Direct characterization of quantum dynamics: General theory

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The characterization of the dynamics of quantum systems is a task of both fundamental and practical importance. A general class of methods which have been developed in quantum information theory to accomplish this task is known as quantum process tomography (QPT). In an earlier paper [M. Mohseni and D. A. Lidar Phys. Rev. Lett. **97**, 170501 (2006)] we presented an algorithm for direct characterization of quantum dynamics (DCQD) of two-level quantum systems. Here we provide a generalization by developing a theory for direct and complete characterization of the dynamics of arbitrary quantum systems. In contrast to other QPT schemes, DCQD relies on quantum error-detection techniques and does not require any quantum state tomography. We demonstrate that for the full characterization of the dynamics of n d-level quantum systems (with d prime), the minimal number of required experimental configurations is reduced quadratically from d^{4n} in separable QPT schemes to d^{2n} in DCQD.

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I. INTRODUCTION

The characterization of quantum dynamical systems is a fundamental problem in quantum physics and quantum chemistry. Its ubiquity is due to the fact that knowledge of quantum dynamics of (open or closed) quantum systems is indispensable in prediction of experimental outcomes. In particular, accurate estimation of an unknown quantum dynamical process acting on a quantum system is a pivotal task in coherent control of the dynamics, especially in verifying and/or monitoring the performance of a quantum device in the presence of decoherence. The procedures for characterization of quantum dynamical maps are traditionally known as quantum process tomography (QPT) [1–3].

In most QPT schemes the information about the quantum dynamical process is obtained indirectly. The quantum dynamics is first mapped onto the state(s) of an ensemble of probe quantum systems, and then the process is reconstructed via quantum state tomography of the output states. Quantum state tomography is itself a procedure for identifying a quantum system by measuring the expectation values of a set of noncommuting observables on identical copies of the system. There are two general types of QPT schemes. The first is standard quantum process tomography (SQPT) [1,4,5]. In SQPT all quantum operations, including preparation and (state tomography) measurements, are performed on the system whose dynamics is to be identified (the "principal" system), without the use of any ancillas. The SQPT scheme has already been experimentally demonstrated in a variety of systems including liquid-state nuclear magnetic resonance (NMR) [6–8], optical [9,10], atomic [11], and solid-state systems [12]. The second type of QPT scheme is known as ancilla-assisted process tomography (AAPT) [13–16]. In AAPT one makes use of an ancilla (auxilliary system). First, the combined principal system and ancilla are prepared in a "faithful" state, with the property that all information about the dynamics can be imprinted on the final state [13,15,16]. The relevant information is then extracted by performing quantum state tomography in the joint Hilbert space of system and ancilla. The AAPT scheme has also been demonstrated experimentally [15,17]. The total number of experimental configurations required for measuring the quantum dynamics of n d-level quantum systems (qudits) is d^{4n} for both SQPT and separable AAPT, where separable refers to the measurements performed at the end. This number can in principle be reduced by utilizing nonseparable measurements, e.g., a generalized measurement [1]. However, the nonseparable QPT schemes are rather impractical in physical applications because they require many-body interactions, which are not experimentally available or must be simulated at high resource cost [3].

Both SQPT and AAPT make use of a mapping of the dynamics onto a state. This raises the natural question of whether it is possible to avoid such a mapping and instead perform a *direct* measurement of quantum dynamics, which does not require any state tomography. Moreover, it seems reasonable that by avoiding the indirect mapping one should be able to attain a reduction in resource use (e.g., the total number of measurements required), by eliminating redundancies. Indeed, there has been a growing interest in the development of direct methods for obtaining specific information about the states or dynamics of quantum systems. Examples include the estimation of general functions of a quantum state [18], detection of quantum entanglement [19], measurement of nonlinear properties of bipartite quantum states [20], reconstruction of quantum states or dynamics from incomplete measurements [21], estimation of the average fidelity of a quantum gate or process [22,23], and universal source coding and data compression [24]. However, these schemes cannot be used directly for a *complete* characterization of quantum dynamics. In Ref. [25] we presented such a scheme, which we called "direct characterization of quantum dynamics" (DCQD).

