

## Adiabatic Preparation of Topological Order

Alioscia Hamma<sup>1</sup> and Daniel A. Lidar<sup>1,2</sup>

<sup>1</sup>*Department of Chemistry, University of Southern California, Los Angeles, California 90089, USA*

<sup>2</sup>*Departments of Electrical Engineering and Physics, University of Southern California, Los Angeles, California 90089, USA*  
(Received 12 October 2007; published 24 January 2008)

Topological order characterizes those phases of matter that defy a description in terms of symmetry and cannot be distinguished in terms of local order parameters. Here we show that a system of  $n$  spins forming a lattice on a Riemann surface can undergo a second order quantum phase transition between a spin-polarized phase and a string-net condensed phase. This is an example of a quantum phase transition between magnetic and topological order. We furthermore show how to prepare the topologically ordered phase through adiabatic evolution in a time that is upper bounded by  $O(\sqrt{n})$ . This provides a physically plausible method for constructing and initializing a topological quantum memory.

DOI: [10.1103/PhysRevLett.100.030502](https://doi.org/10.1103/PhysRevLett.100.030502)

PACS numbers: 03.67.Lx, 05.50.+q, 73.43.Nq

The notion of topological order [1] can explain phase transitions that depart from the standard Landau theory of symmetry breaking, long range correlations, and local order parameters. It plays a key role in condensed matter theory of strongly correlated electrons; e.g., it provides a means to understand the different phases arising in the fractional quantum Hall effect [2]. These phases have exactly the same symmetries. No local order parameter can distinguish them. The internal order that characterizes these phases is topological and can be characterized in terms of a nonlocal order parameter, e.g., the expectation value of string operators (operators formed by tensor products of local operators taken along a string).

Another arena in which topological order has found profound applications is quantum computation, in particular, in the context of systems exhibiting natural fault tolerance [3,4]. Central to this application is the ability to prepare certain topologically ordered states, which are the ground states of a Hamiltonian describing a spin system occupying lattice links on a Riemann surface of genus  $g$ . In a topologically ordered phase the ground state degeneracy depends on  $g$ . Because any two orthogonal states in the ground state manifold  $\mathcal{L}$  are coupled only by loop operators with loops that wind around the holes of the Riemann surface, this encoding is robust against any local perturbation. Such a system constitutes a very robust memory register [3].

Here we address the problem of preparation of topologically ordered states via adiabatic evolution. Our motivation for studying this problem is at least threefold. First, while quantum phase transitions (QPTs) between deconfined and confined phases have been studied extensively in quantum chromodynamics [5], this has not been the case in condensed matter physics. Here we study a second order QPT between a magnetically ordered phase (deconfined, topologically disordered) and a string-net condensed phase [6] (confined, topologically ordered). Second, we are motivated by the aforementioned need for topologically ordered states as the input to topological quantum computers. In Ref. [7] it was shown how to prepare a

ground state of the toric code via  $O(\sqrt{n})$  repeated syndrome measurements, where  $n \sim L^2$  is the number of spins on a lattice of linear dimension  $L$ . Such ground state preparation is an essential step in constructing a topologically fault tolerant quantum memory. However, preparation via syndrome measurements has certain drawbacks, most notably that the measurements are assumed to be as fast as logic gates and must be fast compared to the decoherence time scale, assumptions that are likely to be hard to satisfy in practice. A system in which the phase transition predicted here can potentially be realized experimentally, as well as used for topological quantum memory, is a Josephson junction array [8]. Third, we believe that the methods presented here will also find applications in the field of adiabatic quantum computing [9], for the preparation time in our adiabatic method is  $O(\sqrt{n})$  (as in the syndrome measurement method [7]), which is optimal in the sense that it saturates the Lieb-Robinson bound [10].

