Direct characterization of quantum dynamics: General theory

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The characterization of the dynamics of quantum systems is a task of both fundamental and practical importance. A general class of methods which have been developed in quantum information theory to accomplish this task is known as quantum process tomography (QPT). In an earlier paper [M. Mohseni and D. A. Lidar Phys. Rev. Lett. \textbf{97}, 170501 (2006)] we presented an algorithm for direct characterization of quantum dynamics (DCQD) of two-level quantum systems. Here we provide a generalization by developing a theory for direct and complete characterization of the dynamics of arbitrary quantum systems. In contrast to other QPT schemes, DCQD relies on quantum error-detection techniques and does not require any quantum state tomography. We demonstrate that for the full characterization of the dynamics of $n$-d-level quantum systems (with $d$ prime), the minimal number of required experimental configurations is reduced quadratically from $d^{4n}$ in separable QPT schemes to $d^{2n}$ in DCQD.

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I. INTRODUCTION

The characterization of quantum dynamical systems is a fundamental problem in quantum physics and quantum chemistry. Its ubiquity is due to the fact that knowledge of quantum dynamics of (open or closed) quantum systems is indispensable in prediction of experimental outcomes. In particular, accurate estimation of an unknown quantum dynamical process acting on a quantum system is a pivotal task in coherent control of the dynamics, especially in verifying and/or monitoring the performance of a quantum device in the presence of decoherence. The procedures for characterization of quantum dynamical maps are traditionally known as quantum process tomography (QPT) \cite{1,2,3}. In most QPT schemes the information about the quantum dynamical process is obtained indirectly. The quantum dynamics is first mapped onto the state(s) of an ensemble of probe quantum systems, and then the process is reconstructed via quantum state tomography of the output states. Quantum state tomography is itself a procedure for identifying a quantum system by measuring the expectation values of a set of noncommuting observables on identical copies of the system. There are two general types of QPT schemes. The first is standard quantum process tomography (SQPT) \cite{1,4,5}. In SQPT all quantum operations, including preparation and (state tomography) measurements, are performed on the system whose dynamics is to be identified (the “principal” system), without the use of any ancillas. The SQPT scheme has already been experimentally demonstrated in a variety of systems including liquid-state nuclear magnetic resonance (NMR) \cite{6,7,8}, optical \cite{9,10}, atomic \cite{11}, and solid-state systems \cite{12}. The second type of QPT scheme is known as ancilla-assisted process tomography (AAPT) \cite{13,14,15,16}. In AAPT one makes use of an ancilla (auxiliary system). First, the combined principal system and ancilla are prepared in a “faithful” state, with the property that all information about the dynamics can be imprinted on the final state \cite{13,15,16}. The relevant information is then extracted by performing quantum state tomography in the joint Hilbert space of system and ancilla. The AAPT scheme has also been demonstrated experimentally \cite{15,17}. The total number of experimental configurations required for measuring the quantum dynamics of $n$-d-level quantum systems (qudits) is $d^{4n}$ for both SQPT and separable AAPT, where separable refers to the measurements performed at the end. This number can in principle be reduced by utilizing nonseparable measurements, e.g., a generalized measurement \cite{1}. However, the nonseparable QPT schemes are rather impractical in physical applications because they require many-body interactions, which are not experimentally available or must be simulated at high resource cost \cite{3}.

Both SQPT and AAPT make use of a mapping of the dynamics onto a state. This raises the natural question of whether it is possible to avoid such a mapping and instead perform a \textit{direct} measurement of quantum dynamics, which does not require any state tomography. Moreover, it seems reasonable that by avoiding the indirect mapping one should be able to attain a reduction in resource use (e.g., the total number of measurements required), by eliminating redundancies. Indeed, there has been a growing interest in the development of direct methods for obtaining specific information about the states or dynamics of quantum systems. Examples include the estimation of general functions of a quantum state \cite{18}, detection of quantum entanglement \cite{19}, measurement of nonlinear properties of bipartite quantum states \cite{20}, reconstruction of quantum states or dynamics from incomplete measurements \cite{21}, estimation of the average fidelity of a quantum gate or process \cite{22,23}, and universal source coding and data compression \cite{24}. However, these schemes cannot be used directly for a \textit{complete} characterization of quantum dynamics. In Ref. \cite{25} we presented such a scheme, which we called “direct characterization of quantum dynamics” (DCQD).

In trying to address the problem of \textit{direct} and \textit{complete} characterization of quantum dynamics, we were inspired by the observation that quantum error detection (QED) \cite{1} provides a means to directly obtain partial information about the...
nature of a quantum process, without ever revealing the state of the system. In general, however, it is unclear if there is a fundamental relationship between QED and QPT, namely whether it is possible to completely characterize the quantum dynamics of arbitrary quantum systems using QED. And, providing the answer is affirmative, how the physical resources scale with system size. Moreover, one would like to understand whether entanglement plays a fundamental role, and what potential applications emerge from such a theory linking QPT and QED. Finally, one would hope that this approach may lead to new ways of understanding and/or controlling quantum dynamical systems. We addressed these questions for the first time in Ref. [25] by developing the DCQD algorithm in the context of two-level quantum systems. In DCQD—see Fig. 1—the state space of an ancilla is utilized such that experimental outcomes from a Bell-state measurement provide direct information about specific properties of the underlying dynamics. A complete set of probe states is then used to fully characterize the unknown quantum dynamics via application of a single Bell-state measurement device [3,25].

Here we generalize the theory of Ref. [25] to arbitrary open quantum systems undergoing an unknown, completely positive (CP) quantum dynamical map. In the generalized DCQD scheme, each probe qudit (with $d$ prime) is initially entangled with an ancillary qudit system of the same dimension, before being subjected to the unknown quantum process. To extract the relevant information, the corresponding measurements are devised in such a way that the final (joint) probability distributions of the outcomes are directly related to specific sets of the dynamical superoperator’s elements. A complete set of probe states can then be utilized to fully characterize the unknown quantum dynamical map. The preparation of the probe systems and the measurement schemes are based on QED techniques, however, the objective and the details of the error-detection schemes are different from those appearing in the protection of quantum systems against decoherence (the original context of QED). More specifically, we develop error-detection schemes to directly measure the coherence in a quantum dynamical process, represented by off-diagonal elements of the corresponding superoperator. We explicitly demonstrate that for characterizing a dynamical map on $n$ qubits, the number of required experimental configurations is reduced from $d^{4n}$ in SQPT and separable AAPT, to $d^{2n}$ in DCQD. A useful feature of DCQD is that it can be efficiently applied to partial characterization of quantum dynamics [25,26]. For example, it can be used for the task of Hamiltonian identification, and also for simultaneous determination of the relaxation time $T_1$ and the dephasing time $T_2$.

This paper is organized as follows. In Sec. II, we provide a brief review of completely positive quantum dynamical maps, and the relevant QED concepts such as stabilizer codes and normalizers. In Sec. III, we demonstrate how to determine the quantum dynamical populations, or diagonal elements of a superoperator, through a single (ensemble) measurement. In order to further develop the DCQD algorithm and build the required notations, we introduce some lemmas and definitions in Sec. IV, and then we address the characterization of quantum dynamical coherences, or off-diagonal elements of a superoperator, in Sec. V. In Sec. VI, we show that measurement outcomes obtained in Sec. V provide $d^2$ linearly independent equations for estimating the coherences in a process, which is in fact the maximum amount of information that can be extracted in a single measurement. A complete characterization of the quantum dynamics, however, requires obtaining $d^4$ independent real parameters of the superoperator (for nontrace preserving maps). In Sec. VII, we demonstrate how one can obtain complete information by appropriately rotating the input state and repeating the above algorithm for a complete set of rotations. In Secs. VIII and IX, we address the general constraints on input stabilizer codes and the minimum number of physical qudits required for the encoding. In Sec. X and Sec. XI, we define a standard notation for stabilizer and normalizer measurements and then provide an outline of the DCQD algorithm for the case of a single qudit. For convenience, we provide a brief summary of the entire DCQD algorithm in Sec. XII. We conclude with an outlook in Sec. XIII. In the Appendix, we generalize the scheme for arbitrary open quantum systems. For a discussion of the experimental feasibility of DCQD see Ref. [25], and for a detailed and comprehensive comparison of the required physical resources in different QPT schemes see Ref. [3].

II. PRELIMINARIES

In this section we introduce the basic concepts and notation from the theory of open quantum system dynamics and quantum error detection, required for the generalization of the DCQD algorithm to qudits.

A. Quantum dynamics

The evolution of a quantum system (open or closed) can, under natural assumptions, be expressed in terms of a completely positive quantum dynamical map $\mathcal{E}$, which can be represented as [1]

$$
\mathcal{E}(\rho) = \sum_{m,n=0}^{d^2-1} \chi_{mn} E_m \rho E_n^\dagger.
$$

(1)

Here $\rho$ is the initial state of the system, and the $\{E_m\}$ are a set of (error) operator basis elements in the Hilbert-Schmidt space of the linear operators acting on the system. That is, any arbitrary operator acting on a $d$-dimensional quantum system can be expanded over an orthonormal and unitary error operator basis $\{E_0, E_1, \ldots , E_{d^2-1}\}$, where $E_0 = 1$ and $\text{tr}(E_i E_j) = d \delta_{ij}$ [27]. The $\{\chi_{mn}\}$ are the matrix elements of the superoperator $\chi$, or “process matrix,” which encodes all the information about the dynamics, relative to the basis set $\{E_m\}$.
Therefore, for any two elements \( E_i, E_j \) of the (error) operator basis can be expressed as \( E_i E_j = \sum a_{ij} E_k \), where \( i, j, k = 0, 1, \ldots, d^2 - 1 \). However, we use the “very nice” (error) operator basis in which \( E_i E_j = a_{ij} E_j E_i \), \( \det E_i = 1 \). \( a_{ij} \) is a \( d \times d \) root of unity, and the operation \( * \) induces a group on the indices [27]. This provides a natural generalization of the Pauli group to higher dimensions. Any element \( E_i \) can be generated from appropriate products of \( X_d \) and \( Z_d \), where \( X_d(k) = |k+1\rangle \), \( Z_d(k) = \omega^k |k\rangle \), and \( X_d Z_d = \omega^{-1} Z_d X_d \) [27,28]. Therefore, for any two elements \( E_{i=q,q+p} = \omega^{pq} X_d Z_d \) and \( E_{i=q,q+p} = \omega^{pq} X_d Z_d \) (where \( 0 \leq q, p < d \)) of the single-qudit Pauli group, we always have

\[
E_i E_j = \omega^{pq' - qp'} E_i E_i,
\]

where \( pq' - qp' = k(\text{mod} d) \).