In trying to address the problem of *direct* and *complete* characterization of quantum dynamics, we were inspired by the observation that quantum error detection (QED) [1] provides a means to directly obtain partial information about the

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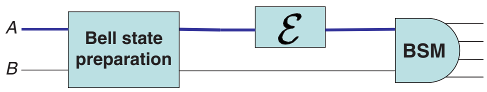

FIG. 1. (Color online) Schematic of DCQD for a single qubit, consisting of Bell-state-type preparations, application of the unknown quantum map, $\mathcal{E}$, and Bell-state measurement (BSM).

nature of a quantum process, without ever revealing the state of the system. In general, however, it is unclear if there is a fundamental relationship between QED and QPT, namely whether it is possible to completely characterize the quantum dynamics of arbitrary quantum systems using QED. And, providing the answer is affirmative, how the physical resources scale with system size. Moreover, one would like to understand whether entanglement plays a fundamental role, and what potential applications emerge from such a theory linking QPT and QED. Finally, one would hope that this approach may lead to new ways of understanding and/or controlling quantum dynamical systems. We addressed these questions for the first time in Ref. [25] by developing the DCQD algorithm in the context of two-level quantum systems. In DCQD—see Fig. 1—the state space of an ancilla is utilized such that experimental outcomes from a Bell-state measurement provide direct information about specific properties of the underlying dynamics. A complete set of probe states is then used to fully characterize the unknown quantum dynamics via application of a single Bell-state measurement device [3,25].

Here we generalize the theory of Ref. [25] to arbitrary open quantum systems undergoing an unknown, completely positive (CP) quantum dynamical map. In the generalized DCQD scheme, each probe qudit (with d prime) is initially entangled with an ancillary qudit system of the same dimension, before being subjected to the unknown quantum process. To extract the relevant information, the corresponding measurements are devised in such a way that the final (joint) probability distributions of the outcomes are directly related to specific sets of the dynamical superoperator's elements. A complete set of probe states can then be utilized to fully characterize the unknown quantum dynamical map. The preparation of the probe systems and the measurement schemes are based on QED techniques, however, the objective and the details of the error-detection schemes are different from those appearing in the protection of quantum systems against decoherence (the original context of QED). More specifically, we develop error-detection schemes to directly measure the coherence in a quantum dynamical process, represented by off-diagonal elements of the corresponding superoperator. We explicitly demonstrate that for characterizing a dynamical map on n qudits, the number of required experimental configurations is reduced from d^{4n}, in SQPT and separable AAPT, to d^{2n} in DCQD. A useful feature of DCQD is that it can be efficiently applied to *partial* characterization of quantum dynamics [25,26]. For example, it can be used for the task of Hamiltonian identification, and also for simultaneous determination of the relaxation time T_1 and the dephasing time T_2.

This paper is organized as follows. In Sec. II, we provide a brief review of completely positive quantum dynamical maps, and the relevant QED concepts such as stabilizer codes and normalizers. In Sec. III, we demonstrate how to determine the quantum dynamical populations, or diagonal elements of a superoperator, through a single (ensemble) measurement. In order to further develop the DCQD algorithm and build the required notations, we introduce some lemmas and definitions in Sec. IV, and then we address the characterization of quantum dynamical coherences, or off-diagonal elements of a superoperator, in Sec. V. In Sec. VI, we show that measurement outcomes obtained in Sec. V provide d^2 linearly independent equations for estimating the coherences in a process, which is in fact the maximum amount of information that can be extracted in a single measurement. A complete characterization of the quantum dynamics, however, requires obtaining d^4 independent real parameters of the superoperator (for nontrace preserving maps). In Sec. VII, we demonstrate how one can obtain complete information by appropriately rotating the input state and repeating the above algorithm for a complete set of rotations. In Secs. VIII and IX, we address the general constraints on input stabilizer codes and the minimum number of physical qudits required for the encoding. In Sec. X and Sec. XI, we define a standard notation for stabilizer and normalizer measurements and then provide an outline of the DCQD algorithm for the case of a single qudit. For convenience, we provide a brief summary of the entire DCQD algorithm in Sec. XII. We conclude with an outlook in Sec. XIII. In the Appendix , we generalize the scheme for arbitrary open quantum systems. For a discussion of the experimental feasibility of DCQD see Ref. [25], and for a detailed and comprehensive comparison of the required physical resources in different QPT schemes see Ref. [3].