*The toric code model.*—Consider an (irregular) lattice on a Riemann surface of genus  $g$ . At every link of the lattice we place a spin  $1/2$ . If  $n$  is the number of links, the Hilbert space of such a system is  $\mathcal{H} = \mathcal{H}_1^{\otimes n}$ , where  $\mathcal{H}_1 = \text{Span}\{|0\rangle, |1\rangle\}$  is the Hilbert space of a single spin. In the usual computational basis we define a reference basis vector  $|0\rangle \equiv |0\rangle_1 \otimes \dots \otimes |0\rangle_n$ , i.e., all spins up. This string-free state is the vacuum state for strings. We define the Abelian group  $X$  as the group of all spin flips on  $\mathcal{H}$ . Its elements can be represented in terms of the standard Pauli matrices as  $x_\alpha = (\sigma_1^x)^{\alpha_1} \otimes \dots \otimes (\sigma_n^x)^{\alpha_n}$ , where  $\alpha_j \in \{0, 1\}$  and  $\alpha = \{\alpha_j\}_{j=1}^n$ . Every  $x_\alpha \in X$  squares to identity  $\mathbb{1}$ ,  $\dim \mathcal{H} = |X| = 2^n$ , and any computational basis vector can be written as  $|\alpha\rangle = x_\alpha |0\rangle$ .

Now we associate a geometric interpretation with the elements of  $X$ , as in Fig. 1.

To every string  $\gamma$  connecting any two vertices of the lattice we can associate the string operator  $x_\gamma$  operating with  $\sigma^x$  on all the spins covered by  $\gamma$ . The product of two string operators is a string-net operator:  $x_\gamma \equiv x_{\gamma_1} x_{\gamma_2}$ , and we can view  $X$  as the group of string-net operators on the

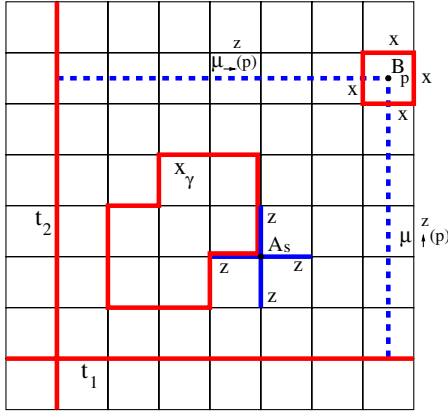


FIG. 1 (color online). The square lattice on a torus. Closed  $x$  strings [solid (red) lines] commute with the star operator  $A_s$  because they have an even number of links in common. The elementary closed  $x$  string is the plaquette  $B_p$ . Dashed  $z$  strings connect plaquettes  $p$ . The two incontractible  $x$  strings are denoted  $t_{1,2}$ . The dual string variable [dashed (blue) lines] connects the reference lines  $t_1, t_2$  with the vertices  $p$  in the dual lattice. Thus the dual operator  $\mu_{z-(p)}^z$  anticommutes with the plaquette variable  $\mu^x(p) \equiv B_p$ .

lattice. A particularly interesting subgroup of  $X$  is formed by the products of all closed strings; we denote this closed string-nets group [1] by  $\bar{X}$ . If the Riemann surface has genus  $g$ , we can also draw  $2g$  incontractible strings around the holes in the surface:  $T = \langle t_1, \dots, t_{2g} \rangle$ ; see Fig. 1. The group of contractible closed strings is denoted by  $\mathcal{B}$  and is generated by the elementary closed strings:  $B_p = \prod_{j \in \partial p} \sigma_j^x$ , where  $j$  labels all the spins located on the boundary  $\partial p$  of a plaquette  $p$ . A star operator is defined as acting with  $\sigma^z$  on all the spins coming out of the vertex  $s$ :  $A_s = \prod_{j \in s} \sigma_j^z$ . Then  $\bar{X}$  splits into  $2g$  topological sectors labeled by the  $t_i$ :  $\bar{X} = T \cdot \mathcal{B}$  [11].