The operators \( E_i \) and \( E_j \) commute if and only if \( k=0 \). Henceforth, all algebraic operations are performed in \( \text{mod}(d) \) arithmetic, and all quantum states and operators, respectively, belong to and act on a \( d \)-dimensional Hilbert space. For simplicity, from now on we drop the subscript \( d \) from the operators.

**B. Quantum error detection**

In the last decade the theory of quantum error correction (QEC) has been developed as a general method for detecting and correcting quantum dynamical errors acting on multiqubit systems such as a quantum computer [1]. QEC consists of three steps: preparation, quantum error detection (QED) or syndrome measurements, and recovery. In the preparation step, the state of a quantum system is encoded into a subspace of a larger Hilbert space by entangling the principal system with some other quantum systems using unitary operations. This encoding is designed to allow detection of arbitrary errors on one or more physical qubits of a code by performing a set of QED measurements. The measurement strategy is to map different possible sets of errors only to orthogonal and undeformed subspaces of the total Hilbert space, such that the errors can be unambiguously discriminated. Finally, the detected errors can be corrected by applying the required unitary operations on the physical qubits during the recovery step. A key observation relevant for our purposes is that by performing QED one can actually obtain partial information about the dynamics of an open quantum system.

For a qudit in a general state \( |\phi_\psi\rangle \) in the code space, and for arbitrary error basis elements \( E_m \) and \( E_n \), the Knill-Lafllame QEC condition for degenerate codes is

\[
\langle \phi_\psi | E^*_m E^n_m | \phi_\psi \rangle = \lambda_{nm}, \quad \lambda_{nm} \text{ is a Hermitian matrix of complex numbers} [1].
\]

For nondegenerate codes, the QEC condition reduces to \( \langle \phi_\psi | E^*_m E^n_m | \phi_\psi \rangle = \delta_{nm} \); i.e., in this case the errors always take the code space to orthogonal subspaces. The difference between nondegenerate and degenerate codes is illustrated in Fig. 2. In this work, we concentrate on a large class of error-correcting codes known as stabilizer codes [29]; however, in contrast to QEC, we restrict our attention almost entirely to degenerate stabilizer codes as the initial states. Moreover, by definition of our problem, the recovery and/or correction step is not needed or used in our analysis.

A stabilizer code is a subspace \( \mathcal{H}_C \) of the Hilbert space of \( n \) qubits that is an eigenspace of a given Abelian subgroup \( S \) of the \( n \)-qudit Pauli group with the eigenvalue \( +1 \) [1,29]. In other words, for \( |\psi_i\rangle \in \mathcal{H}_C \) and \( S \in S \), we have \( S_j |\psi_i\rangle = |\psi_i\rangle \), where \( S_j \) is a stabilizer generator and \( \{S_j, S_j\} = 0 \). Consider the action of an arbitrary error operator \( \hat{E} \) on the stabilizer code \( |\phi_\psi\rangle, E |\phi_\psi\rangle \). The detection of such an error will be possible if the error operator anticommutes with (at least one of) the stabilizer generators, \( \{S_j, E\} = 0 \). That is, by measuring all generators of the stabilizer and obtaining one or more negative eigenvalues we can determine the nature of the error unambiguously as

\[
S_j (E |\phi_\psi\rangle) = - E (S_j |\phi_\psi\rangle) = - (E |\phi_\psi\rangle).
\]

A stabilizer code \([n, k, d]\) represents an encoding of \( k \) logical qubits into \( n \) physical qubits with code distance \( d \), such that an arbitrary error on any subset of \( t=(d-1)/2 \) or fewer qubits can be detected by QED measurements. A stabilizer group with \( n-k \) generators has \( d^{n-k} \) elements and the code space is \( d^k \) dimensional. Note that this is valid when \( d \) is a power of a prime [28]. The unitary operators that preserve the stabilizer group by conjugation, i.e., \( USU^*=S \), are called the normalizer of the stabilizer group, \( N(S) \). Since the normalizer elements preserve the code space they can be used to perform certain logical operations in the code space.
However, they are insufficient for performing arbitrary quantum operations [1].

Similarly to the case of a qubit [25], the DCQD algorithm for the case of a qudit system consists of two procedures: (i) a single experimental configuration for characterization of the quantum dynamical populations, and (ii) \( d^2-1 \) experimental configurations for characterization of the quantum dynamical coherences. In both procedures we always use two physical qudits for the encoding, the principal system \( A \) and the ancilla \( B \), i.e., \( n = 2 \). In procedure (i)—characterizing the diagonal elements of the superoperator—the stabilizer group has two generators. Therefore it has \( d^2 \) elements and the code space consists of a single quantum state (i.e., \( k = 0 \)). In procedure (ii)—characterizing the off-diagonal elements of the superoperator—the stabilizer group has a single generator, thus it has \( d \) elements, and the code space is two dimensional. That is, we effectively encode a logical qudit (i.e., \( k = 1 \)) into two physical qudits. In the next sections, we develop the procedures (i) and (ii) in detail for a single qudit with \( d \) being a prime, and in the Appendix we address the generalization to systems with \( d \) being an arbitrary power of a prime.

### III. CHARACTERIZATION OF QUANTUM DYNAMICAL POPULATION

To characterize the diagonal elements of the superoperator, or the population of the unitary error basis, we use a nondegenerate stabilizer code. We prepare the principal qudit, \( A \), and an ancilla qudit, \( B \), in a common +1 eigenstate \( |\phi_k\rangle \) of the two unitary operators \( E^A_i \) and \( E^B_i \), such that \([E^A_i, E^B_j]=0 \) [e.g., \( X^A X^B \) and \( Z^A (Z^B)^{d-1} \)]. Therefore, simultaneous measurement of these stabilizer generators at the end of the dynamical process reveals arbitrary single qudit errors on the system \( A \). The possible outcomes depend on whether a specific operator in the operator-sum representation of the quantum dynamics commutes with \( E^A_i \) and \( E^B_j \), with the eigenvalue +1, or with one of the eigenvalues \( \omega, \omega^2, \ldots, \omega^{d-1} \). The projection operators corresponding to outcomes \( \omega^k \) and \( \omega^{k'} \), where \( k, k' = 0, 1, \ldots, d-1 \), have the form \( P_k = \frac{1}{d} \sum_{i=0}^{d-1} \omega^{-ik} (E^A_i E^B_i) \) and \( P_{k'} = \frac{1}{d} \sum_{i=0}^{d-1} \omega^{-i k'} (E^A_i E^B_i) \). The joint probability distribution of the commuting Hermitian operators \( P_k \) and \( P_{k'} \) on the output state \( \mathcal{E}(\rho) \) is \( \sum_{m,n} \chi_{mn} E^i_m E^j_n \rho E^i_n \), where \( \rho = |\phi_k\rangle \langle \phi_k| \), is

\[
\text{Tr}[P_k P_{k'} \mathcal{E}(\rho)] = \frac{1}{d} \sum_{m,n=0}^{d-1} \chi_{mn} \sum_{i=0}^{d-1} \sum_{j=0}^{d-1} \omega^{-ik} \omega^{-ij} \times \text{Tr}[E^A_i (E^A_i)^\dagger (E^B_j)^\dagger E^B_n E^A_n (E^B_j) (E^A_i)^\dagger \rho].
\]

Using \( E^A E^B = \omega^a E^A E^i \) and the relation \( (E^A_i E^B_j)^\dagger (E^A_i E^B_j)^\dagger \rho = \rho \), we obtain

\[
\text{Tr}[P_k P_{k'} \mathcal{E}(\rho)] = \frac{1}{d} \sum_{m,n=0}^{d-1} \chi_{mn} \sum_{i=0}^{d-1} \sum_{j=0}^{d-1} \omega^{-i(k-k')} \omega^{-i(k-k')} \delta_{mn},
\]

where we have used the QED condition for nondegenerate codes.

\[
\mathcal{E} (|\psi\rangle \langle \psi|, \rho) = \delta_{mn}
\]

i.e., the fact that different errors should take the code space to orthogonal subspaces, in order for errors to be unambiguously detectable, see Fig. 3. Now, using the discrete Fourier transform identities \( \sum_{t=0}^{d-2} \omega^{im} \omega^{i'k} = d \delta_{mk} \) and \( \sum_{t=0}^{d-2} \omega^{i(m-k')} = d \delta_{m,k} \), we obtain

\[
\text{Tr}[P_k P_{k'} \mathcal{E}(\rho)] = \sum_{m=0}^{d^2-1} \chi_{mn} \delta_{mk} \delta_{m,k'} = \chi_{mn} \rho_{mn}.
\]

Here, \( m_0 \) is defined through the relations \( i_{m_0} = k \) and \( i'_{m_0} = k' \), i.e., \( E_{m_0} \) is the unique error operator that anticommutes with the stabilizer operators with a fixed pair of eigenvalues \( \omega^k \) and \( \omega^{k'} \), corresponding to the experimental outcomes \( k \) and \( k' \). Since each \( P_k \) and \( P_{k'} \) operator has \( d \) eigenvalues, we have \( d^2 \) possible outcomes, which gives us \( d^2 \) linearly independent equations. Therefore, we can characterize all the diagonal elements of the superoperator with a single ensemble measurement and \( 2d \) detectors.