II. PRELIMINARIES

In this section we introduce the basic concepts and notation from the theory of open quantum system dynamics and quantum error detection, required for the generalization of the DCQD algorithm to qudits.

A. Quantum dynamics

The evolution of a quantum system (open or closed) can, under natural assumptions, be expressed in terms of a completely positive quantum dynamical map $\mathcal{E}$, which can be represented as [1]

$$\mathcal{E}(\rho) = \sum_{m,n=0}^{d^2-1} \chi_{mn} E_m \rho E_n^{\dagger}. \tag{1}$$

Here ρ is the initial state of the system, and the $\{E_m\}$ are a set of (error) operator basis elements in the Hilbert-Schmidt space of the linear operators acting on the system. That is, any arbitrary operator acting on a d-dimensional quantum system can be expanded over an orthonormal and unitary error operator basis $\{E_0, E_1, \ldots, E_{d^2-1}\}$, where $E_0 = I$ and $\mathrm{tr}(E_i^{\dagger} E_j) = d\delta_{ij}$ [27]. The $\{\chi_{mn}\}$ are the matrix elements of the superoperator $\boldsymbol{\chi}$, or "process matrix," which encodes all the information about the dynamics, relative to the basis set $\{E_m\}$

[1]. For an n-qudit system, the number of independent matrix elements in $\boldsymbol{\chi}$ is d^{4n} for a non-trace-preserving map and $d^{4n}-d^{2n}$ for a trace-preserving map. The process matrix $\boldsymbol{\chi}$ is positive and Tr $\boldsymbol{\chi} \leq 1$. Thus $\boldsymbol{\chi}$ can be thought of as a density matrix in the Hilbert-Schmidt space, when we often refer to its diagonal and off-diagonal elements as "quantum dynamical population" and "quantum dynamical coherence," respectively.

In general, any successive operation of the (error) operator basis can be expressed as $E_iE_j=\sum_k \omega^{i,j,k}E_k$, where $i,j,k=0,1,\ldots,d^2-1$. However, we use the "very nice (error) operator basis" in which $E_iE_j=\omega^{i,j}E_{i*j}$, $\det E_i=1$, $\omega^{i,j}$ is a dth root of unity, and the operation $*$ induces a group on the indices [27]. This provides a natural generalization of the Pauli group to higher dimensions. Any element E_i can be generated from appropriate products of X_d and Z_d, where $X_d|k\rangle=|k+1\rangle$, $Z_d|k\rangle=\omega^k|k\rangle$, and $X_dZ_d=\omega^{-1}Z_dX_d$ [27,28]. Therefore, for any two elements $E_{i=\{a,q,p\}}=\omega^a X_d^q Z_d^p$ and $E_{j=\{a',q',p'\}}=\omega^{a'} X_d^{q'} Z_d^{p'}$ (where $0\leq q,p<d$) of the single-qudit Pauli group, we always have

$$E_iE_j=\omega^{pq'-qp'}E_jE_i, \tag{2}$$

where

$$pq'-qp' \equiv k (\mathrm{mod}\, d). \tag{3}$$

The operators E_i and E_j commute if and only if $k=0$. Henceforth, all algebraic operations are performed in mod(d) arithmetic, and all quantum states and operators, respectively, belong to and act on a d-dimensional Hilbert space. For simplicity, from now on we drop the subscript d from the operators.

B. Quantum error detection

In the last decade the theory of quantum error correction (QEC) has been developed as a general method for detecting and correcting quantum dynamical errors acting on multiqubit systems such as a quantum computer [1]. QEC consists of three steps: preparation, quantum error detection (QED) or syndrome measurements, and recovery. In the preparation step, the state of a quantum system is encoded into a subspace of a larger Hilbert space by entangling the principal system with some other quantum systems using unitary operations. This encoding is designed to allow detection of arbitrary errors on one (or more) physical qubits of a code by performing a set of QED measurements. The measurement strategy is to map different possible sets of errors only to orthogonal and undeformed subspaces of the total Hilbert space, such that the errors can be unambiguously discriminated. Finally the detected errors can be corrected by applying the required unitary operations on the physical qubits during the recovery step. A key observation relevant for our purposes is that by performing QED one can actually obtain partial information about the dynamics of an open quantum system.