*Topological order in the toric code model.*—Consider the Hamiltonian  $H(g, U) = -g \sum_p B_p - U \sum_s A_s$ . For  $g = U = 1$  this is the Hamiltonian introduced by Kitaev [3], and we refer to it as the Kitaev Hamiltonian. It is an example of a  $Z_2$ -spin liquid, a model that features string condensation in the ground state. The operators  $A_s, B_p$  all commute, so the model is exactly solvable, and the ground state manifold is given by  $\mathcal{L} = \{|\phi\rangle \in \mathcal{H} | A_s |\phi\rangle = B_p |\phi\rangle = |\phi\rangle\}$ . The vector in the trivial topological sector of  $\mathcal{L}$  is explicitly given by [11]

$$|\phi_0\rangle = |\mathcal{B}|^{-(1/2)} \sum_{x \in \mathcal{B}} x|0\rangle. \quad (1)$$

On a torus the ground state manifold  $\mathcal{L}$  is fourfold degenerate. A vector in  $\mathcal{L}$  can be written as the superposition of the four orthogonal ground states in different topological sectors  $|\phi_k\rangle = t_1^i t_2^j |\phi_0\rangle$ ,  $i, j \in \{0, 1\}$ ,  $k = i + 2j$ . On an arbitrary lattice (regular or irregular) on a generic Riemann surface of genus  $g$  there are  $2g$  such operators  $t_i$  and thus

the degeneracy of the ground state manifold is, in general,  $2^{2g}$ . This is a manifestation of topological order [1].

*A quantum phase transition.*—Consider the Hamiltonian obtained by applying a magnetic field in the  $z$  direction to all the spins:

$$H_\xi = -\xi \sum_{j=1}^n \sigma_j^z + \xi n I. \quad (2)$$

The zero-energy ground state of this Hamiltonian is the magnetically ordered, spin-polarized string-vacuum state  $|0\rangle$ . In the presence of  $H_\xi$  a string state  $x_\gamma|0\rangle$ , with  $x_\gamma = \prod_{j \in \gamma} \sigma_j^x \neq \mathbb{1}$ , pays an excitation energy that depends only on the string length  $l_\gamma$ , namely  $\langle 0 | x_\gamma H_\xi x_\gamma | 0 \rangle = 2\xi l_\gamma$ . Thus  $H_\xi$  is a tension term. If we add the star term  $H_U = -U \sum_s A_s$  to  $H_\xi$ , the ground state is still  $|0\rangle$  [its energy is  $E_g(U) = -U n_s$ ], but the spectrum is quite different. Now there is a drastic difference between open and closed  $x$ -type strings. Closed strings commute with all the  $A_s$  and hence only pay the tension, whereas open strings also pay an energy of  $2U$  for every spin that is flipped due to anticommutation ( $-U \langle 0 | \sigma_j^x A_s \sigma_j^x | 0 \rangle = +U$ ). So every open string pays a total energy of  $4U$ . In the  $U \gg \xi$  limit, open strings are energetically forbidden. On the other hand, the plaquette operators  $B_p$  deform  $x$  strings: the plaquette term  $H_g = -g \sum_p B_p$  acts as a kinetic energy for the (closed) strings and induces them to fluctuate. In light of these considerations the total model  $H = H_\xi + H(g, U)$  has two different phases. The first phase is the spin-polarized phase described above, when  $U \gg \xi \gg g \sim 0$ . Here the string fluctuations are energetically suppressed by the tension, so the ground state is  $|0\rangle$ . The other phase arises when  $U \gg g \gg \xi \sim 0$ . Now the tension is too weak to prevent the closed loops from fluctuating strongly. Nevertheless, open  $x$ -type strings cost the large energy  $4U$ , so are forbidden in the ground state. The ground state consists of the superposition with equal amplitude of all possible closed strings,  $\mathcal{L}$ . This is the string-net condensed phase. As the tension decreases, and the fluctuations increase, the ground state becomes a superposition of an increasingly larger number of closed strings. Open strings cannot be present in the ground state because their energy is too large. In the thermodynamic limit, for some critical value of the ratio between tension and fluctuations  $\xi/g$ , the gap between the ground state and the first excited state closes and the system undergoes a second order QPT [12]. Notice that the gap between the ground and the first excited states is given only by the interplay between tension and fluctuations, and is unaffected by  $H_U$ . The second phase we have described cannot be characterized by a local order parameter, such as magnetization: we have phases without symmetry breaking. This is an example of topological order. The low-energy sector has energy much smaller than  $U$ , and it thus comprises closed strings:  $\mathcal{L}_{\text{low}} = \text{span}\{x|0\rangle, x \in \bar{X}\}$ . This subspace also splits into four sectors labeled by the elements of  $T$ :  $\mathcal{L}_{\text{low}}^{ij} = \text{span}\{x t_1^i t_2^j | 0\rangle, x \in \mathcal{B}; i, j = 0, 1\}$ . Thus  $\dim \mathcal{L}_{\text{low}} = \dim(\mathcal{H}) / (2^{n/2+1})$ .