In order to investigate the properties of the pure state \( |\phi_k\rangle \), we note that the code space is one dimensional (i.e., it has only one vector) and can be Schmidt decomposed as \( |\phi_k\rangle \sim \sum_{j=0}^{d-1} \lambda_j |k\rangle \langle j| \), where \( \lambda_j \) are non-negative real numbers. Suppose \( Z|k\rangle = \omega^k |k\rangle \); without loss of generality the two stabilizer generators of \( |\phi_k\rangle \) can be chosen to be \( (X^A X^B)^{d-1} \) and \( (Z^A (Z^B)^{d-1})^p \). We then have \( \langle \phi_k | (X^A X^B)^{d-1} |\phi_k\rangle = 1 \) and \( \langle \phi_k | (Z^A (Z^B)^{d-1})^p |\phi_k\rangle = 1 \) for any \( q \) and \( p \), where \( 0 \leq q, p < d \). This results in the set of equations \( \sum_{j=0}^{d-1} \lambda_j \delta_{jq} = 1 \) for all \( q \), which have only one positive real solution, \( \lambda_q = \lambda_1 = \cdots = \lambda_k = 1/\sqrt{d} \); i.e., the stabilizer state, \( |\phi_k\rangle \), is a maximally entangled state in the Hilbert space of the two qudits.

In the remaining parts of this paper, we first develop an algorithm for extracting optimal information about the dynamical coherence of a \( d \)-level quantum system (with \( d \) being a prime), through a single experimental configuration, in Secs. IV–VI. Then, we further develop the algorithm to obtain complete information about the off-diagonal elements of the superoperator by repeating the same scheme for different input states, Sec. VII. In the Appendix, we address the generalization of the DCQD algorithm for qudit systems with \( d \).
being a power of a prime. In the first step, in the next section, we establish the required notation by introducing some lemmas and definitions.

IV. BASIC LEMMAS AND DEFINITIONS

Lemma 1. Let 0 ≤ q, p, q', p' < d, where d is prime. Then, for given q, p, q' and k(mod d), there is a unique p' that solves pq' - qp' = k(mod d).

Proof. We have pq' - qp' = k(mod d) = k + td, where t is an integer. The possible solutions for p' are indexed by t as p'(t) = ((pq' - k - td)/q). We now show that if p'(t1) is a solution for a specific value t1, there exists no other integer t2 ≠ t1 such that p'(t2) is another independent solution to this equation, i.e., p'(t2) ≠ p'(t1)(mod d). First, note that if p'(t2) is another solution then we have p'(t1) = p'(t2) + (t2 - t1)/d. Since d is prime, there are two possibilities: (a) q divides (t2 - t1), then (t2 - t1)d/q = ±nd, where n is a positive integer, therefore we have p'(t2) = p'(t1)(mod d), which contradicts our assumption that p'(t2) is an independent solution from p'(t1). (b) q does not divide (t2 - t1), then (t2 - t1)d/q is not a integer, which is unacceptable. Thus, we have t2 = t1, i.e., the solution p'(t) is unique.

Note that the above argument does not hold if d is not prime, and therefore, for some q' there could be more than one p' that satisfies pq' - qp' = k(mod d). In general, the validity of this lemma relies on the fact that Z_d is a field only for prime d.

Lemma 2. For any unitary operator basis E_i acting on a Hilbert space of dimension d, where d is a prime and i=0,1, ..., d-1, there are d unitary operator basis elements, E_j, that anticommute with E_i with a specific eigenvalue \omega^k, i.e., E_i E_j = \omega^k E_j E_i, where k=0, ..., d-1.

Proof. We have E_i E_j = \omega^{pq' - qp'} E_j E_i, where 0 ≤ q, p, q', p' < d, and pq' - qp' = k(mod d). Therefore, for fixed q, p, and k (mod d) we need to show that there are d solutions (q', p'). According to Lemma 1, for any q' there is only one p' that satisfies pq' - qp' = k(mod d); but q' can have d possible values, therefore there are d possible pairs of (q', p').

Definition 1. We introduce d different subsets, W_k, k = 0, 1, ..., d-1, of a unitary operator basis \{E_i\} (i.e., W_k \subseteq \{E_i\}). Each subset contains d members which all anticommute with a particular basis element E_i, where i = 0, 1, ..., d-1, with fixed eigenvalue \omega^k. The subset W_0 which includes E_0 and E_i is in fact an Abelian subgroup of the single-qudit Pauli group, G_i.

V. CHARACTERIZATION OF QUANTUM DYNAMICAL COHERENCE

For characterization of the coherence in a quantum dynamical process acting on a qudit system, we prepare a two-qudit quantum system in a nonseparable eigenstate |\phi_i⟩ of a unitary operator S_ij = E_i^A E_j^B. We then subject the qudit A to the unknown dynamical map, and measure the sole stabilizer operator \bar{S}_j at the output state. Here, the state |\phi_i⟩ is in fact a degenerate code space, since all the operators E_m^A that anticommute with E_i^A, with a particular eigenvalue \omega^k, perform the same transformation on the code space and cannot be distinguished by the stabilizer measurement. If we express the spectral decomposition of S_ij = E_i^A E_j^B as S_ij = \sum_k \omega^k P_k, the projection operator corresponding to the outcome \omega^k can be written as P_k = \frac{1}{d} \sum_l \omega^{k l} (E_i^A E_j^B)^l. The post-measurement state of the system, up to a normalization factor, will be

\[ P_k \bar{\mathcal{E}}(\rho) P_k = \frac{1}{d} \sum_{l=0}^{d-1} \omega^{-kl} \sum_{l' = 0}^{d-1} \sum_{m,n=0}^{d-1} \omega^{l'm} \omega^{l'n} \sum_{m,n=0}^{d-1} \mathcal{X}_{mm} \mathcal{E}_{mn} \mathcal{P}_{ln}. \]

Using the relations \mathcal{E}_{mn} = \omega^{n/k} \mathcal{E}_{mn}, \mathcal{E}_{nm} = \omega^{n/k} \mathcal{E}_{nm}, and \mathcal{E}_{nm} = \omega^{n/k} \mathcal{E}_{nm}, we have

\[ P_k \bar{\mathcal{E}}(\rho) P_k = \frac{1}{d} \sum_{l=0}^{d-1} \sum_{l' = 0}^{d-1} \sum_{m,n=0}^{d-1} \mathcal{X}_{mm} \mathcal{E}_{mn} \mathcal{P}_{ln}. \]

Now, using the discrete Fourier transform properties \sum_{l=0}^{d-1} \omega^{l(k - d - 1)} = d \delta_{n,k} and \sum_{l=0}^{d-1} \omega^{l(k - d - 1)} = d \delta_{n,k}, we obtain

\[ P_k \bar{\mathcal{E}}(\rho) P_k = \sum_{m} \mathcal{X}_{mm} \mathcal{E}_{mn}^A \mathcal{E}_{mn}^B + \sum_{m<n} (\mathcal{X}_{mn} \mathcal{E}_{mn}^A \mathcal{E}_{mn}^B + \mathcal{X}_{nm} \mathcal{E}_{nm}^A \mathcal{E}_{nm}^B). \]

Here, the summation runs over all \mathcal{E}_{kn}^A and \mathcal{E}_{kn}^B that belong to the same W_k, see Lemma 2. That is, the summation is over all unitary operator basis elements \mathcal{E}_{m,n}^A and \mathcal{E}_{m,n}^B that anticommute with \mathcal{E}_{m,n}^A with a particular eigenvalue \omega^k. Since the number of elements in each W_k is d, the state of the two-qudit system after the projective measurement comprises d + 2[d(d - 1)/2] = d^2 terms. The probability of getting the outcome \omega^k is

\[ \text{Tr}[P_k \bar{\mathcal{E}}(\rho) P_k] = \sum_{m} \mathcal{X}_{mm} + 2 \sum_{m<n} \text{Re} \{\mathcal{X}_{mn} \text{Tr}(\mathcal{E}_{mn}^A \mathcal{E}_{mn}^B)\}. \]

Therefore, the normalized post-measurement states are \rho_k = P_k \bar{\mathcal{E}}(\rho) P_k / \text{Tr}[P_k \bar{\mathcal{E}}(\rho)] and these d equations provide us with information about off-diagonal elements of the superoperator if and only if Tr([(E_{kn}^A)^* E_{mn}^B]) ≠ 0. Later we will derive some general properties of the state \rho such that this condition can be satisfied.

Next we measure the expectation value of any other unitary operator basis element T_{rs} = \mathcal{E}_{rs}^B on the output state, such that \mathcal{E}_{rs}^A ≠ I, \mathcal{E}_{rs}^B ≠ I, T_{rs} ∈ \mathcal{S}(S), and T_{rs} \neq (S_{ij})^\ast, where 0 ≤ a < d. Let us write the spectral decomposition of T_{rs} as T_{rs} = \sum_{l'} \omega^{l'} \mathcal{P}_{l'}. The joint probability distribution of the commuting Hermitian operators \mathcal{P}_E and \mathcal{P}_F on the output state \mathcal{E}(\rho) is Tr[P_E P_F \bar{\mathcal{E}}(\rho)] and the average of these joint probability distributions of P_k and P_{k'} over different values of k' becomes \sum_{k'} \omega^{k'} Tr[P_k P_{k'} \bar{\mathcal{E}}(\rho)] = Tr[T_{rs} P_k \bar{\mathcal{E}}(\rho)] = Tr[T_{rs} P_k], which can be explicitly written as
VI. LINEAR INDEPENDENCE AND OPTIMALITY OF MEASUREMENTS

Before presenting the proof of linear independence of the functions \( \text{Tr}(T_{rs} \rho_k) \) and of the optimality of the DCQD algorithm, we need to introduce the following lemmas and definitions.

**Lemma 3.** If a stabilizer group, \( S \), has a single generator, the order of its normalizer group, \( N(S) \), is \( d^2 \).

**Proof.** Let us consider the sole stabilizer generator \( S_{12} \)

\[
S_{12} = E_1^A E_2^B,
\]

and a typical normalizer element \( T_{12} = E_1^A E_2^B \),

where \( E_1^A = X_1^A Z_1^A \), \( E_2^A = X_2^A Z_2^A \), \( E_1^B = X_1^B Z_1^B \), and \( E_2^B = X_2^B Z_2^B \). Since \( S_{12} \) and \( T_{12} \) commute, we have \( S_{12} T_{12} = \omega^{i a q} \rho_{12} a q \rho_{12}^* \) for \( a = 0, 1, 2, \ldots, d-1 \) and \( q = 0, 1, 2, \ldots, d-1 \). Since any particular code with a single stabilizer generator, all \( q, p_1, q_1, p_2 \) are fixed. Now, by Lemma 1, for given values of \( q, p_1, q_1, p_2 \), there is only one value for \( p_2 \) that satisfies the above equation. However, each \( q, p_1, q_1, p_2 \) can have \( d \) different values. Therefore, there are \( d^4 \) different normalizer elements \( T_{12} \).

