mathcal{L}(t)$. Thus, if the initial density matrix is associated to a certain set of instantaneous Jordan blocks, it will remain associated to the same instantaneous set at all times. Note that, if there is an overall growing exponential in the right-hand side of Eq. (3), then the adiabatic behavior takes place over a finite time interval and, afterwards, disappears. In this case, which is an exclusive feature of open systems, *we have the existence of a privileged time for adiabaticity.* Having determined the adiabatic time interval, the performance of the algorithm can be understood from the adiabatic density operator $\rho_a(s, \lambda_i, T)$, where $\lambda_i$ are the system-bath coupling constants. This operator is obtained by solving the adiabatic master equation (1), where we disregard any coupling among Jordan blocks associated to distinct eigenvalues.

The final result, coming from $\rho_a(1, \lambda_i, T)$, will then depend on a competition between the adiabatic runtime $T$ and the coupling constants $\lambda_i$. On the one hand, the adiabatic approximation is favored for a certain time interval. On the other hand, decohering processes tend to progressively destroy the performance of the algorithm over time (intuitively, decoherence causes broadening of the energy levels, until they overlap). This compromise between adiabaticity and decoherence generates an optimal runtime for the algorithm, which provides the optimal success probability for given system-bath coupling strength. In agreement with this picture, *an optimal time has indeed been detected* in the experimental NMR AQC algorithm reported in Ref. [3]. Here, we provide a general explanation for such an optimal time in terms of the decoupling of the Jordan blocks of $\mathcal{L}(s)$.

*Constancy of the gap.*—An important condition for the decoupling of the Jordan blocks is the existence of gaps in the spectrum of LJ eigenvalues $\{\gamma_\alpha\}$. This is relevant for AQC, where a major concern is the scaling of the gap with problem input size. In fact, there have been indications that AQC may take an exponential time to solve certain hard instances of NP-complete problems due to vanishingly small gaps [10]. A physical interpretation for the exponential delay has been proposed in terms of the quantum tunneling of a large spin during the computation [11]. In the closed-systems case, it is in principle possible *to keep the gap constant throughout the execution of AQC* via a ''unitary interpolation'' scheme [5]. However, the trade-off in using this method is that, in general, it may require many-body interactions. In spite of this difficulty, schemes with a constant gap are an interesting possibility, since they constitute a favorable situation for closed-systems AQC. Thus, it is natural to ask whether a constant gap setting may also be implemented within an open-systems context. Our methodology for AQC can be used to answer this question. In fact, we will see that this possibility persists in the open-systems setting only under very special conditions. We emphasize that the general approach we introduced above applies to all interpolation schemes, in particular, to standard, linear interpolation AQC [1,6]. The latter has the

advantage of avoiding the many-body interactions associated with unitary interpolation [4,5].

Let us first show that if the Hamiltonian changes by a unitary transformation, then the corresponding superoperator $\mathcal{H}(s)$ also changes by a unitary transformation. The eigenvalues of $\mathcal{H}(s)$ are given by the set of energy differences $\{\epsilon_{mn}(s) = E_m(s) - E_n(s)\}$ and the eigenvectors by the set $\{|\psi_m(s)\rangle\langle\psi_n(s)|\}$. Therefore, if the Hamiltonian giving rise to $\mathcal{H}(s)$ changes by a unitary transformation $U(s)$, then the eigenvectors of $\mathcal{H}(s)$ evolve as $|\psi_m(s)\rangle\langle\psi_n(s)| = U^\dagger(s)|\psi_m(0)\rangle\langle\psi_n(0)|U(s)$. Expressing them as vectors $|\rho_{mn}(s)\rangle\rangle$ in Hilbert-Schmidt space, we have $|\rho_{mn}(s)\rangle\rangle = \mathcal{V}^\dagger(s)|\rho_{mn}(0)\rangle\rangle$, with $\mathcal{V}(s)\mathcal{V}^\dagger(s) = I$, which follows from the orthonormality of $|\rho_{mn}(s)\rangle\rangle$. Hence, from $\mathcal{H}(s)|\rho_{mn}(s)\rangle\rangle = \epsilon_{mn}(s)|\rho_{mn}(s)\rangle\rangle$, we obtain $\mathcal{H}(s) = \mathcal{V}^\dagger(s)\mathcal{H}(0)\mathcal{V}(s)$.

