Quantum Tensor Product Structures are Observable Induced

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It is argued that the partition of a quantum system into subsystems is dictated by the set of operationally accessible interactions and measurements. The emergence of a multipartite tensor product structure of the state space and the associated notion of quantum entanglement are then relative and observable induced. We develop a general algebraic framework aimed to formalize this concept.

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Suppose one is given a four-state quantum system. How does one decide whether such a system supports entanglement or not? In other words, should the given Hilbert space ($\mathbb{C}^4$) be viewed as bipartite ($\cong \mathbb{C}^2 \otimes \mathbb{C}^2$) or irreducible? In the former case, there exists a tensor product structure (TPS) that supports two entangleable qubits. In this case, one finds a sharp dichotomy between the quantum and classical realms, as perhaps most dramatically exemplified in quantum information processing [1]. In the irreducible case there is no entanglement and, hence, none of the advantages associated with efficient quantum information processing [2].

Here we propose that a partitioning of a given Hilbert space is induced by the experimentally accessible observables (interactions and measurements) (see also Refs. [3–5]). Thus, it is meaningless to refer to a state such as the Bell state $|\Phi^+\rangle = (|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle)/\sqrt{2}$ as entangled [6], without specifying the manner in which one can manipulate and probe its constituent physical degrees of freedom. In this sense entanglement is always relative to a particular set of experimental capabilities. Before introducing a formalization, let us illustrate these ideas with a simple example.

Example 0: Bell basis.—Let $|x\rangle \otimes |y\rangle \equiv |x, y\rangle$ $(x, y \in \{0, 1\})$ be the standard product basis for a two-qubit system. Each qubit forms a subsystem. With respect to (wrt) this bipartition, the Bell-basis states $|\Phi^\pm\rangle = (|00\rangle \pm |11\rangle)/\sqrt{2}$ and $|\Psi^\pm\rangle = (|01\rangle \pm |10\rangle)/\sqrt{2}$ are maximally entangled. Now note that these can be rewritten as $|\chi^\lambda\rangle := |\chi\rangle \otimes |\lambda\rangle$, where $\chi = \Phi, \Psi$ and $\lambda = +, -$. With respect to this new bipartition the Bell states are by definition product states, and the subsystems are the χ and λ degrees of freedom. On the other hand, some separable superpositions of the states $|x, y\rangle$ are now entangled and can be used for entanglement-based quantum information protocols such as teleportation [1]. This striking difference can be highlighted by considering the SWAP operator S, which is nonentangling in the usual x, y bipartition, but, in the χ, λ bipartition, one has $S|\chi, \lambda\rangle = (-1)^{\chi\lambda}|\chi, \lambda\rangle$. Thus, S realizes a controlled phase shift over $|11\rangle := |\Psi^-\rangle$, and in the new decomposition SWAP is a maximally entangling operator. Which then is the correct characterization of the TPS and the associated entanglement? The answer depends on the set of accessible interactions and measurements. In stating that the Bell states are entangled, one is implicitly assuming that there is experimental access to (local) observables of the form $\{\sigma^\alpha \otimes \mathbb{1}\}$ and $\{\mathbb{1} \otimes \sigma^\beta\}$ (where $\alpha, \beta \in \{x, y, z\}$ and σ are the Pauli matrices). But this assumption may not always be justified. For example, in quantum dot quantum computing proposals utilizing electron spins [7], it is more convenient to manipulate exchange interactions than to control single spins [8,9]. In such cases the accessible interactions may be nonlocal, and this is precisely the situation that favors the χ, λ bipartition that then acquires the same operational status as the standard x, y one.

