

# One-Spin Quantum Logic Gates from Exchange Interactions and a Global Magnetic Field

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It has been widely assumed that one-qubit gates in spin-based quantum computers suffer from severe technical difficulties. We show that one-qubit gates can, in fact, be generated using only modest and presently feasible technological requirements. Our solution uses only global magnetic fields and controllable Heisenberg exchange interactions, thus circumventing the need for single-spin addressing.

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The Heisenberg exchange interaction features prominently in several of the promising solid state proposals for quantum computation (QC) [1–7]. For this reason, extensive studies have been made of the problem of constructing a universal set of quantum logic gates [8] that employs this interaction. The difficulty is that as long as each qubit is represented by a single-spin it appears necessary to supplement the Heisenberg interaction with single-spin addressing, thus requiring either strongly localized magnetic fields [9] or  $g$ -factor engineering [10], both of which are technically highly demanding. This problem can be alleviated by encoding a logical qubit into the state of at least three spins, in which case the Heisenberg interaction by itself can be shown to be universal [11–14], or by using a combination of Heisenberg interactions with inhomogeneous, delocalized magnetic fields and an encoding into at least two spins [15–17]. Another option is to supplement the Heisenberg interaction with certain two-spin measurements [18]. These alternatives to single-spin addressing each come with space and time overhead [11–17] or a probabilistic protocol that needs to be repeated several times until it converges [18].

Here we show that, in fact, it is not necessary to introduce any encoding overhead when using the Heisenberg interaction in conjunction with inhomogeneous, delocalized, time-dependent magnetic fields. Furthermore, unlike previous proposals we do not require spin-resonance techniques [16] (which involve the difficult demand of modulating the interaction strength at high frequency) and do not need the recoupling assumption [15] that the exchange coupling can be made much greater than the difference in Zeeman energies. These considerations overall amount to a significant simplification compared to various aspects of previously proposed methods [11–17]. Instead, one can generate the single-spin gates needed for universality by pulsing the Heisenberg interaction together with global magnetic fields, under relaxed controllability assumptions.

*Interactions.*—Let us now define the basic interactions involved in our method. We consider  $N$  spins interact-

ing via a controllable Heisenberg exchange interaction  $J_{ij}(t)\mathbf{S}_i \cdot \mathbf{S}_j$  that generates a unitary evolution

$$U_{ij}(\xi_{ij}) = \exp(-i\xi_{ij}\mathbf{S}_i \cdot \mathbf{S}_j). \quad (1)$$

Here  $\xi_{ij} = \frac{1}{\hbar} \int_{\tau}^{\tau'} J_{ij}(t)dt$  is a fully controllable rotation angle,  $\mathbf{S}_i = (S_i^x, S_i^y, S_i^z)$  and  $S_i^\alpha = \frac{1}{2}\sigma_i^\alpha$  ( $\alpha = x, y, z$ ) are the spin-1/2 matrices acting on qubit  $i$ . In addition, we assume the presence of a *global* (i.e., delocalized) magnetic field that can be switched on/off fast on the time scale of spin decoherence. Recent theoretical estimates [19] of dephasing times for electron spins in GaAs quantum dots are on the order of 50  $\mu$ s, while experimental data [20] for isotopically pure  $^{28}\text{Si:P}$  at 7 K indicates  $T_2 \gtrsim 60$  ms. We assume that the field is either spatially inhomogeneous, or there is an inherent  $g$ -factor inhomogeneity, or both. In either case, when the magnetic field is switched on it generates the unitary transformation

$$V^\alpha(\theta_i, \theta_j) = \prod_{k=1}^N e^{-i\theta_k S_k^\alpha}, \quad (2)$$

via the Zeeman effect, where  $\theta_k = (g_k \mu_B / 2\hbar) \times \int_{\tau}^{\tau'} B_k^\alpha(t)dt$ . We have explicitly introduced the indices  $i, j$  in Eq. (2) to denote the pair of spins that is coupled by the Heisenberg interaction in order to generate a controllable rotation of spin  $i$  [see Fig. 1(a)]. Other notation that we use is  $\mu_B = 9.27 \times 10^{-24} \text{ JT}^{-1}$  for the Bohr magneton and  $B_i^\alpha$  for the value of the global magnetic field  $\alpha$  component at spin  $i$ . We emphasize that the only way in which we assume the Zeeman angles  $\theta_i$  to be controllable is through the time dependence of the magnetic field, *which is the same for all spins*:  $B_i(t) = f(t)B_i$ , where  $f(t)$  does not depend on the spin index  $i$  and  $B_i$  does not depend on time. The inhomogeneity assumption means that  $\theta_i \neq \pm\theta_{i+1}$ .