*Theorem 1.*—Consider a Lindblad superoperator $\mathcal{L}(s) = \mathcal{H}(s) + \mathcal{R}(s)$, where $\mathcal{H}(s)$ [$\mathcal{R}(s)$] denotes the Hamiltonian [decohering] component. If the Hamiltonian changes as $\mathcal{H}(s) = \mathcal{V}^\dagger(s)\mathcal{H}(0)\mathcal{V}(s)$ with $\mathcal{V}$ unitary, then a sufficient condition for a constant spectrum of $\mathcal{L}(s)$ is $\mathcal{R}(s) = \mathcal{V}^\dagger(s)\mathcal{R}(0)\mathcal{V}(s)$. If the Jordan form of $\mathcal{L}(s)$ contains just one-dimensional Jordan blocks, this is also a necessary condition. In the case of time-independent $\mathcal{R}(s)$, this simplifies to $[\mathcal{R}, \mathcal{V}(s)] = 0$ or $[\mathcal{R}, \mathcal{V}^\dagger(s)] = 0$. Under these conditions open-systems AQC with unitary interpolation is possible

*Proof.*—Sufficiency: By assumption we have $\mathcal{L}(s) = \mathcal{V}^\dagger(s)[\mathcal{H}(0) + \mathcal{V}(s)\mathcal{R}(s)\mathcal{V}^\dagger(s)]\mathcal{V}(s)$. Therefore, if $\mathcal{R}(0) = \mathcal{V}(s)\mathcal{R}(s)\mathcal{V}^\dagger(s)$ then $\mathcal{L}(s) = \mathcal{V}^\dagger(s)\mathcal{L}(0)\mathcal{V}(s)$. By inserting this equation into the right-eigenvector equation $\mathcal{L}(s)|\mathcal{D}_\alpha(s)\rangle\rangle = \gamma_\alpha(s)|\mathcal{D}_\alpha(s)\rangle\rangle$, we obtain that the eigenvalues of $\mathcal{L}(s)$ are independent from $s$. The simplification in the case of time-independent $\mathcal{R}(s)$ is immediate. Necessity: Assuming, in the eigenvector equation $\mathcal{L}(s)|\mathcal{D}_\alpha(s)\rangle\rangle = \gamma_\alpha(s)|\mathcal{D}_\alpha(s)\rangle\rangle$, that $\mathcal{L}(s)$ has a constant spectrum, we obtain $[\mathcal{H}(s) + \mathcal{R}(s)]|\mathcal{D}_\alpha(s)\rangle\rangle = \gamma_\alpha(0)|\mathcal{D}_\alpha(s)\rangle\rangle \Rightarrow \mathcal{V}(s)\mathcal{R}(s)\mathcal{V}^\dagger(s)|\tilde{\mathcal{D}}_\alpha(s)\rangle\rangle = [\gamma_\alpha(0)I - \mathcal{H}(0)]|\tilde{\mathcal{D}}_\alpha(s)\rangle\rangle$, where $|\tilde{\mathcal{D}}_\alpha(s)\rangle\rangle = \mathcal{V}(s)|\mathcal{D}_\alpha(s)\rangle\rangle$. But then, if the Jordan form of $\mathcal{L}(s)$ contains just one-dimensional Jordan blocks, we have that the set $\{|\mathcal{D}_\alpha(s)\rangle\rangle\}$ is complete and constitutes a basis in Hilbert-Schmidt space. It follows that $\mathcal{V}(s)\mathcal{R}(s)\mathcal{V}^\dagger(s) = \gamma_\alpha(0)I - \mathcal{H}(0) = \mathcal{R}(0)$.■

Theorem 1 implies that constant gaps in the spectrum of $\mathcal{L}(s)$ are *a priori nongeneric*. This places a limit on AQC with constant gap in the presence of decoherence.