Within each sector, low level excitations have an energy  $E_{\text{low}} \leq g$  and become gapless only at the critical point in the thermodynamic limit.

*Adiabatic evolution.*—We now show how to prepare the topologically ordered state  $|\phi_0\rangle$  through adiabatic evolution. The idea is to adiabatically interpolate between an initial Hamiltonian  $H(0)$  whose ground state is easily preparable, and a final Hamiltonian  $H(1)$  whose ground state is the desired one. The interpolation between two such Hamiltonians has been used as a paradigm for adiabatic quantum computation [9]. This process must be accomplished such that the error  $\delta$  between the actual final state and the desired (ideal adiabatic) ground state of  $H(1)$ , is as small as possible. The problem is that real and virtual excitations can mix excited states with the desired final state, thus lowering the fidelity [13]. Rigorous criteria are known [14,15], and used below, which improve on the first-order approximation typically encountered in quantum mechanics textbooks. For smooth interpolations and one relevant excited state, it has been proven that an arbitrarily small error  $\delta$  is obtained when the evolution time  $T$  obeys  $T = O(1/\Delta E_{\text{min}})$ , where  $\Delta E_{\text{min}}$  is the minimum energy gap encountered along the adiabatic evolution [16].

Consider the following time-dependent Hamiltonian:

$$H(\tau) = H_U + [1 - f(\tau)]H_\xi + f(\tau)H_g, \quad (3)$$

where  $U \gg g$ ,  $\xi \sim 1$ , and  $f: \tau \in [0, 1] \mapsto [0, 1]$  such that  $f(0) = 0$  and  $f(1) = 1$ ,  $\tau = t/T$  is a dimensionless time. At  $\tau = 0$  the Hamiltonian is  $H(0) = H_U + H_\xi$  whose ground state is the spin-polarized phase  $|0\rangle$ . At  $\tau = 1$  the Hamiltonian is the Kitaev Hamiltonian  $H(1) = H_U + H_g$  whose ground state is in  $\mathcal{L}$ , the string-net condensed phase. The tension strength is given by  $\xi[1 - f(\tau)]$ , while the fluctuations have strength  $gf(\tau)$ . Adiabatic evolution from the initial state  $|0\rangle$  will cause the fluctuations to prepare the desired ground state  $|\phi_0\rangle$ . Notice that because of the interplay between the tension term  $H_\xi$  and the large term  $H_U$ , at every  $\tau$  the ground state is in  $\mathcal{L}_{\text{low}}^{00}$ . As the tension decreases, the gap between the ground state and states in the other sectors  $\mathcal{L}_{\text{low}}^{ij}$  ( $i + j > 0$ ) diminishes. Beyond the critical point, the gap scales down exponentially with the number of spins. Nevertheless, this does not cause transitions: irrespective of how small the gap is, the Hamiltonian does not couple between different sectors. Indeed, the adiabatic theorem states that the probability of a transition between the instantaneous eigenstate  $|\psi_j(\tau)\rangle$  the system occupies and another eigenstate  $|\psi_i(\tau)\rangle$  [with respective energies  $E_i(\tau)$ ,  $E_j(\tau)$ ] is given by  $p_{ij} \leq |\langle \psi_i(\tau) | \dot{H} | \psi_j(\tau) \rangle / [E_i(\tau) - E_j(\tau)]|^2$ . In our case,  $\langle \psi_i(\tau) | \dot{H} | \psi_j(\tau) \rangle \equiv 0$  whenever  $|\psi_i(\tau)\rangle$ ,  $|\psi_j(\tau)\rangle$  belong to two different sectors [17]. Thus transitions between different  $\mathcal{L}_{\text{low}}$  sectors are completely forbidden, and the adiabatic evolution keeps the ground state in the sector it starts in. The situation is different in the presence of an arbitrary  $k$ -local ( $k = 1, 2, 3$ ) perturbation  $V$ : time evolution can then, in principle, couple the different sectors, and this