**Lemma 4.** Each Abelian subgroup of a normalizer, which includes the stabilizer group \( \{ S_{ij} \} \) as a proper subgroup, has order \( d^2 \).

**Proof.** Suppose \( T_{rs} \) is an element of \( N(S) \), i.e., it commutes with \( S_{ij} \). Moreover, all unitary operators of the form \( T_{rs} S_{ij} \) where \( 0 \leq r, s < d \), also commute. Therefore, any Abelian subgroup of the normalizer, \( A \subset N(S) \), which includes \( \{ S_{ij} \} \) as a proper subgroup, is at least of order \( d^2 \). Now let \( T_{rs} \) be any other normalizer element, i.e., \( T_{rs} \neq T_{rs} S_{ij} \) with \( 0 \leq r, s < d \), which belongs to the same Abelian subgroup \( A \). In this case, any operator of the form \( T_{rs} T_{rs} S_{ij} \) would also belong to \( A \). Then all elements of the normalizer should commute with \( A = N(S) \), which is unacceptable. Thus, either \( T_{rs} = T_{rs} S_{ij} \) or \( T_{rs} \notin A \), i.e., the order of the Abelian subgroup \( A \) is at most \( d^2 \).

**Lemma 5.** There are \( d+1 \) Abelian subgroups, \( A \), in the normalizer \( N(S) \).

**Proof.** Suppose that the number of Abelian subgroups which includes the stabilizer group as a proper subgroup is \( n \). Using Lemmas 3 and 4, we have \( d^2 = nd^2 - (n-1)d \), where the term \( (n-1)d \) has been subtracted from the total number of elements of the normalizer due to the fact that the elements of the stabilizer group are common to all Abelian subgroups. Solving this equation for \( n \), we find that \( n = d^2 - d + 1 \).

**Lemma 6.** The basis of eigenvectors defined by \( d+1 \) Abelian subgroups of \( N(S) \) are mutually unbiased.

**Proof.** It has been shown [30] that if a set of \( d^2 - 1 \) traceless and mutually orthogonal \( d \times d \) unitary matrices can be partitioned into \( d+1 \) subsets of equal size, such that the \( d - 1 \) unitary operators in each subset commute, then the basis of eigenvectors corresponding to these subsets is mutually unbiased. We note that, based on Lemmas 3, 4, and 5, and in the code space (i.e., up to multiplication by the stabilizer elements \( \{ S_{ij} \} \)), the normalizer \( N(S) \) has \( d^2 - 1 \) nontrivial elements, and each Abelian subgroup \( A \), has \( d - 1 \) nontrivial commuting operators. Thus, the bases of eigenvectors defined by \( d+1 \) Abelian subgroups of \( N(S) \) are mutually unbiased.

**Lemma 7.** Let \( C \) be a cyclic subgroup of \( A \), i.e., \( C \subset A \subset N(S) \). Then, for any fixed \( T \in A \), the number of distinct left (right) cosets, \( TC(CT) \), in each \( A \) is \( d \).

**Proof.** We note that the order of any cyclic subgroup
C C A, such as \( T^b_{rs} \) with \( 0 \leq b < d \), is \( d \). Therefore, by Lemma 4, the number of distinct cosets in each \( A \) is \( \frac{d^2}{2} \).

**Definition 2.** We denote the cosets of an (invariant) cyclic subgroup, \( C_{d} \), of an Abelian subgroup of the normalizer, \( A_{s} \), by \( A_s/C_{d} \), where \( v = 1, 2, \ldots , d+1 \). We also represent generic members of \( A_s/C_{d} \) as \( T^b_{rs} S^p_{r} \), where 0 \( \leq a, b < d \). The members of a specific coset \( A_s/C_{d} \) are denoted as \( T^b_{rs} S^p_{r} \), where \( a_0 \) represents a fixed power of stabilizer generator \( S_{r} \), that labels a particular coset \( A_s/C_{d} \), and \( b (0 \leq b < d) \) labels different members of that particular coset.

**Lemma 8.** The elements of a coset, \( T^b_{rs} S^p_{r} \) (where \( T^b_{rs} = E^b_{rs} = \bar{E}^b_{rs} \), \( S^p_{r} \) \( = \bar{E}^p_{rs} \), and \( 0 \leq b < d \) anticommute with \( E^b_{rs} \) with different eigenvalues \( \omega \). That is, there are no two different members of a coset, \( A_s/C_{d} \), that anticommute with \( E^b_{rs} \) with the same eigenvalue.

**Proof.** First we note that for each \( T^b_{rs} = (E^b_{rs})^p \), the unitary operators acting only on the principal subspace, \( (E^b_{rs})^p \), must satisfy either (a) \( E^b_{rs} E^p_{rs} = E^p_{rs} \) or (b) \( E^b_{rs} E^p_{rs} = \bar{E}^b_{rs} \). In the case (a), and due to \( T^b_{rs} S^p_{r} = 0 \), we should also have \( E^b_{rs} = \bar{E}^b_{rs} \), which results in \( T^b_{rs} = S^p_{r} \); i.e., \( T^b_{rs} \) is a stabilizer and not a normalizer. This is unacceptable. In the case (b), in particular for \( b = 1 \), we have \( E^b_{rs} = \pm E^b_{rs} \). Therefore, for arbitrary \( b \) we have \( E^b_{rs} E^p_{rs} = \pm E^b_{rs} \). Since \( 0 \leq b < d \), we conclude that \( \omega^{b r_i n} \neq \omega^{b' r_i n} \) for any two different values of \( b \) and \( b' \).

As a consequence of this, different \( (E^b_{rs})^p \), for \( 0 \leq b < d \), belong to different \( W^b_k \).

**Lemma 9.** For any fixed unitary operator \( E^s \in W^b_k \), where \( k \neq 0 \), and any other independent operators \( E^m \) and \( E^n \) that belong to the same \( W^b_k \), we always have \( \omega^{s n} \neq \omega^{n s} \), where \( E^s_{E^m} = \omega^{s n} E^m E^n \) and \( E^n_{E^m} = \omega^{n s} E^m E^n \).

**Proof.** We need to prove for operators \( E^s \), \( E^m \), \( E^n \) \( \in \mathcal{W}^b_k \) (where \( k \neq 0 \)), that always have \( E^s \neq E^n \) \( \Rightarrow \omega^{s n} \neq \omega^{n s} \). Let us prove the converse, \( \omega^{s n} = \omega^{n s} \Rightarrow E^s = E^n \). We define \( E^s = X^{s \in Z_p^q} E^s = X^{s \in Z_p^q} \), \( E^m = X^{m \in Z_p^q} \), \( E^n = X^{n \in Z_p^q} \). Based on the definition of subsets \( W^b_k \), with \( k \neq 0 \), we have \( p(\omega^{s n} - \omega^{n s}) = m \). Let us prove the converse, \( \omega^{s n} = \omega^{n s} \Rightarrow E^s = E^n \).

**Theorem 1.** The expectation values of normalizer elements on a post-measurement state, \( \rho_b \), are linearly independent if these elements are the \( d-1 \) nontrivial members of a coset \( A_s/C_{d} \). That is, for two independent operators \( T^b_{rs}, T^{b'}_{r's'} \), we have \( \text{Tr}(T^b_{rs} \rho_b) \neq c \text{Tr}(T^{b'}_{r's'} \rho_b) \), where \( c \) is an arbitrary complex number.

**Proof.** We know that the elements of a coset can be written as \( T^b_{rs} S^p_{r} = (E^b_{rs})^p \), where \( b = 1, 2, \ldots , d-1 \). We also proved that \( (E^b_{rs})^p \) belongs to different \( W^b_k \) \( (k \neq 0) \) for different values of \( b \) (see Lemma 8). Therefore, according to Lemma 9 and regardless of the outcome of \( k \) (after measuring the stabilizer \( S_{r} \)), there exists one member in the coset \( A_s/C_{d} \) that has different eigenvalues \( \omega^{s n} \) with all (independent) members \( E^m_{E^s} \in W^b_k \). The expectation value of \( T^b_{rs} S^p_{r} \) is

\[
\text{Tr}(T^b_{rs} S^p_{r} \rho_b) = \sum_{m} \chi_{mm} \text{Tr}(E^m_{E^b_{rs}}) S^p_{r} \rho_b,
\]

\[
+ \sum_{m<n} \chi_{mm} \text{Tr}(E^m_{E^s_{rs}} S^p_{r} \rho_b) + \chi_{mm} \text{Tr}(A^m_{E^b_{rs}} S^p_{r} \rho_b),
\]

\[
(10)
\]

\[
\text{Tr}(T^b_{rs} \rho_b) = \sum_{m} \omega^{b r_i n} \chi_{mm} \text{Tr}(E^m_{E^b_{rs}}) \rho_b,
\]

\[
+ \sum_{m<n} \omega^{b r_i n} \chi_{mm} \text{Tr}(E^m_{E^s_{rs}} \rho_b) + \omega^{b r_i n} \chi_{mm} \text{Tr}(E^m_{E^b_{rs}} \rho_b),
\]

\[
(11)
\]

where \( \omega^{b r_i n} \neq \omega^{m r_i n} \) for all elements \( E^m_{E^s_{rs}} \); that belong to a specific \( W^b_k \). Therefore, for two independent members of a coset denoted by \( b \) and \( b' \) (i.e., \( b \neq b' \)), we have \( \omega^{b r_i n} \neq \omega^{b r_i n} \) for all values of \( 0 \leq b, b' < d \), and any complex number \( c \). We also note that we have \( \text{Tr}(E^m_{E^b_{rs}} \rho_b) \neq c \text{Tr}(E^{m_{s}}_{E^b_{rs}} \rho_b) \), since \( T^b_{rs} = \bar{E}^b_{rs} \) is a normalizer, not a stabilizer element, and its action on the state cannot be expressed as a global phase. Thus, for any two independent members of a coset \( A_s/C_{d} \), we always have \( \text{Tr}(T^b_{rs} \rho_b) \neq c \text{Tr}(T^{b'}_{r's'} \rho_b) \).