For a qudit in a general state $|\phi_c\rangle$ in the code space, and for arbitrary error basis elements E_m and E_n, the Knill-Laflamme QEC condition for degenerate codes is

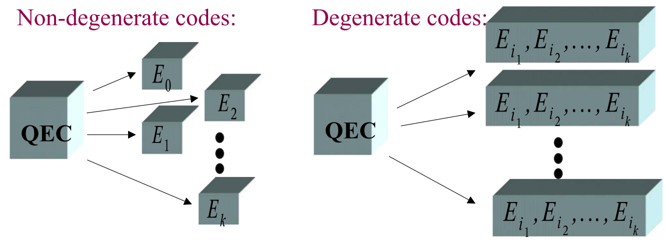

FIG. 2. (Color online) A schematic diagram of quantum error detection (QED). The projective measurements corresponding to eigenvalues of stabilizer generators are represented by arrows. For a nondegenerate QEC code, after the QED, the wave function of the multiqubit system collapses into one of the orthogonal subspaces each of which is associated with a single error operator. Therefore, all errors can be unambiguously discriminated. For degenerate codes, by performing QED the code space also collapses into a set orthogonal subspaces. However, each subspace has multiple degeneracies among k error operators in a subset of the operator basis, i.e., $\{E_m\}_{m=1}^{k} \subset \{E_i\}_{i=0}^{d^2-1}$. In this case, one cannot distinguish between different operators within a particular subset $\{E_m\}_{m=1}^{k_0}$.

$\langle\phi_c|E_n^\dagger E_m|\phi_c\rangle=\alpha_{nm}$, where α_{nm} is a Hermitian matrix of complex numbers [1]. For nondegenerate codes, the QEC condition reduces to $\langle\phi_c|E_n^\dagger E_m|\phi_c\rangle=\delta_{nm}$; i.e., in this case the errors always take the code space to orthogonal subspaces. The difference between nondegenerate and degenerate codes is illustrated in Fig. 2. In this work, we concentrate on a large class of error-correcting codes known as stabilizer codes [29]; however, in contrast to QEC, we restrict our attention almost entirely to degenerate stabilizer codes as the initial states. Moreover, by definition of our problem, the recovery and/or correction step is not needed or used in our analysis.

A stabilizer code is a subspace $\mathcal{H}_C$ of the Hilbert space of n qubits that is an eigenspace of a given Abelian subgroup $\mathcal{S}$ of the n-qubit Pauli group with the eigenvalue $+1$ [1,29]. In other words, for $|\phi_c\rangle \in \mathcal{H}_C$ and $S_i \in \mathcal{S}$, we have $S_i|\phi_c\rangle=|\phi_c\rangle$, where S_i's are the stabilizer *generators* and $[S_i,S_j]=0$. Consider the action of an arbitrary error operator E on the stabilizer code $|\phi_c\rangle$, $E|\phi_c\rangle$. The detection of such an error will be possible if the error operator anticommutes with (at least one of) the stabilizer generators, $\{S_i,E\}=0$. That is, by measuring all generators of the stabilizer and obtaining one or more negative eigenvalues we can determine the nature of the error unambiguously as

$$S_i(E|\phi_c\rangle) = -E(S_i|\phi_c\rangle) = -(E|\phi_c\rangle).$$

A stabilizer code $[n,k,d_c]$ represents an encoding of k logical qudits into n physical qudits with code distance d_c, such that an arbitrary error on any subset of $t=(d_c-1)/2$ or fewer qudits can be detected by QED measurements. A stabilizer group with $n-k$ generators has d^{n-k} elements and the code space is d^k dimensional. Note that this is valid when d is a power of a prime [28]. The unitary operators that preserve the stabilizer group by conjugation, i.e., $USU^\dagger=S$, are called the normalizer of the stabilizer group, $N(S)$. Since the normalizer elements preserve the code space they can be used to perform certain logical operations in the code space.

However, they are insufficient for performing arbitrary quantum operations [1].