*Adiabatic implementation of the Deutsch-Jozsa (DJ) algorithm under dephasing.*—Given a binary function $f$ which is promised to be either balanced or constant, the Deutsch problem is to determine which type the function is [12]. Here we construct an adiabatic implementation for the optimized version of the algorithm [13]. The input state is $|\psi(0)\rangle = |+_1\rangle \otimes \cdots \otimes |+_N\rangle$, where $|\pm_i\rangle = (|0_i\rangle \pm |1_i\rangle)/\sqrt{2}$, with $\{|0_i\rangle, |1_i\rangle\}$ being the computational basis for the $i$th qubit (eigenstates of the Pauli matrix $\sigma_z$). The initial Hamiltonian is chosen such that its ground state is $|\psi(0)\rangle$, i.e., $H(0) = \omega\sum_{i=1}^N |-_i\rangle\langle-_i|$, where $\omega$ is the energy scale. The Deutsch problem can be solved by a single computation of the function $f$ through the unitary transformation $U|x\rangle = (-1)^{f(x)}|x\rangle$ ($x \in \{0, 1\}^N$) [13], so that in the $\{|x\rangle\}$ (computational) basis $U$ is represented by the diagonal matrix $U = \mathrm{diag}[(-1)^{f(0)}, \ldots, (-1)^{f(2^N-1)}]$. An adiabatic implementation requires a final Hamiltonian $H(1)$ such that its ground state is $|\psi(1)\rangle = U|\psi(0)\rangle$. This is accomplished by a unitary transformation on $H(0)$, i.e., $H(1) = UH(0)U^\dagger$ [5]. Then the final Hamiltonian encodes the solution of the Deutsch problem in its ground state, which can be extracted via a measurement of the qubits in the basis $\{|+\rangle, |-\rangle\}$. A suitable interpolation between $H(0)$ and $H(1)$, which preserves the spectral gaps, can be defined by $H(s) = \tilde{U}(s)H(0)\tilde{U}^\dagger(s)$, where $\tilde{U}(s) = \exp(i\frac{\pi}{2}sU)$. The runtime of the closed-system version of the algorithm can be determined from the standard adiabatic theorem, yielding $T \gg \pi/2\omega$. This result is independent of $N$, as required.

We now analyze the effect of dephasing in the computational basis $\{|0\rangle, |1\rangle\}^{\otimes N}$. For simplicity, we consider the case of a single qubit, i.e., $N = 1$. Dephasing is modeled by the Lindblad operator $\Gamma = \lambda\sqrt{\omega}\sigma_z$, where $\lambda$ is a dimensionless parameter denoting the strength of the dephasing and the factor $\sqrt{\omega}$ is introduced to make the energy scale explicit. Thus, expanding the coherence vector $|\rho\rangle\rangle$ in the Pauli basis $\{I, \sigma_x, \sigma_y, \sigma_z\}$, the Lindblad superoperator for the master equation (2) is found to be

$$\mathcal{L}(s) = \omega\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -2\lambda^2 & 0 & q(s) \\ 0 & 0 & -2\lambda^2 & -r(s) \\ 0 & -q(s) & r(s) & 0 \end{pmatrix}, \qquad (4)$$