In summary, after the action of the unknown dynamical process, we measure the eigenvalues of the stabilizer generator, \( (E^b_{rs})^p \), that has \( d \) eigenvalues for \( k = 0, 1, \ldots , d-1 \) and provides \( d \) linearly independent equations for the real and imaginary parts of \( \chi_{mm} \). This is due to the fact that the outcomes corresponding to different eigenvalues of a unitary operator are independent. We also measure expectation values of all the \( d-1 \) independent and commuting normalizer operators \( T^b_{rs} S^p_{r} \in A_s/C_{d} \) on the post-measurement state \( \rho_b \), which provides \( (d-1) \) linearly independent equations for each outcome \( k \) of the stabilizer measurements. Overall, we obtain \( d + (d-1) = 2d \) linearly independent equations for characterization of the real and imaginary parts of \( \chi_{mm} \) by a single ensemble measurement. In the following, we show
that the above algorithm is optimal. That is, within the $d^2$ Hilbert space of principal system and ancilla, there does not exist any other possible strategy that can provide more than $d^2$ linearly independent equations by a single measurement on the output state $\mathcal{E}(\rho)$.

B. Optimality

**Theorem 2.** The maximum number of commuting normalizer elements that can be measured simultaneously to provide independent equations for the joint distribution functions $\text{Tr}(T^{b}_{rs} \rho)$ is $d-1$.

**Proof.** Any Abelian subgroup of the normalizer has order $d^2$ (see Lemma 4). Therefore, the desired normalizer operators should all belong to a particular $A$, and are limited to $d^2$ members. We already showed that the outcomes of measurements for $d-1$ elements of a coset $A_i/C_{w}$, represented by $T^{b}_{rs}$ (with $b \neq 0$), are independent (see Theorem 1). Now we show that measuring any operator, $T^{b}_{rs'}$, from any other coset $A_j/C_{w'}$, results in linearly dependent equations for the functions $w = \text{tr}(T^{b}_{rs} \rho_{k})$ and $w' = \text{tr}(T^{b}_{rs'} \rho_{k})$ as the following:

$$
w = \text{Tr}(T^{b}_{rs} \rho_{k}) = \sum_{m} \chi_{mn} \text{Tr}(E_{m}^{\dagger} T^{b}_{rs} E_{m} \rho_{k}) + \sum_{m < n} \chi_{mn} \text{Tr}(E_{n}^{\dagger} T^{b}_{rs} E_{m} \rho_{k}) + \sum_{m < n} \chi_{mn} \text{Tr}(E_{n}^{\dagger} T^{b}_{rs} E_{m} \rho_{k})
$$

$$
+ \sum_{m < n} \chi_{mn} \text{Tr}(E_{n}^{\dagger} T^{b}_{rs} E_{m} \rho_{k}) + \sum_{m < n} \chi_{mn} \text{Tr}(E_{n}^{\dagger} T^{b}_{rs} E_{m} \rho_{k}) + \sum_{m < n} \chi_{mn} \text{Tr}(E_{n}^{\dagger} T^{b}_{rs} E_{m} \rho_{k})
$$

$$
w' = \text{Tr}(T^{b}_{rs'} \rho_{k}) = \sum_{m} \chi_{mn} \text{Tr}(E_{m}^{\dagger} T^{b}_{rs'} E_{m} \rho_{k}) + \sum_{m < n} \chi_{mn} \text{Tr}(E_{n}^{\dagger} T^{b}_{rs'} E_{m} \rho_{k}) + \sum_{m < n} \chi_{mn} \text{Tr}(E_{n}^{\dagger} T^{b}_{rs'} E_{m} \rho_{k})$$

Using the commutation relations $T^{b}_{rs} S^{i}_{ij} = \omega^{b} T^{b}_{rs} S^{i}_{ij}$, we obtain

$$
w = \sum_{m} \omega^{b} \chi_{mn} \text{Tr}(T^{b}_{rs} \rho) + \sum_{m < n} \chi_{mn} \text{Tr}(E_{n}^{\dagger} T^{b}_{rs} E_{m} \rho) + \sum_{m < n} \chi_{mn} \text{Tr}(E_{n}^{\dagger} T^{b}_{rs} E_{m} \rho) + \sum_{m < n} \chi_{mn} \text{Tr}(E_{n}^{\dagger} T^{b}_{rs} E_{m} \rho)
$$

$$
+ \sum_{m < n} \chi_{mn} \text{Tr}(E_{n}^{\dagger} T^{b}_{rs} E_{m} \rho) + \sum_{m < n} \chi_{mn} \text{Tr}(E_{n}^{\dagger} T^{b}_{rs} E_{m} \rho) + \sum_{m < n} \chi_{mn} \text{Tr}(E_{n}^{\dagger} T^{b}_{rs} E_{m} \rho)
$$

$$
w' = \sum_{m} \omega^{b} \chi_{mn} \text{Tr}(T^{b}_{rs} \rho) + \sum_{m < n} \chi_{mn} \text{Tr}(E_{n}^{\dagger} T^{b}_{rs} E_{m} \rho) + \sum_{m < n} \chi_{mn} \text{Tr}(E_{n}^{\dagger} T^{b}_{rs} E_{m} \rho) + \sum_{m < n} \chi_{mn} \text{Tr}(E_{n}^{\dagger} T^{b}_{rs} E_{m} \rho)
$$

$$
+ \sum_{m < n} \chi_{mn} \text{Tr}(E_{n}^{\dagger} T^{b}_{rs} E_{m} \rho) + \sum_{m < n} \chi_{mn} \text{Tr}(E_{n}^{\dagger} T^{b}_{rs} E_{m} \rho) + \sum_{m < n} \chi_{mn} \text{Tr}(E_{n}^{\dagger} T^{b}_{rs} E_{m} \rho)
$$

Thus, we have $w' = \omega^{(b-n)k} w$, and consequently the measurements of operators from other cosets $A_i/C_{w'}$ do not provide any new information about $\chi_{mn}$ beyond the corresponding measurements from the coset $A_i/C_{w}$.

For another proof of the optimality, based on fundamental limitation of transferring information between two parties given by the Holevo bound see Ref. [26]. In principle, one can construct a set of non-Abelian normalizer measurements, different from $A_i$, where $v=1,2,\ldots,d+1$, to obtain information about the off-diagonal elements $\chi_{mn}$. However, determining the eigenvalues of a set of noncommuting operators cannot be done via a single measurement. Moreover, as mentioned above, by measuring the stabilizer and $d-1$ commuting normalizer elements, one can in principle transfer log2 $d^2$ bits of classical information between two parties, which is the maximum allowed by the Holevo bound [31]. Therefore, other strategies involving non-Abelian, or a mixture of Abelian and non-Abelian normalizer measurements, cannot improve the above scheme. It should be noted that there are several possible alternative sets of Abelian normalizers that are equivalent for this task. We address this issue in the next lemma.

**Lemma 10.** The number of alternative sets of Abelian normalizer measurements that can provide optimal information about quantum dynamics, in one ensemble measurement, is $d^2$.

**Proof.** We have $d+1$ Abelian normalizers $A_i$ (see Lemma 5). However, there are $d$ of them that contain unitary operators that act nontrivially on both qudit systems $A$ and $B$, i.e., $T^{b}_{rs} = (E_{i}^{A} E_{j}^{B})^{b}$, where $E_{i}^{A} \neq I$, $E_{j}^{B} \neq I$. Moreover, in each $A_i$, we have $d$ cosets (see Lemma 5) that can be used for optimal characterization of $\chi_{mn}$. Overall, we have $d^3$ possible sets of Abelian normalizers that are equivalent for our purpose.

In the next section, we develop the algorithm further to obtain complete information about the off-diagonal elements of the superoperator by repeating the above scheme for different input states.

VII. REPEATING THE ALGORITHM FOR OTHER STABILIZER STATES

We have shown that by performing one ensemble measurement one can obtain $d^2$ linearly independent equations for $\chi_{mn}$. However, a complete characterization of quantum dynamics requires obtaining $d^2-d^2$ independent real parameters of the superoperator (or $d^4$ for nontrace preserving maps). We next show how one can obtain complete information by appropriately rotating the input state and repeating the above algorithm for a complete set of rotations.

**Lemma 11.** The number of independent eigenkets for the error operator basis $\{E_{j}\}$, where $j=1,2,\ldots,d^2-1$, is $d+1$. These eigenkets are mutually unbiased.
Proof. We have $d^2 \chi$ unitary operators, $E_i$. We note that the operators $E_i$ for all values of $1 \leq a \leq d^2 - d$ commute and have a common eigenket. Therefore, overall we have $(d^2 - 1)/(d^2 - d) = d^2 + 1$ independent eigenkets. Moreover, it has been shown [30] that if a set of $d^2 - 1$ traceless and mutually orthogonal $d \times d$ unitary matrices can be partitioned into $d + 1$ subsets of equal size, then the $d$ unitary operators in each subset commute, then the basis of eigenvectors defined by these subsets are mutually unbiased.

Let us construct a set of $d + 1$ stabilizer operators $E_i A_j$, such that the following conditions hold: (a) $E_i A_j E_i^\dagger \neq I$, (b) $(E_i A_j)^{a-b} = E_i$ for $i \neq a$, and $1 \leq a \leq d - 1$. Then, by preparing the eigenstates of these $d + 1$ independent stabilizer operators, one at a time, and measuring the eigenvalues of $S_{ji}$ and its corresponding $d - 1$ normalizer operators $T_{ij}^a S_{ij}^a A_i / C_a$, one can obtain $(d + 1)d^2$ linearly independent equations to characterize the superoperator’s off-diagonal elements. The linear independence of these equations can be understood by noting that the eigenstates of all operators $E_i A_j$ of the $d + 1$ stabilizer operator $S_{ji}$ are mutually unbiased (i.e., the measurements in these unbiased bases are maximally noncommuting). For example, the bases $\{|0\rangle, |1\rangle\}$, $\{|\pm\rangle, |\mp\rangle\}$ and $\{+\rangle, Y\rangle$ (the eigenstates of the Pauli operators $Z$, $X$, and $Y$) are mutually unbiased, i.e., the inner products of each pair of elements in these bases have the same magnitude. Then measurements in these bases are maximally noncommuting [32].