Similarly to the case of a qubit [25], the DCQD algorithm for the case of a qudit system consists of two procedures: (i) a single experimental configuration for characterization of the quantum dynamical populations, and (ii) d^2-1 experimental configurations for characterization of the quantum dynamical coherences. In both procedures we always use two physical qudits for the encoding, the principal system A and the ancilla B, i.e., $n=2$. In procedure (i)–characterizing the diagonal elements of the superoperator—the stabilizer group has two generators. Therefore it has d^2 elements and the code space consists of a *single* quantum state (i.e., $k=0$). In procedure (ii)–characterizing the off-diagonal elements of the superoperator–the stabilizer group has a single generator, thus it has d elements, and the code space is two dimensional. That is, we effectively encode a logical qudit (i.e., $k=1$) into two physical qudits. In the next sections, we develop the procedures (i) and (ii) in detail for a single qudit with d being a prime, and in the Appendix we address the generalization to systems with d being an arbitrary power of a prime.

III. CHARACTERIZATION OF QUANTUM DYNAMICAL POPULATION

To characterize the diagonal elements of the superoperator, or the population of the unitary error basis, we use a nondegenerate stabilizer code. We prepare the principal qudit, A, and an ancilla qudit, B, in a common $+1$ eigenstate $|\phi_c\rangle$ of the two unitary operators $E_i^A E_j^B$ and $E_{i'}^A E_{j'}^B$, such that $[E_i^A E_j^B, E_{i'}^A E_{j'}^B]=0$ [e.g., $X^A X^B$ and $Z^A (Z^B)^{d-1}$]. Therefore, simultaneous measurement of these stabilizer generators at the end of the dynamical process reveals arbitrary single qudit errors on the system A. The possible outcomes depend on whether a specific operator in the operator-sum representation of the quantum dynamics commutes with $E_i^A E_j^B$ and $E_{i'}^A E_{j'}^B$, with the eigenvalue $+1$, or with one of the eigenvalues $\omega, \omega^2, \ldots, \omega^{d-1}$. The projection operators corresponding to outcomes ω^k and $\omega^{k'}$, where $k, k'=0,1,\ldots,d-1$, have the form $P_k = \frac{1}{d}\sum_{l=0}^{d-1}\omega^{-lk}(E_i^A E_j^B)^l$ and $P_{k'} = \frac{1}{d}\sum_{l'=0}^{d-1}\omega^{-l'k'}(E_{i'}^A E_{j'}^B)^{l'}$. The joint probability distribution of the commuting Hermitian operators P_k and $P_{k'}$ on the output state $\mathcal{E}(\rho) = \sum_{m,n}\chi_{mn}E_m \rho E_n^\dagger$, where $\rho=|\phi_c\rangle\langle\phi_c|$, is

$$\mathrm{Tr}[P_k P_{k'}\mathcal{E}(\rho)] = \frac{1}{d^2}\sum_{m,n=0}^{d^2-1}\chi_{mn}\sum_{l=0}^{d-1}\sum_{l'=0}^{d-1}\omega^{-lk}\omega^{-l'k'} \times \mathrm{Tr}[E_n^\dagger (E_i^A)^l (E_{i'}^A)^{l'} E_m (E_j^B)^l (E_{j'}^B)^{l'}\rho].$$

Using $E_i E_m = \omega^{i_m} E_m E_i$ and the relation $(E_i^A E_j^B)^l (E_{i'}^A E_{j'}^B)^{l'}\rho = \rho$, we obtain

$$\mathrm{Tr}[P_k P_{k'}\mathcal{E}(\rho)] = \frac{1}{d^2}\sum_{m,n=0}^{d^2-1}\chi_{mn}\sum_{l=0}^{d-1}\sum_{l'=0}^{d-1}\omega^{(i_m-k)l}\omega^{(i'_m-k')l'}\delta_{mn},$$

where we have used the QED condition for nondegenerate codes

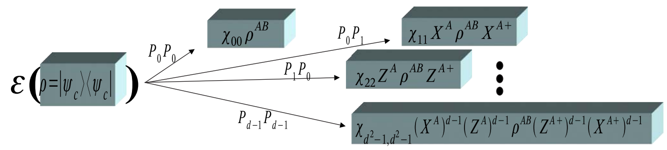

FIG. 3. (Color online) A diagram of the error-detection measurement for estimating quantum dynamical population. The arrows represent the projection operators $P_k P_{k'}$ corresponding to different eigenvalues of the two stabilizer generators S and S'. These projective measurements result in a projection of the wave function of the two-qudit systems, after experiencing the dynamical map, into one of the orthogonal subspaces each of which is associated to a specific error operator basis. By calculating the joint probability distribution of all possible outcomes, $P_k P_{k'}$, for $k, k'=0,\ldots,d$, we obtain all d^2 diagonal elements of the superoperator in a single ensemble measurement.