General framework.—We now lay down a conceptual framework aimed to capture in its generality and relativity the notion of "induced tensoriality" of subsystems. Our definitions are observable based and mostly involve algebraic objects [10]. Let us consider a quantum system with finite-dimensional state space $\mathcal{H}$, a subspace $C \subseteq \mathcal{H}$, and a collection $\{\mathcal{A}_i\}_{i=1}^n$ of subalgebras of $\mathrm{End}(C)$ satisfying the following three axioms: (i) Local accessibility: Each $\mathcal{A}_i$ corresponds to a set of controllable observables. (ii) Subsystem independence: $[\mathcal{A}_i, \mathcal{A}_j] = 0$ $(\forall\, i \neq j)$. (iii) Completeness: $\vee_{i=1}^n \mathcal{A}_i \cong \otimes_{i=1}^n \mathcal{A}_i \cong \mathrm{End}(C)$.

Notice that the standard case of N qudits (d-level systems) $C = \mathcal{H} = (\mathbb{C}^d)^{\otimes N}$ is the case $\mathcal{A}_i \cong M_d$ $\forall\, i$ acting as the identity over all factors (subsystems) but the ith one. Now we discuss the physical meaning of the axioms (i)–(iii).

Axiom (i) simply defines the basic algebraic objects at our disposal. These objects are controllable observables (Hamiltonians with tunable parameters and measurements). Axiom (ii) addresses separability. In order to claim that a system is composite it must be possible to perform operations manipulating a well-defined set of degrees of freedom while leaving all the others unaffected. Typically this is achieved by having individually addressable, spatially separated subsystems i (e.g., a single excess electron per quantum dot [7]), but as we shall see this is certainly not the only possibility. Axiom (iii) is the crucial one in order to ensure that our observable-based definition of multipartiteness induces a corresponding one at the state-space level. Its meaning will follow from Proposition 1 below: all the operations not affecting the state of a subsystem (its symmetries) are realized by operators corresponding to nontrivial operations only over the degrees of freedom of the other subsystems. All symmetries are then physical operations, and no superselection rules [11] are present when a suitable state space C is chosen. When C is a proper subspace of $\mathcal{H}$, we are dealing with an "encoding," a notion that has proved useful, e.g., in quantum error correction and avoidance [12,14,15] and encoded universality [8,9]. Generalizing Ref. [3] we have the following central result.

Proposition 1.—A set of subalgebras $\mathcal{A}_i$ satisfying Axioms (i)–(iii) induces a TPS $C = \otimes_{i=1}^{n}\mathcal{H}_i$. We call such a multipartition an induced TPS.

The proof is given in Ref. [16].

Example 1.—Assume that one is given the following set of independently controllable two-body interactions $\{\sigma^y \otimes \sigma^z, \sigma^z \otimes \sigma^z, \sigma^x \otimes \sigma^y, \sigma^x \otimes \sigma^x\}$. These interactions generate the following subalgebras: $\mathcal{A}_\chi := \{\mathbb{1}, \sigma^x \otimes \mathbb{1}, \sigma^y \otimes \sigma^z, \sigma^z \otimes \sigma^z\}$, $\mathcal{A}_\lambda := \{\mathbb{1}, \mathbb{1} \otimes \sigma^z, \sigma^x \otimes \sigma^y, \sigma^x \otimes \sigma^x\}$. These satisfy Axioms (i)–(iii) (with $C = \mathbb{C}^4$) and act, respectively, as local identity and Pauli x, y, z matrices on the χ and degrees of freedom considered above. Thus, by Proposition 1, $\mathcal{A}_\chi$ and $\mathcal{A}_\lambda$ induce a TPS $\mathbb{C}^4 \cong \mathbb{C}^2 \otimes \mathbb{C}^2$, namely, the χ, λ bipartition.

Superselection.—An important example for which one is led to consider nonstandard TPSs is a system exhibiting *superselection* rules [11]. There the only allowed physical operations correspond to operators commuting with a set of superselection charges $\{Q_l\}_{l=1}^{M}$, e.g., particle numbers, which generate an Abelian algebra $\mathcal{Q}$. Denoting by $\Pi_{\mathcal{Q}}$ the projector over the commutant of $\mathcal{Q}$, the physically realizable subsystem operations are $\Pi_{\mathcal{Q}}(\mathcal{A}_i)$ $(i = 1, \ldots, n)$. These projected algebras typically either (a) define a new invariant subspace C' with a new induced TPS or (b) do not satisfy axioms (ii), (iii) anymore and therefore fail to induce a proper TPS. The associated notion of entanglement and entanglement-based protocols then must be reconsidered [11].