To obtain complete information about the quantum dynamical coherence, we again prepare the eigenkets of the above $d + 1$ stabilizer operators $E_i A_j$, but after the stabilizer measurement we calculate the expectation values of the operators $T_{ij}^\dagger S_{ij}^\dagger$ belonging to other Abelian subgroups $A_i / C_a$ of the normalizer, i.e., $A_i \neq A_j$. According to Lemma 6 the bases of different Abelian subgroups of the normalizer are mutually unbiased, therefore, the expectation values of $T_{ij}^\dagger S_{ij}^\dagger$ and $T_{kl}^\dagger S_{kl}^\dagger$ from different Abelian subgroups $A_i$ and $A_j$ are independent. In order to make the stabilizer measurements also independent we choose a different superposition of logical basis in the preparation of $d + 1$ possible stabilizer state in each run. Therefore in each of these measurements we can obtain at most $d^2$ linearly independent equations. By repeating these measurements for $d - 1$ different $A_i$ all $d + 1$ possible input stabilizer states, we obtain $(d + 1)(d - 1)d^2 = d^2 - d^2$ linearly independent equations, which suffice to fully characterize all independent parameters of the superoperator’s off-diagonal elements. In the next section, we address the general properties of these $d + 1$ stabilizer states.

VIII. GENERAL CONSTRAINTS ON THE STABILIZER STATES

The restrictions on the stabilizer states $\rho$ can be expressed as follows:

Condition 1. The state $\rho = |\phi_i \rangle \langle \phi_i |$ is a nonseparable pure state in the Hilbert space of the two-qubit system $\mathcal{H}$. That is, $|\phi_i \rangle \langle \phi_i | \neq |\phi_i \rangle \langle \phi_i | \otimes |\phi_i \rangle \langle \phi_i |$.

Condition 2. The state $|\phi_i \rangle$ is a stabilizer state with a sole stabilizer generator $S_{ji} = E_i^\dagger A_j^\dagger$. That is, it satisfies $S_{ji} |\phi_i \rangle = \omega^a |\phi_i \rangle$, where $k \in \{0, 1, \ldots, d - 1\}$ denotes a fixed eigenvalue of $S_{ji}$, and $a = 1, \ldots, d - 1$ labels $d - 1$ nontrivial members of the stabilizer group.

The second condition specifies the stabilizer subspace, $V_S$, that the state $\rho$ lives in, which is the subspace fixed by all the elements of the stabilizer group with fixed eigenvalues $k$. More specifically, an arbitrary state in the entire Hilbert space $\mathcal{H}$ can be written as $|\phi\rangle = \sum_{a=0}^{d^2-1} \alpha_{a} |u_{a}| |u_{a}\rangle$ where $|\{u\rangle\}$ and $|\{u\rangle\}$ are bases for the Hilbert spaces of the qudits $A$ and $B$, such that $X_{a} |\langle u_{a}| = |u_{a}+\rangle\rangle$ and $Z_{a} |\langle u_{a}| = \omega^{a\langle u_{a}|\rangle |(u_{a})\rangle}$. However, we can expand $|\phi\rangle$ in another basis as $|\phi\rangle = \sum_{a=0}^{d^2-1} \beta_{a} \langle v_{a}| |v_{a}\rangle$, such that $X_{a} |\langle v_{a}| = \omega^{a\langle v_{a}|\rangle |(v_{a})\rangle}$ and $Z_{a} |\langle v_{a}| = |(v_{a})\rangle $. Let us consider a stabilizer state fixed under the action of a unitary operator $U_{i} A_{j} = (X_{i}^a) (X_{j}^b) (Z_{i}^p) (Z_{j}^p) |\phi\rangle$ with eigenvalue $\omega^k$. Regardless of the basis chosen to expand $|\phi\rangle$, we should always have $S_{ji} |\phi\rangle = \omega^k |\phi\rangle$. Consequently, we have the constraints $pu \langle p^* | k = k$, for the stabilizer subspace of $V_S$ spanned by the $|\{u\rangle\}$ and $|\{u\rangle\}$ bases, where $p$ is addition of $d$. From these relations, and also using the fact that the bases $|\{u\rangle\}$ and $|\{u\rangle\}$ are related by a unitary transformation, one can find the general properties of $V_S$ for a given stabilizer generator $E_i A_j$ and a given $k$.

We have already shown that the stabilizer states $\rho$ should also satisfy the set of conditions $\text{Tr} (\rho E_i A_j) \neq 0$ and $\text{Tr} (\rho E_i A_j E_i A_j) \neq 0$ for all operators $E_i A_j$ belonging to the same $W_k$, where $W_k (0 < b \leq d - 1)$ are the members of a particular coset $A_i / C_a$ of an Abelian subgroup, $A_i$, of the normalizer $\mathcal{N}(S)$. These relations can be expressed more compactly as follows:

Condition 3. For stabilizer state $\rho = |\phi_i \rangle \langle \phi_i | \neq |\phi_i \rangle \langle \phi_i |$ and for all $E_i A_j \neq W_k$, we have

$$\langle \phi_i | E_i A_j E_i A_j \langle \phi_i \rangle = 0,$$

where $0 \leq b \leq d - 1$.

Before developing the implications of the above formula for the stabilizer states we give the following definition and lemma.

Definition 3. Let $|\{\rangle\}$ be the logical basis of the code space that is fixed by the stabilizer generator $E_i A_j$. The stabilizer state in that basis can be written as $|\phi_i \rangle = \sum_{a=0}^{d^2-1} \alpha_{a} |\{a\rangle\}$, and all the normalizer operators, $T_{un}$, can be generated from tensor products of logical basis $X$ and $Z$ defined as $X = Z \otimes I$, and $X = I \otimes X$, where $X|k\rangle = |k+1\rangle$ and $Z|k\rangle = \omega^k |k\rangle$.

Lemma 12. For a stabilizer generator $E_i A_j$ and all unitary operators $E_i A_j \neq W_k$, we always have $E_i A_j E_i A_j = |\phi_i \rangle \langle \phi_i |$, where $Z$ is the logical $Z$ operation acting on the code space and $a$ and $c$ are integers.

Proof. Let us consider $E_i A_j = X_{i}^a Z_{j}^a$, and two generic operators $E_{i}^a$ and $E_{j}^a$ that belong to $W_k^a$, $E_{i}^a = X_{i}^a Z_{j}^a = X_{i}^a Z_{j}^a$. From the definition of $W_k^a$ (see Definition 1) we have two equations to get $g_{m} q_{n} = g_{m} q_{n} / (k+t)$ and $p_{m} p_{n} = p_{m} p_{n} / (k+t)$. We also define $p_{m} p_{n} = p_{m} p_{n} = k+t$. Therefore, we obtain $g_{m} q_{n} = q_{a} a$ and $p_{m} p_{n} = p_{a}$, where we have introduced
Moreover, we have $E^A_{n} = X'^{d-q_m}Z'^{d-p_q}p_q$ for some other integer $t''$. Then we get $E^A_{n} = X'^{d-q_m}Z'^{d-p_q}p_q$ $= X'^{d-q_m}Z'^{d-p_q}p_q$ $= (X'^{d}Z^{d})^{c}$, where $c=(t''d-p_q)(t''d+q_m-p_q)$. However, $X'^{d}Z^{d}I$ acts as logical $Z$ on the code subspace, which is the eigenstate of $E^A_{n}$. Thus, we obtain $E^A_{n} = (X'^{d}Z^{d})^{c}$.

Based on the above lemma, for the case of $b=0$ we obtain $\langle \phi_i | E^A_{n} | \phi_i \rangle = \omega^{\frac{d}{2}} \langle \phi_i | Z^{d} | \phi_i \rangle = \omega^{\frac{d}{2}} \sum_{j=0}^{d-1} 2^{-d} | \alpha_i |^2$. Therefore, our constraint in this case becomes $\sum_{j=0}^{d-1} \omega^{\frac{d}{2}} | \alpha_i |^2 \neq 0$, which is not satisfied if the stabilizer state is maximally entangled. For $b \neq 0$, we note that $T^b_\alpha$ are in fact the normalizers. By considering the general form of the normalizer elements as $T^b_\alpha = (X'^{d}Z^{d})^{c}$. If the stabilizer code is nondegenerate each of these errors should correspond to an orthogonal $d^d$-dimensional subspace; but if the code is uniformly $g$-fold degenerate (i.e., with respect to all possible errors), then each set of $g$ errors can be fit into an orthogonal $d^d$-dimensional subspace. All these subspaces must be fit into the entire $d^d$-dimensional Hilbert space. This leads to the following inequality:

$$\sum_{j=0}^{d-1} \binom{n_j}{d-1} \frac{d^d}{g} \leq d^n.$$

We are always interested in finding the errors on one physical qubit. Therefore, we have $n_j=1$, $j=0,1$ and $\binom{d}{d-1}=1$, and each $\binom{d}{d-1}=1$ is automatically satisfied. In order to characterize diagonal elements, we use a nondegenerate stabilizer code with $n=2$, $g=1$, and we have $\sum_{j=0}^{d-1} \frac{d^d}{g} \leq d^n$. For off-diagonal elements, we use a degenerate stabilizer code with $n=2$, $g=1$, and $d=2$, and we have $\sum_{j=0}^{d-1} \frac{d^d}{g} \leq d^2$. Therefore, in both cases the upper-bound of the quantum Hamming bound is satisfied by our codes. Note that if instead we use $n=k$, i.e., if we encode $n$ logical qubits into $k$ separable physical qudits, we get $\sum_{j=0}^{d-1} \frac{d^d}{g} \leq 1$. This can only be satisfied if $g=d^2$, in which case we cannot obtain any information about the errors. The above argument justifies condition (i) of the stabilizer state being nonseparable. Specifically, it explains why alternative encodings such as $n=k=2$ and $n=k=1$ are excluded from our discussions. However, if we encode zero logical qubits into one physical qubit, i.e., $n=1$, $k=0$, then, by using a $d$-fold degenerate code, we can obtain $\sum_{j=0}^{d-1} \frac{d^d}{g} = d^2$ which satisfies the quantum Hamming bound and could be useful for characterizing off-diagonal elements. For this to be true, the code $| \phi_j \rangle$ should also satisfy the set of conditions $\langle \phi_i | E^A_{n} | \phi_i \rangle \neq 0$ and $\langle \phi_i | E^A_{n} | T^b_\alpha \phi_i \rangle \neq 0$. Due to the $d$-fold degeneracy of the code, the condition $\langle \phi_i | E^A_{n} | \phi_i \rangle \neq 0$ is automatically satisfied. However, the condition $\langle \phi_i | E^A_{n} | T^b_\alpha \phi_i \rangle \neq 0$ can never be satisfied, since the code space is one-dimensional, i.e., $d^d=1$, and the normalizer operators cannot be defined. That is, there does not exist any nontrivial unitary operator $T^b_\alpha$ that can perform logical operations on the one-dimensional code space.