$$\mathrm{Tr}[E_n^\dagger E_m \rho] = \langle\phi_c|E_n^\dagger E_m|\phi_c\rangle = \delta_{mn}$$

i.e., the fact that different errors should take the code space to orthogonal subspaces, in order for errors to be unambiguously detectable, see Fig. 3. Now, using the discrete Fourier transform identities $\sum_{l=0}^{d-1}\omega^{(i_m-k)l} = d\delta_{i_m,k}$ and $\sum_{l'=0}^{d-1}\omega^{(i'_m-k')l'} = d\delta_{i'_m,k'}$, we obtain

$$\mathrm{Tr}[P_k P_{k'}\mathcal{E}(\rho)] = \sum_{m=0}^{d^2-1}\chi_{mm}\delta_{i_m,k}\delta_{i'_m,k'} = \chi_{m_0 m_0}. \tag{4}$$

Here, m_0 is defined through the relations $i_{m_0}=k$ and $i'_{m_0}=k'$, i.e., E_{m_0} is the unique error operator that anticommutes with the stabilizer operators with a fixed pair of eigenvalues ω^k and $\omega^{k'}$ corresponding to the experimental outcomes k and k'. Since each P_k and $P_{k'}$ operator has d eigenvalues, we have d^2 possible outcomes, which gives us d^2 linearly independent equations. Therefore, we can *characterize all the diagonal elements of the superoperator with a single ensemble measurement* and $2d$ detectors.

In order to investigate the properties of the pure state $|\phi_c\rangle$, we note that the code space is one dimensional (i.e., it has only one vector) and can be Schmidt decomposed as $|\phi_c\rangle = \sum_{k=0}^{d-1}\lambda_k|k\rangle_A|k\rangle_B$, where λ_k are non-negative real numbers. Suppose $Z|k\rangle=\omega^k|k\rangle$; without loss of generality the two stabilizer generators of $|\phi_c\rangle$ can be chosen to be $(X^A X^B)^q$ and $[Z^A(Z^B)^{d-1}]^p$. We then have $\langle\phi_c|(X^A X^B)^q|\phi_c\rangle=1$ and $\langle\phi_c|[Z^A(Z^B)^{d-1}]^p|\phi_c\rangle=1$ for any q and p, where $0\le q,p<d$. This results in the set of equations $\sum_{k=0}^{d-1}\lambda_k\lambda_{k+q}=1$ for all q, which have only one positive real solution, $\lambda_0=\lambda_1=\cdots=\lambda_k=1/\sqrt{d}$; i.e., the stabilizer state, $|\phi_c\rangle$, is a *maximally entangled state* in the Hilbert space of the two qudits.

In the remaining parts of this paper, we first develop an algorithm for extracting optimal information about the dynamical coherence of a d-level quantum system (with d being a prime), through a single experimental configuration, in Secs. IV–VI. Then, we further develop the algorithm to obtain complete information about the off-diagonal elements of the superoperator by repeating the same scheme for different input states, Sec. VII. In the Appendix, we address the generalization of the DCQD algorithm for qudit systems with d

being a power of a prime. In the first step, in the next section, we establish the required notation by introducing some lemmas and definitions.

IV. BASIC LEMMAS AND DEFINITIONS

Lemma 1. Let $0\leq q,p,q',p'<d$, where d is prime. Then, for given q, p, q' and $k(\mathrm{mod}\,d)$, there is a unique p' that solves $pq'-qp'=k(\mathrm{mod}\,d)$.