Irreducible representations.—A prototypical way for obtaining an encoded bipartite TPS is to consider the decomposition of $\mathcal{H}$ into irreducible representations (irreps) of a $*$-subalgebra $\mathcal{A}$ [3]. In that case,

$$\mathcal{H} \cong \oplus_J \mathbb{C}^{n_J} \otimes \mathcal{H}_J, \tag{1}$$

where the $\mathcal{H}_J$ are the d_J-dimensional irreps of $\mathcal{A}$ and n_J their multiplicities. The algebra (commutant) can then be written as $\mathcal{A} \cong \oplus_J \mathbb{1}_{n_J} \otimes M_{d_J}$ $(\mathcal{A}' \cong \oplus_J M_{n_J} \otimes \mathbb{1}_{d_J}$ [15]. Upon restriction to a particular J sector, one has $\mathcal{A} \vee \mathcal{A}' \cong M_{n_J} \otimes M_{d_J} \cong \mathcal{A} \otimes \mathcal{A}'$. Then, according to Proposition 1, $\mathcal{A}$ and $\mathcal{A}'$ induce an (encoded) bipartite TPS in each irreducible block.

Example 2: Encoded tensoriality.—As an example of the above construction, let $\mathcal{H}_N := (\mathbb{C}^2)^{\otimes N}$ denote an N-qubit space, $\mathcal{A}_1$ the algebra of totally symmetric operators in End$(\mathcal{H}_N)$, and $\mathcal{A}_2$ the algebra of permutations exchanging the qubits. $\mathcal{A}_1$ is generated by the collective spin operators, i.e., $\mathcal{A}_1 = \mathbb{C}\{S^\alpha := \sum_{i=1}^{N} \sigma_i^\alpha | \alpha = x, y, z\}$, and $\mathcal{A}_2 = \mathcal{A}_1'$ is generated by Heisenberg exchange interactions, $\mathcal{A}_2 = \mathbb{C}\{\sigma_i \cdot \sigma_j\}$ $[\sigma = (\sigma^x, \sigma^y, \sigma^z)]$. In the context of decoherence-free subspaces and subsystems [14,15], $\mathcal{A}_1$ is the algebra of error operators (system-bath interactions), and $\mathcal{A}_2$ is the algebra of allowed quantum computational operations. Here our perspective is quite different: we view both as algebras of accessible interactions that induce a TPS. This is, in fact, an encoded TPS, since one has (for even N) the Hilbert space decomposition (1) with $J = 0, \ldots, N/2$, $\mathcal{H}_J = \mathbb{C}^{d_J}$, $d_J = 2J + 1$, and $n_J(N) = (2J+1)N!/[(N/2+J+1)!(N/2-J)!]$. Each summand in Eq. (1) is a code subspace with a bipartite TPS. We stress the unusual feature of this example: the two "qudits" (i.e., subsystems) composing the TPS need not have the same dimension (though they do for $J = N/2 - 1$), and are manipulable by interactions of a physically distinct nature. The left (right) qudit is manipulated by tuning only Heisenberg exchange couplings (global magnetic fields). This example, therefore, has implications for spin-based quantum computation [7], where single-spin addressing is technically very demanding.