*Circuits.*—With these definitions we are ready to present our analytically derived pulse sequence that generates arbitrary single-spin operations. It is given as a quantum circuit in Fig. 1(a). It is simple to verify the validity of the circuit by direct matrix multiplication. The

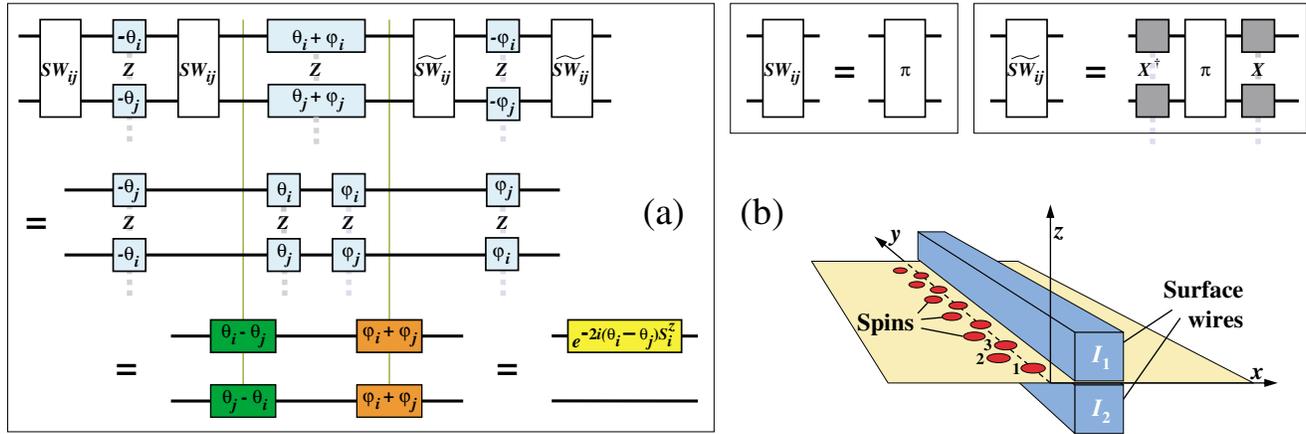


FIG. 1 (color online). (a) Circuit for a one-spin gate. Time proceeds from left to right. The Heisenberg interaction couples spins  $i$  and  $j$ , carried by the long horizontal lines. Large rectangular boxes with  $\pi$ , also denoted by  $SW_{ij}$  for “swap,” represent  $U_{ij}(\pi)$ . The action of  $SW_{ij}$  is also a swap, as explained in the text. Small boxes with angles  $\theta_i$ ,  $\theta_j$  or  $\phi_i$ ,  $\phi_j$  inside denote a global magnetic field inducing evolution according to Eq. (2), and small boxes with  $-\theta_i$ ,  $-\theta_j$  (or  $-\phi_i$ ,  $-\phi_j$ ) denote  $V^\alpha(\theta_i, \theta_j)^\dagger$  [or  $V^\alpha(\phi_i, \phi_j)^\dagger$ ]. Consider an arbitrary single spin rotation  $\exp(-2i\theta S_i^z)$ , where the rotation angle  $\theta \equiv \theta_i - \theta_j$  is constructed from global rectangular pulses of amplitude  $A$  and duration  $T$ . Neighboring spins experience different amplitudes:  $A_i = Aa_i$ , where the  $a_i$  are device constants. Typically,  $A$  will be limited by physical considerations, as in the device of (b), so that  $T$  controls the pulse area. Thus,  $T = \theta/A(a_i - a_j)$ , and the rotation angles appearing in the circuit are given by  $-\theta_i = -Aa_iT$ . The parameters  $\phi_i$  and  $\phi_j$  are defined through  $\theta_i - \theta_j = \phi_i + \phi_j$ , so that  $\phi_k = Aa_kT'$ , where  $T' = (a_i - a_j)/(a_i + a_j)T$ . The magnetic field direction  $\alpha$  for each gate is indicated by  $X$  or  $Z$  between the boxes. Dark boxes with  $\alpha = X$  or  $Z$  between them denote  $V^\alpha(\theta, \theta + \pi)$  with  $\theta$  arbitrary, while dark boxes with  $X^\dagger$  or  $Z^\dagger$  denote  $V^\alpha(\theta, \theta + \pi)^\dagger$ . The circuit yields the gate  $e^{-2i(\theta_i - \theta_j)S_i^z} \otimes I_j$  ( $I$  is the identity operation). Vertical light dotted lines serve as a reminder that the global magnetic field acts on all spins, not just spins  $i$  and  $j$ . However, our pulse sequence is constructed in such a way that the action on all other spins cancels out at the end of the sequence. This follows from its symmetry and the fact that the Heisenberg interaction couples only spins  $i$  and  $j$ . The circuit uses eleven elementary steps, four of which are Heisenberg interactions and seven of which are global magnetic field pulses. (b) Zigzag spin arrangement, with symmetrically positioned wires. The two current orientations,  $I_1 = I_2$  and  $I_1 = -I_2$ , enable the complete control of the global field gradients in the  $x$  and  $z$  directions.