We have demonstrated how we can characterize quantum dynamics using the most general form of the relevant stabilizer states and generators. In the next section, we choose a...
standard form of stabilizers, in order to simplify the algorithm and to derive a standard form of the normalizer.

X. STANDARD FORM OF STABILIZER AND NORMALIZER OPERATORS

Let us choose the set \( \{ |0\rangle, |1\rangle, \ldots, |k-1\rangle \} \) as a standard basis, such that \( Z|k\rangle = \omega^k |k\rangle \) and \( X|k\rangle = |k+1\rangle \). In order to characterize the quantum dynamical population, we choose the standard stabilizer generators to be \((X^a Z^b)^d\) and \((Z^a X^b)^d\). Therefore, the maximally entangled input states can be written as \( |\phi\rangle = \frac{1}{\sqrt{2}} \sum_{i=0}^{d-1} |i\rangle |A\rangle |B\rangle \). In order to characterize the quantum dynamical coherence we choose the sole stabilizer operator as \( E_i (E_i^*)^{d-1} \), which has an eigenket of the form \( |\phi\rangle = \sum_{i=0}^{d-1} \alpha_i |i\rangle |A\rangle |B\rangle \), where \( E_i |i\rangle = \omega_i |i\rangle \) and \( |\phi\rangle \) represents one of \( d+1 \) mutually unbiased basis states in the Hilbert space of one qubit. The normalizer elements can be written as \( T_{qp} = (X^P Z^Q)^d \in A_{\alpha}, \) for all \( 0 < b \leq d-1 \), where \( X = E_1 \otimes E_2 \otimes E_3 \), \( Z = E_1 \otimes E_0 \otimes I \), \( E_1 |i\rangle = |i+1\rangle \) and \( E_1 |i\rangle = \omega_i |i\rangle \), and \( A_{\alpha} / C_{\alpha} \) represents a fixed coset of a particular Abelian subgroup, \( A_{\alpha} / C_{\alpha} \) of the normalizer \( N(S) \). For example, for a stabilizer generator of the form \( E_i (E_i^*)^{d-1} \), \( Z^a X^b \), we prepare its eigenket \( |\phi\rangle = \sum_{i=0}^{d-1} \alpha_i |i\rangle |A\rangle |B\rangle \), and the normalizers become \( T_{qp} = (X^P Z^Q)^d \), where \( X = X \otimes X \) and \( Z = Z \otimes I \). Using this notation for stabilizer and the normalizer operators, we provide an overall outline for the DCQD algorithm in the next section.

XI. ALGORITHM: DIRECT CHARACTERIZATION OF QUANTUM DYNAMICS

The DCQD algorithm for the case of a qudit system is summarized as follows (see also Figs. 5 and 6):

**Inputs.** (1) An ensemble of two-qudit systems, \( A \) and \( B \), prepared in the state \( |0\rangle_A \otimes |0\rangle_B \). (2) An arbitrary unknown CP quantum dynamical map \( \mathcal{E} \), whose action can be expressed by \( \mathcal{E}(\rho) = \sum_{m=0}^{d-1} \rho_{mn} E_m \rho E_m^\dagger \), where \( \rho \) denotes the state of the primary system and the ancilla.

**Output.** \( \mathcal{E} \), given by a set of measurement outcomes in the procedures (a) and (b) below.

**Procedure (a).** Characterization of quantum dynamical population (diagonal elements \( \chi_{mn} \) of \( \mathcal{E} \)), see Fig. 5.

1. Prepare \( |\phi_0\rangle = |0\rangle_A \otimes |0\rangle_B \), a pure initial state.
2. Transform it to \( |\phi\rangle = \frac{1}{\sqrt{2}} \sum_{i=0}^{d-1} |i\rangle |A\rangle |B\rangle \), a maximally entangled state of the two qubits. This state has the stabilizer operators \( (E_i (E_i^*)^{d-1})^d \). (3) Apply the unknown quantum dynamical map to the qudit \( A \), \( \mathcal{E}(\rho) \), where \( \rho = |\phi\rangle \langle \phi| \). (4) Perform a projective measurement \( P_k \mathcal{E}(\rho) P_k \) for \( k' = 0, 1, \ldots, d-1 \), corresponding to eigenvalues of the stabilizer operators \( S \) and \( S' \) with eigenvalues \( \omega_a \) and \( \omega_b \), respectively. The number of ensemble measurements for procedure (a) is one.

**Procedure (b).** Characterization of quantum dynamical coherence (off-diagonal elements \( \chi_{mn} \) of \( \mathcal{E} \), see Fig. 6.

1. Prepare \( |\phi_0\rangle = |0\rangle_A \otimes |0\rangle_B \), a pure initial state.
2. Transform it to \( |\phi\rangle = \frac{1}{\sqrt{2}} \sum_{i=0}^{d-1} |i\rangle |A\rangle |B\rangle \), a nonmaximally entangled state of the two qubits. This state has stabilizer operators \( (E_i (E_i^*)^{d-1})^d \).

3. Apply the unknown quantum dynamical map to the qudit \( A \): \( \mathcal{E}(\rho) = \sum_{m=0}^{d-1} \rho_{mn} E_m \rho E_m^\dagger \), where \( \rho = |\phi\rangle \langle \phi| \).
4. Perform a projective measurement

\[
P_k \mathcal{E}(\rho) P_k \rightarrow P_k \mathcal{E}(\rho) P_k = \sum_m \chi_{mn} E_m \rho E_m^\dagger + \sum_m (\chi_{mn} \rho E_m \rho E_m^\dagger) + \chi_{mn} E_m \rho E_m^\dagger
\]

where \( P_k = \frac{1}{\sqrt{d-1}} \sum_{i=0}^{d-1} \omega^{-ik} (E_i (E_i^*)^{d-1})^d \), and \( \chi_{mn} \) and \( \chi_{mn} \rho \) are calculated as the joint probability distributions of the outcomes \( k \) and \( k' \).

\[
\text{Tr}[P_k \mathcal{E}(\rho)] = \chi_{mn}.
\]

Number of ensemble measurements for procedure (a), 1.

**FIG. 5.** (Color online) Procedure (a): Measuring the quantum dynamical population (diagonal elements \( \chi_{mn} \)). The arrows indicate direction of time. (1) Prepare \( |\phi_0\rangle = |0\rangle_A \otimes |0\rangle_B \), a pure initial state. (2) Transform it to \( |\phi\rangle = \frac{1}{\sqrt{2}} \sum_{i=0}^{d-1} |i\rangle |A\rangle |B\rangle \), a maximally entangled state of the two qubits. This state has the stabilizer operators \( S = X^a Z^b \) and \( S' = Z^a X^b \). (3) Apply the unknown quantum dynamical map to the qudit \( A \), \( \mathcal{E}(\rho) \), where \( \rho = |\phi\rangle \langle \phi| \). (4) Perform a projective measurement \( P_k \mathcal{E}(\rho) \) for \( k, k' = 0, 1, \ldots, d-1 \), corresponding to eigenvalues of the stabilizer operators \( S \) and \( S' \) with eigenvalues \( \omega_a \) and \( \omega_b \), respectively. The number of ensemble measurements for procedure (a) is one.

**FIG. 6.** (Color online) Procedure (b): Measuring the quantum dynamical coherence (off-diagonal elements \( \chi_{mn} \)).
when \( d \) is prime. In the Appendix, we demonstrate that the DCQD algorithm can be generalized to other \( N \)-dimensional quantum systems with \( N \) being a power of a prime.

**XII. SUMMARY**

For convenience, we provide a summary of the DCQD algorithm. The DCQD algorithm for a qudit, with \( d \) being a prime, was developed by utilizing the concept of an error operator basis. An arbitrary operator acting on a qudit can be expanded over an orthonormal and unitary operator basis \( \{E_0, E_1, \ldots, E_{d^2-1}\} \), where \( E_0 = I \) and \( \text{tr}(E_i E_j) = d \delta_{ij} \). Any element \( E_i \) can be generated from tensor products of \( X \) and \( Z \), where \( X |k \rangle = |k+1 \rangle \) and \( Z |k \rangle = \omega^k |k \rangle \), such that the relation \( XYZ = \omega^{-1} ZX \) is satisfied [28]. Here \( \omega \) is a \( d \)-th root of unity and \( X \) and \( Z \) are the generalizations of Pauli operators to higher dimension.

Characterization of dynamical population. A measurement scheme for determining the quantum dynamical population, \( \chi_{mm} \), in a single experimental configuration. Let us prepare a maximally entangled state of the two qudits \( |\phi \rangle = \frac{1}{\sqrt{d}} \sum_k |k \rangle |k \rangle \). This state is stabilized under the action of stabilizer operators \( S = X^i Z^j \) and \( S' = Z^l (Z^b)^{-1} \), and it is referred to as a stabilizer state [1,28]. After applying the quantum map to the qudit \( A, \mathcal{E}(\rho) \), where \( \rho = |\phi \rangle \langle \phi | \), we can perform a projective measurement \( P_i P_k \mathcal{E}(\rho) \mathcal{P}_k \mathcal{P}_i \), where \( P_k = \frac{1}{d} \sum_{l=0}^{d-1} \omega^{-l k} |l \rangle \langle l | \). This state is stabilized under the action of stabilizer operators \( S = X^i Z^j \) and \( S' = Z^l (Z^b)^{-1} \), and \( \omega = e^{2\pi i/d} \). Then, we calculate the joint probability distributions of the outcomes \( k \) and \( k' \): \( \text{Tr}[P_k P_{k'} \mathcal{E}(\rho)] \equiv \chi_{mm} \), where the elements \( \chi_{mm} \) represent the population of error operators that anticommute with stabilizer generators \( S \) and \( S' \) with eigenvalues \( \omega^k \) and \( \omega^{k'} \), respectively. Therefore, with a single experimental configuration we can identify all diagonal elements of the superoperator.