Nested subalgebra chains.—The commutant construction illustrated above provides a general way to realize an encoded bipartite TPS. In order to obtain encoded TPSs with more than two subsystems, we consider a nested chain of subalgebras:

$$\mathcal{B}_0 \supset \mathcal{B}_1 \supset \cdots \supset \mathcal{B}_n. \tag{2}$$

We assume that $\mathcal{B}_0$ acts irreducibly over $\mathcal{H}$. Then $\mathcal{H}$ typically will be reducible wrt $\mathcal{B}_{i\geq 1}$. In particular, wrt $\mathcal{B}_2$: $\mathbb{C}^{d_{J_1}} \cong \oplus_{J_2}\mathbb{C}^{n_{J_2}} \otimes \mathbb{C}^{d_{J_2}}$ and $\mathcal{B}_2 \cong \oplus_{J_1,J_2} \mathbb{1}_{n_{J_1}} \otimes \mathbb{1}_{n_{J_2}} \otimes M_{d_{J_2}}$. By iterating over the subalgebra chain one obtains

$$\mathcal{H} \cong \oplus_{J_1,\ldots,J_n} \otimes_{k=1}^{n} \mathbb{C}^{n_{J_k}} \otimes \mathbb{C}^{d_{J_n}}. \tag{3}$$

This is a sum over code subspaces $H(J_1, \ldots, J_n) := \otimes_{k=1}^{n}\mathbb{C}^{n_{J_k}} \otimes \mathbb{C}^{d_{J_n}}$ with a multipartite TPS. The nontrivial ones are those for which at least one $n_{J_k} > 1$. Note that while $\mathcal{B}_2$ has nontrivial action only on $\mathbb{C}^{d_{J_2}}$, $\mathcal{B}_1$ has nontrivial action on $\mathbb{C}^{d_{J_1}} \supset \mathbb{C}^{d_{J_2}}$. So how does one operate on a particular subsystem (qudit), say, $\mathbb{C}^{n_{J_k}}$? We come to our second main result.

Proposition 2.—Given a nested subalgebra chain as in Eqs. (2) and (3), the subsystem algebras are given by

$$\mathcal{A}_i = \mathcal{B}_i' \cap \mathcal{B}_{i-1} \qquad (i = 1, \ldots, n). \tag{4}$$

Conversely, when a set of subsystem algebras $\{\mathcal{A}_i\}_{i=1}^{n}$ is given, the nested chain $\mathcal{B}_i := \vee_{k=i+1}^{n}\mathcal{A}_k$ $(i = 1, \ldots, n)$ results.

The proof is given in Ref. [17].

Example 3: Standard TPS.—The standard qubit-TPS over $\mathcal{H}_N$ corresponds to the chain $\mathcal{B}_i = \mathbb{1}^{2^i} \otimes M_{2^{n-i}}$ ($i = 1, \ldots, n$). In this case all the subalgebras are factors, where one has a single $\mathcal{H}(J_1, \ldots, J_n)$ term in Eq. (3), with multiplicities $n_{J_i} = 2$ and dimensions $d_{J_i} = 2^{n-i}$.

Example 4: Stabilizer codes.—Consider N qubits and the following chain of nested algebras: $\mathcal{B}_0$ acts irreducibly on $(\mathbb{C}^2)^{\otimes N}$, $\mathcal{B}_1$ acts trivially on the first qubit but irreducibly on the rest, etc. To realize such a chain let $\{X_1, \ldots, X_k\}$ be a set of N-qubit, mutually commuting operators, and let $\mathcal{B}_i = [\mathbb{C}\{X_1, \ldots, X_i\}]'$ ($i = 1, \ldots, k$). Further assume that the X_i are unitary, traceless, and square to the identity. Then the corresponding Hilbert space decomposition is $\mathcal{H} \cong (\mathbb{C}^2)^{\otimes i} \otimes (\mathbb{C}^2)^{\otimes(n-i)}$, where the first i $\mathbb{C}^2$ factors correspond to the 2^i possible eigenvalues of $X_1, \ldots, X_i \in \mathcal{B}_i'$. When the X_i's are generators of an Abelian subgroup of the Pauli group, one recovers the stabilizer codes of quantum error correction [12].