circuit of Fig. 1(a) shows that an arbitrary one-spin  $S^z$  operation can be generated in terms of controllable Heisenberg interactions and a global time-dependent magnetic field in 11 elementary steps. Below, we show how to implement the required magnetic field using a simple spin architecture, with two nearby wires [see Fig. 1(b)]. We can also generate an arbitrary  $S^x$  operation, simply by switching the current direction in one of the wires. Since these operations generate  $SU(2)$  on an arbitrary spin, and the Heisenberg interaction can generate entanglement, we have a universal set of quantum logic gates. An explicit circuit for generating a controlled-phase (CP) gate can be given as follows:

$$e^{-i\pi S_i^z S_j^z} = U_{ij}(\pi/2)V^z(\theta, \theta + \pi)^\dagger U_{ij}(\pi/2)V^z(\theta, \theta + \pi),$$

which requires four elementary steps and is equivalent to  $CP = \text{diag}(1, 1, 1, -1)$  up to single-qubit corrections (a circuit that generates CP up to an overall phase takes eleven steps). This procedure is an application of the familiar refocusing technique of NMR (see, e.g., [8]), wherein the terms  $S_i^x S_j^x + S_i^y S_j^y$  that are also part of the Heisenberg interaction are canceled.

Intuitively, the role of the Heisenberg pulses in Fig. 1(a) is to *swap* the states of the spins they act on:

$U_{ij}(\pi)e^{-i(\theta_i S_i^z + \theta_j S_j^z)}U_{ij}(-\pi) = e^{-i(\theta_i S_i^z + \theta_j S_j^z)}$ . For this reason we denote  $U_{ij}(\pi)$  by  $SW_{ij}$  in Fig. 1(a). The operator represented by  $\widetilde{SW}_{ij} = V^x(\theta, \theta + \pi)^\dagger U_{ij}(\pi)V^x(\theta, \theta + \pi)$  in Fig. 1(a) also acts as a kind of swap operator, in that it is easy to show that it transforms  $\widetilde{SW}_{ij}e^{-i(\theta_i S_i^z + \theta_j S_j^z)}\widetilde{SW}_{ij} = ie^{i(\theta_i S_i^z + \theta_j S_j^z)}$ . These swaps are the fundamental operations that allow one to address single spins in the presence of a global magnetic field with a gradient, since they allow a spin to be exposed to different values of the field without needing to localize the field to a particular spin site. More specifically, the swaps allow us to modify the relative phase acquired by the two target spins under the action of the global magnetic field. By appropriately modifying the signs of these phases, as shown in Fig. 1(a), we can completely cancel the evolution of one spin while generating a desired unitary transformation on the other spin. In spirit, the method is again similar to refocusing.