where $r(s) = -\cos\frac{\pi F}{2}s$, $q(s) = \sin\frac{\pi F}{2}s$, with $F \equiv (-1)^{f(0)} - (-1)^{f(1)}$. In our DJ implementation the Hamiltonian superoperator evolves unitarily, i.e., $\mathcal{H}(s) = \mathcal{V}^\dagger(s)\mathcal{H}(0)\mathcal{V}(s)$. Explicit evaluation of $\mathcal{V}(s)$ yields that $[\mathcal{R}, \mathcal{V}(s)] = [\mathcal{R}, \mathcal{V}^\dagger(s)] = 0$. Hence, it follows from Theorem 1 that (nongenerically) the LJ spectral gaps are constant in this example. Interestingly, this property is not restricted to dephasing in this example, but holds also, e.g., for spontaneous emission, where $\Gamma \propto \sigma_-$. Indeed, explicit evaluation of the eigenvalues of $\mathcal{L}(s)$ shows that they are independent from $s$ and given by $\gamma_1 = 0$, $\gamma_2 = -2\omega\lambda^2$, $\gamma_3 = \omega(-\lambda^2 - \sqrt{\lambda^4 - 1})$, and $\gamma_4 = \omega(-\lambda^2 + \sqrt{\lambda^4 - 1})$. These eigenvalues are nondegenerate for $0 < \lambda < 1$ and define four one-dimensional Jordan blocks for the Lindblad superoperator, denoted by $J_\alpha$ ($\alpha \in \{1, \ldots, 4\}$) (thus the condition in Theorem 1 is both necessary and sufficient). Expanding the coherence vector as $|\rho(s)\rangle\rangle = \sum_{\beta=1}^4 p_\beta(s)e^{T\gamma_\beta s}|\mathcal{D}_\beta(s)\rangle\rangle$, where the $|\mathcal{D}_\beta(s)\rangle\rangle$ ($\gamma_\beta$) denote the eigenstates (eigenvalues) of $\mathcal{L}(s)$, and using it in the master Eq. (1), we can show that the block $J_1$ is already decoupled from the others. Therefore, adiabaticity is re-

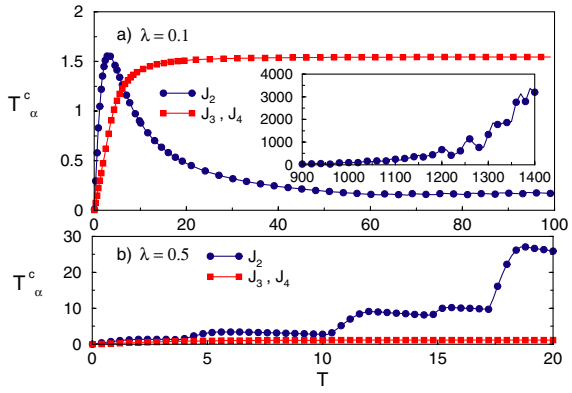


FIG. 1 (color online). $T^c_\alpha$ as a function of $T$ for the Jordan blocks $J_2$, $J_3$, and $J_4$. The inset of (a) shows that $J_2$ has non-negligible couplings for $T \sim 10^3$ (in units such that $\omega = 1$).

lated here to the decoupling of the remaining three Jordan blocks.

Next we compute the crossover times for decoupling of all the Jordan blocks, so as to test for the adiabatic time interval, as defined by the condition $T \gg \max_\alpha\{T^c_\alpha\}$. We work in units such that $\omega = 1$. As anticipated above, one important result is a finite time interval for adiabaticity. This is illustrated in Fig. 1, where we plot the crossover time $T^c_\alpha$ as a function of the evolution time $T$ for two values of $\lambda$. Observe that $T^c_3, T^c_4$ asymptotically approach a constant value, implying the decoupling of blocks $J_3, J_4$ for sufficiently slow evolutions (large $T$) since the condition $T \gg \max_\alpha\{T^c_\alpha\}$ is satisfied. On the other hand, the block $J_2$ can only decouple from the others during a finite interval [see inset of Fig. 1(a)]. While the adiabatic interval $T \gg T^c_2$ is large for $\lambda = 0.1$, it decays rapidly as the dephasing parameter $\lambda$ increases, as shown in Fig. 1(b).