Example 5: Multipartite encoded TPS.—Let us revisit Example 2 and show how a multipartite encoded TPS is induced. Consider $N = n2^K$ qubits, and the chain $\mathcal{B}_0 := \mathrm{End}\mathcal{H}_N$, $\mathcal{B}_i := \mathbb{C}(S_{N/2^{i-1}})^{\times 2^{i-1}}$, $i = 1, \ldots, K$, where S denotes the symmetric group. Conceptually, we have 2^K blocks of n qubits each, and the subalgebra chain corresponds to operating on these blocks with increasing levels of resolution. By Proposition 2 we should find a $K+1$-partite encoded TPS. To see this, recall that the state space $\mathcal{H}_N \cong (\mathbb{C}^2)^{\otimes N}$ of N qubits splits wrt S_N exactly as in the su(2) case (Example 2) except that by the duality between S_N and su(2), the role of n_J and d_J is interchanged, while J remains an su(2) irrep label. For example, for $N = 6$ ($K = 1$ and $n = 3$) we have $\mathcal{H}_6 \cong \oplus_{J=0}^{3} \mathbb{C}^{\tilde{n}_J} \otimes \mathbb{C}^{\tilde{d}_J} \cong H_0 \otimes \mathbb{C}^5 \oplus H_1 \otimes \mathbb{C}^9 \oplus H_2 \otimes \mathbb{C}^5 \oplus H_3 \otimes \mathbb{C}$, where now $\tilde{n}_J = 2J + 1, \tilde{d}_J = n_J(6)$, and $H_J := \mathbb{C}^{2J+1}$, $J = 0, 1, 2, 3$. The chain then consists of $\mathcal{B}_0 = \mathrm{End}(\mathcal{H}_6)$, $\mathcal{B}_1 = \mathbb{C}S_6$, and $\mathcal{B}_2 = \mathbb{C}(S_3 \times S_3)$, i.e., exchanges between the first three $\times$ second three qubits. From Proposition 2 this algebra chain defines the encoded TPSs with algebra subsystems given by $\mathcal{A}_1 := \mathcal{B}_1' =$ totally symmetric operators (recall Example 2) and $\mathcal{A}_2 = \mathcal{B}_1' \cap \mathcal{B}_0$, where $\mathcal{B}_1'$ are block-symmetric operators, so that $\mathcal{A}_1 =$ linear combination of permutations, symmetrized wrt $S_3 \times S_3$, e.g., elements of the form $\boldsymbol{\sigma}_{2+3i} \cdot \boldsymbol{\sigma}_{3+3i} + \boldsymbol{\sigma}_{3+3i} \cdot \boldsymbol{\sigma}_{1+3i}$ $(i = 0, 1)$. Decomposing the $\mathbb{C}^{\tilde{d}_J}$ factors wrt $S_3 \times S_3$ we find, e.g., for the $H_1 \otimes \mathbb{C}^9$ term that it describes a qubit times a qutrit [13].

Returning to the case of K blocks, one can see how an encoded multipartite TPS will emerge. For example, with $n = 3$ and $K = 2$ we have the chain $\mathcal{B}_0 = \mathbb{C}S_{12} \supset \mathcal{B}_1 := \mathbb{C}(S_6 \times S_6) \supset \mathcal{B}_2 := \mathbb{C}(S_3 \times S_3 \times S_3 \times S_3)$. By comparing the decompositions of $\mathcal{H}_{12}$ wrt $\mathcal{B}_1$ and $\mathcal{B}_2$, one can identify the tripartite encoded TPS.