*Parallelism.*—Our scheme is parallelizable in the following sense: *the same gate operation can be applied to all pairs of spins that are not Heisenberg coupled*. This follows simply by virtue of our assumed ability to simultaneously switch the Heisenberg interaction in different locations, coupled with the global coupling to the magnetic field. The limitation to applying the same operation

imposes some constraints in terms of the parallelism required for fault-tolerant quantum error correction; how the fault tolerance threshold [21] is affected under such conditions is an issue we will consider in a future publication. Note, however, that an identical gate operation applied to all qubits is a common feature of quantum algorithms; e.g., a global Hadamard transform is used to create a superposition over all computational basis states starting from the initial state  $|00\cdots 0\rangle$ . Our analysis shows that such a global Hadamard transform would take only eight elementary steps for appropriate qubit geometries.

*Other exchange models.*—Our results are not restricted to the Heisenberg interaction. Consider, e.g., the XY model (which describes a range of QC proposals, e.g., quantum dots in microwave cavities [22]), where the exchange Hamiltonian takes the form  $\sum_{i<j} J_{ij}(S_i^x S_j^x + S_i^y S_j^y)$ . The circuit for generating  $e^{-i\theta_i S_i^z}$  is identical to the one shown in Fig. 1(a), with the Heisenberg Hamiltonian replaced by the XY one. One has  $e^{i2\theta_i S_i^z} = V^x(\theta_i, \theta_j)^\dagger \times e^{i\pi S_i^z} V^x(\theta_i, \theta_j) e^{i\pi S_i^z}$  and similarly for  $S_j^z$ . The following steps will then generate a controlled-phase gate:

$$e^{i(\pi/2)(S_i^z - S_j^z)} e^{-iJ(S_i^x S_j^x + S_i^y S_j^y)} e^{i\pi S_i^z} \times \\ e^{-iJ(S_i^x S_j^x + S_i^y S_j^y)} e^{i\pi S_j^z} e^{-i(\pi/2)(S_i^z - S_j^z)} = e^{-i2tJZ_1 Z_2}. \quad (3)$$

The corresponding circuit takes 32 steps and can certainly be optimized. Another important exchange model is the Heisenberg interaction with time-independent spin-orbit corrections included. The “dressed qubits” method of Ref. [23] is then compatible with the circuits of Fig. 1(a).

*Implementation.*—We now address the technical feasibility of our method. Besides the (conventional) requirement of precise pulse timings, it may appear that the most serious technical challenge in our scheme (as can be seen from Fig. 1) is that we need a very large magnetic field gradient (or  $g$ -factor difference) because of the  $\pi$  difference in terms such as  $V^x(\theta, \theta + \pi)$ . Assuming for simplicity a rectangular  $B$ -field pulse of duration  $\Delta T$ , this condition translates into  $(B_{i+1} - B_i)\Delta T = 18$  mT ns, where we took a uniform  $g$  factor of 2. To achieve a typical gate time [6] of  $\Delta T \simeq 10$  ns, we require a magnetic field increment of  $B_{i+1} - B_i = 1.8$  mT. A uniform field gradient of this magnitude, extending across a scalable device of  $\sim 10^4$  spins, is not physically reasonable: eddy current heating will occur as the large field is switched. Instead, we now suggest a simple scheme to implement the needed gradients, using on-chip patterned wires. Such microfabricated conducting wires are extensively used in the atom optics and Bose-Einstein condensation physics communities, where the technology required to manufacture them has reached maturity [24].

Two superconducting wires are positioned near an array of spins, as shown in Fig. 1(b). The spins are arranged in a zigzag pattern, with half the spins located at  $x = 0$  and the other half at  $x = -100$  nm (this and

other numbers used below should be taken as suggestive estimates; our scheme is general and does not depend on these specific values). Note that for clarity we show only a single row of spins. Our method is fully compatible with a 2D architecture, where the zigzag pattern is repeated parallel to the first row. Inspection of Fig. 1(a) shows that if the exchange interaction is never switched on then the global magnetic field pulses exactly cancel. Thus, to address a specific row of spins one simply does not turn on the exchange interaction in any other row (additional wires may have to be placed between rows of spins as the number of rows increases, due to the  $1/r$  falloff of the field strength). A tunable exchange coupling is established between consecutive spins in the row to be addressed. The wires of cross section  $200 \times 200$  nm are centered on either side of the spin plane at  $x = 200$  nm and  $z = \pm 100$  nm, with a thin insulating layer in between. To avoid resistive heating, the wires should be operated below their superconducting critical currents. For nanopatterned aluminum wires, we can expect [25] critical current densities of order  $2.2 \times 10^{10}$  A/m<sup>2</sup>. Thus, our wires can safely carry currents of size  $I_{1,2} = 0.7$  mA. Magnetic fields arising from such currents range well below the superconducting critical fields [25].