In order to understand the algorithm's performance we still need to analyze the adiabatic solution of the Lindblad equation. Let us select $T$ such that, for given $\lambda$, adiabaticity is a good approximation, i.e., we can disregard the Jordan block couplings. Then, with $\rho(0) = (I + \sigma_x)/2$, the straightforward solution of the Lindblad equation yields $\rho(1) = [I + e^{-2\lambda^2 T}(-1)^{f(0)+f(1)}\sigma_x]/2$. The probabilities $p_\pm$ of finding the system in one of the final states $\{|+\rangle, |-\rangle\}$ are then $p_\pm = [1 \pm e^{-2\lambda^2 T}(-1)^{f(0)+f(1)}]/2$. In the closed-system case ($\lambda = 0$) whether $f$ is constant or balanced is determined, respectively, by $p_+ = 1$ or $p_- = 1$. In the open-system case, for each given value of $\lambda$, we can determine an optimal runtime $T$, provided we impose a certain success probability for the algorithm. For instance, take $\lambda = 0.1$. Then, imposing a certainty of 90% (either $p_+ = 0.9$ or $p_- = 0.9$), we find $T \approx 11$. This result is compatible with the adiabatic interval for $\lambda = 0.1$, where the condition $T \gg \max_\alpha\{T^c_\alpha\}$ for $T = 11$ is relatively well satisfied, with $T^c_2 \approx 0.82$ and $T^c_3 = T^c_4 \approx 1.43$. Therefore, for this dephasing scale, the algorithm has a high probability of success. In order to generalize the results to many qubits, we consider $N$ independent dephasing operators $\Gamma_i = \lambda_i\sqrt{\omega}\sigma^z_i$ acting individually on each qubit. For $N = 2$, it is easy to show that Theorem 1 applies. We conjecture the validity of this result for any $N$ due to the simple diagonal form of $\mathcal{R}(s)$. A multiqubit analysis can then be implemented similarly as done before. For $N$ qubits, the success probability may still be improved by repeating the algorithm execution several times.

*Conclusions.*—We introduced and illustrated a general methodology to analyze the performance of AQC in open quantum systems described by arbitrary convolutionless master equations. We have shown (Theorem 1) that a closed-systems unitary interpolation (constant gap) scheme translates into an open-system constant gap scheme only under specific, nongeneric assumptions. The limited robustness of AQC in an open-systems setting suggests that the development of quantum error correction methods tailored to AQC is an important direction of future research.

We gratefully acknowledge financial support from CNPq-Brazil (to M. S. S.) and the Sloan Foundation (to D. A. L.).

[1] E. Farhi *et al.*, Science **292**, 472 (2001).
[2] W. M. Kaminsky, S. Lloyd, and T. P. Orlando, quant-ph/0403090; M. Grajcar, A. Izmalkov, and E. Il'ichev, Phys. Rev. B **71**, 144501 (2005).
[3] M. Steffen *et al.*, Phys. Rev. Lett. **90**, 067903 (2003).
[4] D. Aharonov *et al.*, quant-ph/0405098.
[5] M. S. Siu, Phys. Rev. A **71**, 062314 (2005).
[6] A. M. Childs, E. Farhi, and J. Preskill, Phys. Rev. A **65**, 012322 (2002).
[7] N. Shenvi, K. R. Brown, and K. B. Whaley, Phys. Rev. A **68**, 052313 (2003); J. Roland and N. J. Cerf, Phys. Rev. A **71**, 032330 (2005); J. Aberg, D. Kult, and E. Sjoqvist, Phys. Rev. A **71**, 060312(R) (2005).
[8] M. S. Sarandy and D. A. Lidar, Phys. Rev. A **71**, 012331 (2005).
[9] R. Alicki and K. Lendi, *Quantum Dynamical Semigroups and Applications* in Lecture Notes in Physics Vol. 286 (Springer-Verlag, Berlin, 1987).
[10] W. van Dam, M. Mosca, and U. Vazirani, *Proceedings of 42nd IEEE Symposium on Foundations of Computational Science* (IEEE, New York, 2001), p. 279; V. N. Smelyanskiy, S. Knysh, and R. D. Morris, Phys. Rev. E **70**, 036702 (2004).
[11] E. Farhi, J. Goldstone, and S. Gutmann, quant-ph/0201031; A. Boulatov and V. N. Smelyanskiy, Phys. Rev. A **68**, 062321 (2003).
[12] D. Deutsch and R. Jozsa, Proc. R. Soc. A **439**, 553 (1992).
[13] D. Collins, K. W. Kim, and W. C. Holton, Phys. Rev. A **58**, R1633 (1998).