Example 6: Tripartite hybrid TPS.—Let us exhibit an unusual example of a TPS, where each factor is of a different physical nature. We consider $\mathcal{H} := (\mathbb{C}^2)^{\otimes 4}$ and $\mathcal{B}_1 = \mathbb{1} \otimes \mathrm{End}(\mathbb{C}^2)^{\otimes 3}$ (full operator space over the last three qubits), $\mathcal{B}_2 = \mathbb{1} \otimes \mathbb{C}S_3$ (permutations exchanging the last three qubits). $\mathcal{B}_1$ is a factor, and one obtains the decomposition $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^8$. The three-qubit space splits wrt S_3 as $\mathbb{C}^4 \otimes \mathbb{C} \oplus \mathbb{C}^2 \otimes \mathbb{C}^2$. It follows that $(\mathbb{C})^{\otimes 4} \cong \mathbb{C}^2 \otimes \mathbb{C}^4 \otimes \mathbb{C} \oplus \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. The last term corresponds to a tripartite system in which the first subsystem is a "standard" qubit, the second is acted upon by collective interactions over the last three "physical" qubits, while the third is acted upon by the algebra of permutations of S_3. Interestingly, this hybrid tripartite system has already been realized experimentally in the context of noiseless subsystems [18].

TPS morphing.—So far we have emphasized kinematics. Next we show that an induced TPS can change dynamically, depending on the algebras of available interactions. Let $\{\mathcal{A}_i\}_{i=1}^{n}$ and $\{\tilde{\mathcal{A}}_i\}_{i=1}^{\tilde{n}}$ define two TPSs over $\mathcal{H}$. Suppose one has the following Hamiltonian:

$$H(\lambda, \mu) = \sum_{i=1}^{n} \sum_{\alpha} \lambda_i^{\alpha} H_{\alpha}^{i} + \sum_{i=1}^{\tilde{n}} \sum_{\beta} \mu_i^{\beta} \tilde{H}_{\beta}^{i}, \qquad (5)$$

where $H_{\alpha}^{i} \in \mathcal{A}_i$, $\tilde{H}_{\beta}^{i} \in \tilde{\mathcal{A}}_i$ ($i = 1, \ldots, n$), and all coupling constants λ_i, μ_i are independently tunable. By setting all the μ_i (λ_i) to zero the first (second) TPS is induced. Therefore, dynamical control of the Hamiltonian allows one to switch among different induced multipartitions, possibly with a different number of subsystems, in a sort of continuous fashion. We call this "TPS morphing." For example, consider three qubits with a controllable Hamiltonian given by $H(\lambda(t), \mu(t)) = \sum_{i,j=1}^{3} \lambda_1^{ij} \sigma_i \cdot \sigma_j + \sum_{\alpha=x,y,z} \lambda_2^{\alpha} S^{\alpha} + \sum_{i=1}^{3} \sum_{\beta=x,y,z} \mu_i^{\beta} \sigma_i^{\beta}$, where $S^{\alpha} = \sum_{i=1}^{3} \sigma_i^{\alpha}$ ($\alpha = x, y, z$). The first two terms induce the (encoded) bipartite TPS described in Example 2, whereas the last term induces the standard tripartite structure.

Stroboscopic entanglement.—A TPS can even be switched on and off under appropriate circumstances. Suppose that the algebra of available interactions does not induce a TPS [e.g., since it is $\cong \mathrm{End}(\mathcal{H})$]. Now suppose that one can turn on an additional interaction that allows one to refocus (see, e.g., [9]) some of these interactions, so that the remaining interactions do induce a TPS. Then at the end of each refocusing period a TPS appears. We call this "stroboscopic entanglement." For instance, and referring back to Example 1, suppose that the controllable Hamiltonian is given by $H = \sum_{X \in \mathcal{A}_{\chi}, Y \in \mathcal{A}_{\lambda}} J_X X + J_Y Y$, where the two-body terms are always on and the one-body terms are controllable. This H mixes the subalgebras $\mathcal{A}_{\chi}$ and $\mathcal{A}_{\lambda}$, so that there is no TPS as long as the two-body terms are present. However, a series of π pulses in terms of $\sigma^x \otimes \mathbb{1}$ ($\mathbb{1} \otimes \sigma^z$) will refocus, i.e., turn off, the two-body terms in the $\mathcal{A}_{\chi}$ ($\mathcal{A}_{\lambda}$) term, thus decoupling the two subalgebras at the end of each refocusing period. In this manner, the χ and λ factors can be separately manipulated; i.e., the TPS has reappeared.