requires a careful analysis. Before topological order is established, different sectors remain gapped because of the tension, and the adiabatic criterion applies. Once in the topologically ordered phase, the exponentially small gap requires us to apply time-dependent perturbation theory to compute the probability of tunneling between different sectors. It turns out that only the  $L/k$ th order in the Dyson series is nonvanishing, and the result is that as long as  $V < \xi$ , these transitions are suppressed exponentially in  $L/k$  at arbitrary times [17]. Namely, the probability of transition between two different sectors  $a$  and  $b$  is

$$W_{b \leftarrow a}(\tau) \leq L[V^2/(U\sqrt{U\xi})]^L. \quad (4)$$

This result is confirmed by numerical analysis [18]: topological order protects the adiabatic evolution from tunneling to other sectors. Note that, in contrast, preparing the ground state by cooling would result in an arbitrary superposition in the ground state manifold. Therefore we are concerned with the gap to the excited states within  $\mathcal{L}_{\text{low}}^{00}$ . We next show that this gap scales as  $\Delta E(n; \tau_c) \sim n^{-1/2}$  at the critical point  $\tau_c$ . To do this, we must understand the lattice gauge theory that represents the low-energy sector of the model.

*Lattice gauge theory and mapping to the Ising model.*—We briefly review the lattice gauge theory invented by Wegner [19], showing that the low-energy sector ( $E \ll U$ ) of Kitaev's model maps onto this theory. The lattice gauge Hamiltonian is given by

$$H_{\text{gauge}} = -\lambda_1 \sum_{s,\hat{k}} \sigma^z(s, \hat{k}) - \lambda_2 \sum_p B_p. \quad (5)$$

A local gauge transformation consists in flipping the phases of all the spins originating from  $s$ , so it can be defined as  $A_s = \prod_{j \in s} \sigma_j^z$ . It is simple to check that  $H_{\text{gauge}}$  is invariant under this local gauge transformation. In a gauge theory the Hilbert space is restricted to the gauge-invariant states. Thus the Hilbert space for this model is given by  $\mathcal{H}_{\text{gauge}} = \{|\psi\rangle \in \mathcal{H} | A_s |\psi\rangle = |\psi\rangle\} \subset \mathcal{H}$ , the Hilbert space spanned by all possible spin configurations. The Hamiltonian (5) clearly resembles our  $H(\tau)$ , apart from the absence of the star term  $H_U$ . Consequently,  $\mathcal{H}_{\text{gauge}}$  is just the low-energy sector  $E \ll U$  of  $H(\tau)$ . In the limit  $U \rightarrow \infty$  the Hamiltonian  $H(\tau)$  maps to  $H_{\text{gauge}}$  with  $\lambda_2/\lambda_1 = gf(\tau)/\xi[1 - f(\tau)]$ . For a critical value of  $\lambda_1/\lambda_2 \sim 0.43$  [18,20], the system undergoes a second order QPT.