Characterization of dynamical coherence. For measuring the quantum dynamical coherence, we create a maximally entangled state of the two qudits \( |\phi \rangle = \frac{1}{\sqrt{d}} \sum_k |k \rangle |k \rangle \). This state has the sole stabilizer operator \( S = E_i^0 (E_j^0)^{d-1} \) (for detailed restrictions on the coefficients \( \alpha_i \) see Sec. VIII). After applying the dynamical map to the qudit \( A, \mathcal{E}(\rho) \), we perform a projective measurement \( P_i = P_j \mathcal{E}(\rho) P_k \), and calculate the probability of the outcome \( k \), \( \text{Tr}[P_k \mathcal{E}(\rho)] = \sum_m \chi_{mm} + 2\sum_{m<n} \text{Re}[\chi_{mm} \text{Tr}(E_i^m E_j^n \mathcal{E}(\rho))] \), where \( E_i^m \) are all the operators in the operator basis, \( \{E_i^m\} \), that anticommute with the operator \( E_i^0 \) with the same eigenvalue \( \omega^k \). We also measure the expectation values of all independent operators \( T_{m,n} = E_i^m \mathcal{E}(\rho) \) of the Pauli group (where \( E_i^0 \neq I \) and \( E_i^m \neq I \)) that simultaneously commute with the stabilizer generator \( S \), \( \text{Tr}(T_{m,n} \rho_0) \). There are only \( d-1 \) such operators \( T_{m,n} \) that are independent of each other, within a multiplication by a stabilizer generator; and they belong to an Abelian subgroup of the normalizer group. The normalizer group is the group of unitary operators that preserve the stabilizer group by conjugation, i.e., \( TST^\dagger = S \). We repeat this procedure \( d+1 \) times, by preparing the eigenvalues of other stabilizer operator \( E_i^s (E_j^0)^{d-1} \) for all \( i \in \{1, 2, \ldots, d+1\} \), such that states \( |\alpha_i \rangle \) in input states belong to a mutually unbiased basis [32]. Also, we can change the measurement basis \( d-1 \) times, each time choosing normal-
izer elements $T_k$ from a different Abelian subgroup of the normalizer, such that their eigenstates form a mutually unbiased basis in the code space. Therefore, we can completely characterize quantum dynamical coherence by $(d+1)(d-1)$ different measurements, and the overall number of experimental configuration for a qudit becomes $d^2$. For $N$-dimensional quantum systems, with $N$ a power of a prime, the required measurements are simply the tensor product of the corresponding measurements on individual qudits—see the Appendix. For quantum system whose dimension is not a power of a prime, the task can be accomplished by embedding the system in a larger Hilbert space whose dimension is a prime.

XIII. OUTLOOK

An important and promising advantage of DCQD is for use in partial characterization of quantum dynamics, where in general, one cannot afford or does not need to carry out a full characterization of the quantum system under study, or when one has some \textit{a priori} knowledge about the dynamics. Using indirect methods of QPT in those situations is inefficient, because one must apply the whole machinery of the scheme to obtain the relevant information about the system. On the other hand, the DCQD scheme has built-in applicability to the task of partial characterization of quantum dynamics. In general, one can substantially reduce the overall number of measurements, when estimating the coherence elements of the superoperator for only specific subsets of the operator basis and/or subsystems of interest. This fact has been demonstrated in Ref. \cite{26} in a generic fashion, and several examples of partial characterization have also been presented. Specifically, it was shown that DCQD can be applied to (single- and two-qubit) Hamiltonian identification tasks. Moreover, it is demonstrated that the DCQD algorithm enables the simultaneous determination of coarse-grained (semiclassical) physical quantities, such as the longitudinal relaxation time $T_1$ and the transversal relaxation (or dephasing) time $T_2$ for a single qubit undergoing a general homogenizing quantum map. The DCQD scheme can also be used for performing generalized quantum dense coding tasks. Other implications and applications of DCQD for partial QPT remain to be investigated and explored.

An alternative representation of the DCQD scheme for higher-dimensional quantum systems, based on generalized Bell-state measurements will be presented in Ref. \cite{33}. The connection of Bell-state measurements to stabilizer and normalizer measurements in DCQD for two-level systems, can be easily observed from Table II of Ref. \cite{3}. Our presentation of the DCQD algorithm assumes ideal (i.e., error-free) quantum state preparation, measurement, and ancilla channels. However, these assumptions can all be relaxed in certain situations, in particular when the imperfections are already known. A discussion of these issues is beyond the scope of this work and will be the subject of a future presentation \cite{33}.

There are a number of other directions in which the results presented here can be extended. One can combine the DCQD algorithm with the method of maximum likelihood estimation \cite{35}, in order to minimize the statistical errors in each experimental configuration invoked in this scheme. Moreover, a new scheme for \textit{continuous} characterization of quantum dynamics can be introduced, by utilizing weak measurements for the required quantum error detections in DCQD \cite{36,37}. Finally, the general techniques developed for direct characterization of quantum dynamics could be further utilized for control of open quantum systems \cite{38}.

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APPENDIX: GENERALIZATION TO ARBITRARY OPEN QUANTUM SYSTEMS

Here, we first demonstrate that the overall measurements for a full characterization of the dynamics of an $n$ qudit systems (with $d$ being a prime) become the tensor product of the required measurements on individual qudits. One of the important examples of such systems is a QIP unit with $r$ qubits, thus having a $2^r$-dimensional Hilbert space. Let us consider a quantum system consisting of $r$ qudits, $\rho=\rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_r$, with a Hilbert space of dimension $N=d^r$. The output state of such a system after a dynamical map becomes $\bar{E}(\rho) = \sum_{m,n=0}^{N^2-1} X_{mn} E_m^\dagger E_n$, where here $\{E_m\}$ are the unitary operator basis elements of an $N$-dimensional Hilbert space. These unitary operator basis elements can be written as $E_m = X^{m_1} Z^{m_2} \cdots X^{m_n} Z^{m_n}$. \cite{34}. Therefore, we have

$$\bar{E}(\rho) = \sum_{m,n=0}^{N^2-1} X_{mn} (X^{m_1} Z^{m_2} \cdots X^{m_n} Z^{m_n}) \rho_1 \otimes \cdots \otimes \rho_r (X^{m_1} Z^{m_2} \cdots X^{m_n} Z^{m_n})^\dagger$$

$$= \sum_{m_1, \ldots, m_r, n_1, \ldots, n_r=0}^{d^2-1} X(m_1, \ldots, m_r)(E_m^\dagger E_{n_1})^\dagger \otimes \cdots \otimes (E_m^\dagger E_{n_r})^\dagger$$

$$= \sum_{m_1, \ldots, m_r, n_1, \ldots, n_r=0}^{d^2-1} X(m_1, \ldots, m_r)(E_m^\dagger E_{n_1})^\dagger \otimes \cdots \otimes (E_m^\dagger E_{n_r})^\dagger$$

where we have introduced $E_m = X^{m_1} Z^{m_2} \cdots X^{m_n} Z^{m_n}$ and $X_{mn} = \chi_{m_1, \ldots, m_r, n_1, \ldots, n_r}$. I.e., $m=(m_1, \ldots, m_r)$ and $n=(n_1, \ldots, n_r)$, and the index $s$ represents a generic qudit. Let us first investigate the tensor product structure of the DCQD algorithm for characterization of the diagonal elements of the superoperator. We prepare the eigenstate of the stabilizer operators $(E_i^A E_j^B)^\dagger \otimes \rho$ and $(E_i^A E_j^B)^\dagger \otimes \rho$. For each qudit, the projection operators corresponding to outcomes $\omega_k$ and $\omega_k^{\perp}$ (where $k, k^{\perp}=0, 1, \ldots, d-1$), have the form $P_k$.
Using the QEC condition for nondegenerate codes, the joint probability distribution of the commuting Hermitian operators $P_{k_1}, P_{k_1'}, P_{k_2}, P_{k_2'}, \ldots, P_{k_s}, P_{k_s'}$ on the output state $\mathcal{E}(\rho)$ is

$$\text{Tr}[(P_k P_{k'})_\mathcal{E}(\rho)] = \sum_{m_1, \ldots, m_n=0}^{d-1} \sum_{n_1, \ldots, n_q=0}^{d-1} \omega^{i_1 m_1 \ldots i_q m_q} \text{Tr}[E_{m_1}^\rho(E_{m_1'}^\rho E_{m_1}^\rho)]_{s}$$

for each qudit, and using the relation $[E_{m_1}^\rho E_{m_1'}^\rho E_{m_1}^\rho]_{s} = \delta_{m_1 m_1'}$ where for each qudit, the index $m_0$ is defined through the relations $i_0 = k$ and $i'_0 = k'$, etc. I.e., $E_{m_0}$ is the unique error operator that anticommutes with the stabilizer operators of each qudit with a fixed pair of eigenvalues $\omega^k$ and $\omega^{k'}$ corresponding to experimental outcomes $k$ and $k'$. Since $P_k$ and $P_{k'}$ operator have $d$ eigenvalues, we have $d^2$ possible outcomes for each qudit, which overall yields $(d^2)^n$ equations that can be used to characterize all the diagonal elements of the superoperator with a single ensemble measurement and $(2d)^n$ detectors. Similarly, the off-diagonal elements of superoperators can be identified by a tensor product of the operations in the DCQD algorithm for each individual qudit, see Ref. [26]. A comparison of the required physical resources for $n$ qudits is given in Table I.

For a $d$-dimensional quantum system where $d$ is neither a prime nor a power of a prime, we can always imagine another $d'$-dimensional quantum system such that $d'$ is prime, and embed the principal qudit as a subspace into that system. For example, the energy levels of a six-level quantum system can be always regarded as the first six energy levels of a virtual seven-level quantum system, such that the matrix elements for coupling to the seventh level are practically zero. Then, by considering the algorithm for characterization of the virtual seven-level system, we can perform only the measurements required to characterize superoperator elements associated with the first six energy levels.
DIRECTIONS OF QUANTUM DYNAMICS: ...