Conclusions.—We have shown that the TPS of quantum mechanics acquires physical meaning relative only to the given set of available interactions and measurements. These induce a TPS through their algebraic structure. The induced TPS may contain factors (qudits) of a different physical nature, and can be dynamical.

A few concluding comments are in order. First, note that while we have given criteria for the appearance of an induced TPS and the associated entanglement, we have deliberately not addressed the issue of efficiency in quantum information processing (QIP) [1], in particular, in relation to the question of resource cost. Indeed, it is simple to construct a set of subalgebras satisfying axioms (i)–(iii), thus inducing a TPS for a "structureless" Hilbert space such as energy levels of a Rydberg atom, while the associated cost of performing a quantum computation scales exponentially in some resource such as spectroscopic resolution [2]. Second, and again in the context of QIP, in order to exploit a given induced TPS for performing quantum computation, one has to be able to implement, along with the local operations $\mathcal{A}_i$, at least one entangling transformation $\mathcal{E}$ in $\mathrm{End}(C) \cong \otimes_i \mathcal{A}_i$. The new set $\{\{\mathcal{A}_i\}, \mathcal{E}\}$, in the prototypical situation of interest in QIP, will be (encoded) universal, i.e., will allow any transformation in $\mathrm{End}(C)$ to be generated by composition of elementary operations involving $\{\{\mathcal{A}_i\}, \mathcal{E}\}$. This will allow access to other TPSs than the original, induced one [e.g., in the case of Example 0 one could argue that access to both the standard and the χ, λ bipartitions is available once all SU(4) transformations can be generated]. The key point is that there is a hierarchy of TPSs: the "natural" one is the one that is induced by the directly accessible observables $\mathcal{A}_i$. The "lower-level" ones are those that are visible only by composition of the elementary observables $\{\{\mathcal{A}_i\}, \mathcal{E}\}$. Third, it is important to emphasize that both interactions and measurements are involved in inducing a TPS, and must be compatible, i.e., induce the same TPS, for this TPS to be both manipulable and observable.