The lattice gauge theory is dual to the  $(2+1)$ -dimensional Ising model [21]. To show this, we define a duality mapping, as follows. With every plaquette of the original lattice we associate a vertex  $p$  on the new lattice, and we define the first dual variable  $\mu^x(p) \equiv B_p$ . In order to obtain the correct mapping, the second dual variable must realize the Pauli algebra with  $\mu^x(p)$ . Consider now the string  $\gamma_{-(\uparrow)}(p)$  from the reference line  $t_2(t_1)$  to the vertex  $p$  on the new lattice; see Fig. 1. With this string we associate the operator that consists of  $\sigma^z$

on every link intersected by  $\gamma_{\rightarrow(\uparrow)}(p)$  and denote it by  $\mu_{\rightarrow(\uparrow)}^z(p)$ . This is the second dual variable. These variables realize the Pauli algebra:  $(\mu^x(p))^2 = (\mu^z(p))^2 = 1$  and  $\{\mu^x(p'), \mu^z(p)\}_+ = \delta_{p'p}$ . In order to write  $H_{\text{gauge}}$  in terms of the dual variables, we note that  $\mu_{\uparrow}^z(p)\mu_{\uparrow}^z(p - \hat{y}) = \sigma^z(s, \hat{x})$  and  $\mu_{\rightarrow}^z(p)\mu_{\rightarrow}^z(p - \hat{x}) = \sigma^z(s, \hat{y})$ . Then the mapping  $(\sigma^x, \sigma^z) \mapsto (\mu^x, \mu^z)$  yields

$$H_g \mapsto H_{\text{Ising}} = -\lambda_2 \sum_p \mu^x(p) - \lambda_1 \sum_{p,i} \mu^z(p) \mu^z(p+i), \quad (6)$$

where  $\mu^z$  is of the  $\rightarrow, \uparrow$  type if  $i = \hat{x}, \hat{y}$ , respectively. We recognize this Hamiltonian as the quantum Hamiltonian for the  $(2+1)$ -dimensional Ising model [21]. This model has two phases with a well understood second order QPT separating them. The gap between the ground state and the first excited state of this model are known to scale at the critical point as  $\Delta(L) \sim L^{-1}$  where  $L = n^{1/2}$  is the size of the system [22]. The first excited state is nondegenerate. Returning to our  $H(\tau)$ , and recalling that its low-energy sector  $E \ll U$  corresponds to the spectrum of the gauge theory Hamiltonian (5), it now follows that also the gap of  $H(\tau)$  to the first excited state scales as  $\Delta(n) \sim n^{-1/2}$  near the critical point. Hence we know that for a smooth interpolation the adiabatic time  $T = O(1/\Delta E_{\text{min}})$  scales as  $n^{1/2}$ , with the final state arbitrarily close to  $|\phi_0\rangle$  [16].

*Error estimates.*—The rest of the spectrum consists of two large bands with gaps of order  $g, U$ . We now estimate how well the actual final state  $|\psi(t=T)\rangle$  [the solution of the time-dependent Schrödinger equation with  $H(\tau)$ ] approximates the desired adiabatic state  $|\phi_0\rangle$  at  $t=T$  [the instantaneous eigenstate of  $H(1)$ ], given that there is “leakage” to the rest of the spectrum. To this end we use the exponential error estimate [14]: for an infinitely differentiable Hamiltonian  $H(\tau)$  the error  $\delta \equiv \|\psi(T) - \phi_0(T)\|$  can be made smaller than any polynomial in  $T\Delta$  (i.e.,  $\delta \leq O[(T\Delta)^{-N}]$  for arbitrary  $N > 0$ ), where  $\Delta$  is the relevant gap. Since  $g \ll U$ , transitions with  $\Delta = g$  into the lower band dominate. Hence an adiabatic time  $T = O(n^{1/2})$  gives an arbitrarily small error  $\delta$  and completely suppresses errors due to leakage to other gapped excited states. We have thus shown that the topologically ordered state  $|\phi_0\rangle$  can be prepared via adiabatic evolution in a time that scales as  $n^{1/2}$ . This is consistent with the strategy devised in [7], where a toric code (the ground state  $|\phi_0\rangle$ ) is prepared via a number of syndrome measurements of order  $L = n^{1/2}$ . This scaling has recently been shown to be optimal using the Lieb-Robinson bound [10], thus proving that the adiabatic scaling found here is optimal too.