We consider two particular current configurations: (a)  $I_1 = I_2 = 0.7$  mA and (b)  $I_1 = -I_2 = 0.7$  mA. Because of symmetry, we find that  $B^x = 0$  in case (a), with a discrete magnetic field gradient of  $|B_{i+1}^z - B_i^z| = 0.28$  mT. For case (b), we obtain  $B^z = 0$  and  $|B_{i+1}^x - B_i^x| = 0.28$  mT. Maximum pulse durations are therefore given by  $\Delta T \simeq 64$  ns. The two current configurations above yield arbitrary single-spin  $S^x$  and  $S^z$  operations. Observing that we need at most  $3 \times 7 = 21$  magnetic field pulses to realize an arbitrary single-spin operation (using the Euler angles method [8]), we deduce a total gate time of about  $1.3 \mu\text{s}$ . Considering that we have not attempted serious optimization and have used limitations imposed by current experiments, this number compares quite favorably with the fault tolerance requirement [21] of  $10^{-3}$ – $10^{-5}$  times the recently estimated decoherence time for impurity-bound spins,  $T_2 \gtrsim 60$  ns [20]. However, we note that since our scheme is somewhat limited in the ability to parallelize single-spin operations (the angles  $\theta_i, \theta_j$  of two spins  $i, j$  involved in an exchange interaction in a given row cannot be independently controlled) and since the exchange interaction enables effective coupling of nearest-neighbor spins only, the effective threshold applicable to our case is likely to be lower [26] than the current optimal estimates [21].

We can also use fault tolerance limits to back out engineering specifications for the wire fabrication. Small inaccuracies in wire positions will result in systematic rotation errors. In this case, fault tolerance suggests an *amplitude* accuracy [27] of about  $\sqrt{10^{-4}}$  for single-qubit rotations, composed of at most 21 magnetic field pulses. In the worst case scenario, pulse errors accumulate coherently, and we will require an accuracy of

$10^{-2}/21$  per pulse. For the wiring scheme of Fig. 1(b), this accuracy level corresponds to fabrication precision of about 1 Å or roughly one atomic monolayer of aluminum. This should be considered a lower bound; in practice, after optimization and considering a more realistic error model, we expect the required fabrication precision to be on the order of several atomic monolayers, which is technologically feasible.

We now show that other potential technical obstacles with the proposed architecture can be overcome. We have estimated the eddy current heating of electrons in the two-dimensional electron gas, due to the rapid switching of magnetic fields of 0.6 mT. We find that only extremely short switching times,  $\Delta T < 2$  ps, are capable of generating enough heat to disturb low-temperature quantum dot experiments [28]. Further, our target switching times of tens of nanoseconds cannot excite the low-temperature superconducting gaps of aluminum or niobium. Considering a conventional gating scheme [6], we estimate that transient voltage fluctuations in the gate electrodes caused by field switching are 7–9 orders of magnitude weaker than normal gate pulses. Additionally, we estimate the  $B$  fields produced by transient currents during the pulsed charging of gate electrodes. For typical pulses of 0.1 V, with 0.1 ns rise times, we obtain  $B$  field fluctuations 3 orders of magnitude smaller than those from the wires and extremely brief compared to gate times.

Finally, we note there already exists an extensive literature on the important issues of spin readout [29,30] and accurate control of the exchange coupling [1–7] needed to implement our scheme.

*Conclusions.*—We have shown how to construct a universal set of quantum logic gates using the Heisenberg exchange interaction between nearest-neighbor spins and global, inhomogeneous magnetic fields. Our scheme bears some similarity to the global addressing schemes developed in the context of quantum cellular automata [31], a connection we plan to explore in a future publication. Optimization of our pulse sequences appears feasible and will be undertaken in a future study. However, our estimates indicate that even before optimization the method we have proposed is feasible using current technology and is compatible with the time-scale requirements of fault-tolerant quantum error correction. We believe that this alternative to previous methods for QC in exchange-Hamiltonian systems offers a significant simplification and a drastic reduction in design constraints. We thus hope to have brought QC with spins a step closer to experimental feasibility.

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