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[10] We summarize algebraic definitions. $\mathrm{End}(\mathcal{H})$ denotes the full operator algebra over the Hilbert state space $\mathcal{H}$ and M_d the algebra of $d \times d$ complex matrices $[\mathrm{End}(\mathbb{C}^d) \cong M_d]$. By (sub)algebra we mean a unital $*$-(sub)algebra. Given a subalgebra $\mathcal{A} \subset \mathrm{End}(\mathcal{H})$ we define its *commutant* as $\mathcal{A}' := \{X | [X, \mathcal{A}] = 0\}$. Given an operator group (set) $\mathcal{G}$, $\mathbb{C}\mathcal{G}$ indicates its group algebra (the algebra generated by $\mathcal{G}$). $A \vee B$ denotes the minimal subalgebra of $\mathrm{End}(\mathcal{H})$ that contains both A and B. If $[A, B] = 0$, then $A \vee B = \mathrm{Span}\{ab | a \in A, b \in B\}$.
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[16] Consider the subalgebras $\mathcal{A}_1^{(i)} := \mathcal{A}_i$ and $\mathcal{A}_2^{(i)} := \vee_{j=i+1}^{n} \mathcal{A}_j$ $(i = 1, \ldots, n-1)$. Let $i = 1$. From (iii) we have $\mathcal{A}_1^{(1)} \vee \mathcal{A}_2^{(1)} \cong \mathcal{A}_1^{(1)} \otimes \mathcal{A}_2^{(1)} \cong \mathrm{End}(C)$. Consider the map ι: $\mathcal{A}_1^{(1)} \otimes \mathcal{A}_2^{(1)} \rightarrow \mathcal{A}_1^{(1)} \vee \mathcal{A}_2^{(1)}: x \otimes y \rightarrow xy$. This is clearly a surjective (onto) algebra homomorphism. In order to have $\mathcal{A}_1^{(1)} \vee \mathcal{A}_2^{(1)} \cong \mathcal{A}_1^{(1)} \otimes \mathcal{A}_2^{(1)}$ it is necessary that $\mathrm{Ker}(\iota) = 0$. In turn, this implies (A) $\mathcal{A}_1^{(1)} \cap \mathcal{A}_2^{(1)} = \mathbb{C}\mathbb{1}$. Indeed, if one had a nontrivial $x \in \mathcal{A}_1^{(1)} \cap \mathcal{A}_2^{(1)}$ then the element $x \otimes \mathbb{1} - \mathbb{1} \otimes x$ would be in $\mathrm{Ker}(\iota)$. Now we consider the irrep decomposition of $\mathcal{A}_1^{(1)}$, i.e., $\mathcal{A}_1^{(1)} \cong \oplus_J \mathbb{1}_{n_J} \otimes M_{d_J}$; since $[\mathcal{A}_2^{(1)}, \mathcal{A}_1^{(1)}]$, $\mathcal{A}_2^{(1)} \in [\mathcal{A}_1^{(1)}]' \cong \oplus_J M_{n_J} \otimes \mathbb{1}_{d_J}$. Then $\mathcal{A}_1^{(1)} \cap \mathcal{A}_2^{(1)} \subset \oplus_J \mathbb{1}_{n_J} \otimes \mathbb{1}_{d_J}$. The only decomposition compatible with (A) is when just one irrep appears, i.e., $\mathcal{A}_1^{(1)} \cong M_{d_J} \otimes \mathbb{1}_{n_J}$. Since $\mathcal{A}_1^{(1)} \vee \mathcal{A}_1^{(2)} \cong \mathrm{End}(\mathcal{H}) \cong M_{n_J d_J}$, one finds that $\mathcal{A}_1^{(1)}$ must be isomorphic to M_{n_J}. In summary, $\exists \mathcal{H}_1^{(i)} \cong \mathbb{C}^{d_J}$, $\mathcal{H}_2^{(i)} \cong \mathbb{C}^{n_J}$ such that $C \cong \otimes_{l=1}^{2} \mathcal{H}_l^{(1)}$ and $\mathcal{A}_l^{(1)} \cong \mathrm{End}(\mathcal{H}_l^{(1)})$ $(l = 1, 2)$. This argument can be iterated over i for $\mathcal{A}_1^{(i)}$ and $\mathcal{A}_2^{(i)}$. The role of C is played by the second state factor obtained for $i-1$. Eventually, one finds $C \cong \otimes_{i=1}^{n} \mathcal{H}_l^{(i)}$, with $\mathcal{A}_i \cong \mathrm{End}(\mathcal{H}_1^{(i)})$.
[17] Upon restriction to a particular code subspace we have $\mathcal{B}_{i-1} \cong \otimes_{k=1}^{i-1} \mathbb{1}_{n_{J_k}} \otimes M_{d_{J_{i-1}}}$ and $\mathcal{B}_i' \cong \otimes_{k=1}^{i} M_{n_{J_k}} \otimes \mathbb{1}_{d_{J_i}}$. Then for $i < n$, $\mathcal{B}_i' \cap \mathcal{B}_{i-1} = \otimes_{k=1}^{i-1} \mathbb{1}_{n_{J_k}} \otimes M_{n_{J_i}} \otimes_{k=i+1}^{n} \mathbb{1}_{n_{J_k}} \otimes \mathbb{1}_{J_{d_n}}$, i.e., has nontrivial action only on the ith subsystem. For $i = n$ we have $\mathcal{B}_n' \cap \mathcal{B}_{n-1} = \otimes_{k=1}^{n} \mathbb{1}_{n_{J_k}} \otimes M_{d_{J_n}}$. For the converse direction, the formula $\mathcal{B}_i := \vee_{k=i+1}^{n} \mathcal{A}_k$ is immediate.
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