*Outlook.*—In this work we have shown how to prepare a topologically ordered state, the ground state of Kitaev’s toric code model, using quantum adiabatic evolution. The adiabatic evolution time scales as  $O(n^{1/2})$  where  $n$  is the number of spins in the system. This method gives a measurement-free physical technique to prepare a topo-

logically ordered state or initialize a topological quantum memory for topological quantum computation [3,4,23].

The robustness of the ground state of the Kitaev model to thermal excitations [3] endows our preparation method with topological protection against decoherence for  $t \geq T$ ; how to extend this robustness to all times  $t$  is another interesting open question, the study of which may benefit from recent ideas merging quantum error correcting codes and dynamical decoupling with adiabatic quantum computation [24]. An example of the robustness of the topological phase studied here, when the spin system interacts with an Ohmic bath, has recently been provided in Ref. [25].

We thank S. Haas, A. Kitaev, M. Freedman, S. Trebst, P. Zanardi, and especially A. Joye, M. B. Ruskai, and X.-G. Wen for important discussions. This work was supported in part by ARO No. W911NF-05-1-0440 and by NSF No. CCF-0523675.

- 
- [1] X.-G. Wen, *Quantum Field Theory of Many-Body Systems* (Oxford University Press, Oxford, 2004).
  - [2] X.-G. Wen and Q. Niu, Phys. Rev. B **41**, 9377 (1990).
  - [3] A. Kitaev, Ann. Phys. (N.Y.) **303**, 2 (2003).
  - [4] M. H. Freedman, A. Kitaev, M. J. Larsen, and Z. Wang, Bull. Am. Math. Soc. **40**, 31 (2003).
  - [5] K. G. Wilson, Phys. Rev. D **10**, 2445 (1974).
  - [6] M. Levin and X.-G. Wen, Phys. Rev. B **71**, 045110 (2005).
  - [7] E. Dennis, A. Kitaev, A. Landahl, and J. Preskill, J. Math. Phys. (N.Y.) **43**, 4452 (2002).
  - [8] L. B. Ioffe *et al.*, Nature (London) **415**, 503 (2002).
  - [9] E. Farhi *et al.*, Science **292**, 472 (2001).
  - [10] S. Bravyi, M. N. Hastings, and F. Verstraete, Phys. Rev. Lett. **97**, 050401 (2006); J. Eisert and T. J. Osborne, Phys. Rev. Lett. **97**, 150404 (2006).
  - [11] A. Hamma, R. Ionicioiu, and P. Zanardi, Phys. Rev. A **71**, 022315 (2005).
  - [12] E. Ardonne, P. Fendley, and E. Fradkin, Ann. Phys. (N.Y.) **310**, 493 (2004).
  - [13] M. S. Sarandy and D. A. Lidar, Phys. Rev. Lett. **95**, 250503 (2005).
  - [14] G. A. Hagedorn and A. Joye, J. Math. Anal. Appl. **267**, 235 (2002).
  - [15] S. Jansen, M.-B. Ruskai, and R. Seiler, J. Math. Phys. (N.Y.) **48**, 102111 (2007).
  - [16] G. Schaller, S. Mostame, and R. Schutzhold, Phys. Rev. A **73**, 062307 (2006).
  - [17] A. Hamma and D. A. Lidar (to be published).
  - [18] A. Hamma, W. Zhang, S. Haas, and D. A. Lidar, arXiv:0705.0026.
  - [19] F. Wegner, J. Math. Phys. (N.Y.) **12**, 2259 (1971).
  - [20] C. Castelnovo and C. Chamon, arXiv:0707.2084.
  - [21] B. Kogut, Rev. Mod. Phys. **51**, 659 (1979).
  - [22] C. J. Hamer, J. Phys. A **33**, 6683 (2000).
  - [23] M. H. Freedman Proc. Natl. Acad. Sci. U.S.A. **95**, 98 (1998).
  - [24] S. P. Jordan, E. Farhi, and P. W. Shor, Phys. Rev. A **74**, 052322 (2006); D. A. Lidar, arXiv:0707.0021.
  - [25] S. Trebst *et al.*, Phys. Rev. Lett. **98**, 070602 (2007).