Proof. We have $pq'-qp'=k(\mathrm{mod}\,d)=k+td$, where t is an integer. The possible solutions for p' are indexed by t as $p'(t)=(pq'-k-td)/q$. We now show that if $p'(t_1)$ is a solution for a specific value t_1, there exists no other integer $t_2\neq t_1$ such that $p'(t_2)$ is another independent solution to this equation, i.e., $p'(t_2)\neq p'(t_1)(\mathrm{mod}\,d)$. First, note that if $p'(t_2)$ is another solution then we have $p'(t_1)=p'(t_2)+(t_2-t_1)d/q$. Since d is prime, there are two possibilities: (a) q divides (t_2-t_1), then $(t_2-t_1)d/q=\pm nd$, where n is a positive integer; therefore we have $p'(t_2)=p'(t_1)(\mathrm{mod}\,d)$, which contradicts our assumption that $p'(t_2)$ is an independent solution from $p'(t_1)$. (b) q does not divide (t_2-t_1), then $(t_2-t_1)d/q$ is not a integer, which is unacceptable. Thus, we have $t_2=t_1$, i.e., the solution $p'(t)$ is unique.

Note that the above argument does not hold if d is not prime, and therefore, for some q' there could be more than one p' that satisfies $pq'-qp'\equiv k(\mathrm{mod}\,d)$. In general, the validity of this lemma relies on the fact that $\mathbb{Z}_d$ is a field only for prime d.

Lemma 2. For any unitary error operator basis E_i acting on a Hilbert space of dimension d, where d is a prime and $i=0,1,\ldots,d^2-1$, there are d unitary error operator basis elements, E_j, that anticommute with E_i with a specific eigenvalue ω^k, i.e., $E_iE_j=\omega^kE_jE_i$, where $k=0,\ldots,d-1$.

Proof. We have $E_iE_j=\omega^{pq'-qp'}E_jE_i$, where $0\leq q,p,q',p'<d$, and $pq'-qp'\equiv k(\mathrm{mod}\,d)$. Therefore, for fixed q, p, and k (mod d) we need to show that there are d solutions (q',p'). According to Lemma 1, for any q' there is only one p' that satisfies $pq'-qp'=k(\mathrm{mod}\,d)$; but q'can have d possible values, therefore there are d possible pairs of (q',p').

Definition 1. We introduce d different subsets, W_k^i, $k=0,1,\ldots,d-1$, of a unitary error operator basis $\{E_j\}$ (i.e., $W_k^i\subset\{E_j\}$). Each subset contains d members which all anticommute with a particular basis element E_i, where $i=0,1,\ldots,d^2-1$, with fixed eigenvalue ω^k. The subset W_0^i which includes E_0 and E_i is in fact an Abelian subgroup of the single-qudit Pauli group, G_1.

V. CHARACTERIZATION OF QUANTUM DYNAMICAL COHERENCE

For characterization of the coherence in a quantum dynamical process acting on a qudit system, we prepare a two-qudit quantum system in a nonseparable eigenstate $|\phi_{ij}\rangle$ of a unitary operator $S_{ij}=E_i^AE_j^B$. We then subject the qudit A to the unknown dynamical map, and measure the sole stabilizer operator S_{ij} at the output state. Here, the state $|\phi_{ij}\rangle$ is in fact a degenerate code space, since all the operators E_m^A that anticommute with E_i^A, with a particular eigenvalue ω^k, perform the same transformation on the code space and cannot be distinguished by the stabilizer measurement. If we express the spectral decomposition of $S_{ij}=E_i^AE_j^B$ as $S_{ij}=\sum_k\omega^kP_k$, the projection operator corresponding to the outcome ω^k can be written as $P_k=\frac{1}{d}\sum_{l=0}^{d-1}\omega^{-lk}(E_i^AE_j^B)^l$. The post-measurement state of the system, up to a normalization factor, will be

$$P_k\mathcal{E}(\rho)P_k=\frac{1}{d^2}\sum_{m,n=0}^{d^2-1}\chi_{mn}\sum_{l=0}^{d-1}\sum_{l'=0}^{d-1}\omega^{-lk}\omega^{l'k}\times[(E_i^AE_j^B)^lE_m\rho E_n^\dagger(E_i^{A\dagger}E_j^{B\dagger})^{l'}].$$

Using the relations $E_iE_m=\omega^{i_m}E_mE_i$, $E_n^\dagger E_i^\dagger=\omega^{-i_n}E_i^\dagger E_n^\dagger$ and $(E_i^AE_j^B)^l\rho(E_i^{A\dagger}E_j^{B\dagger})^{l'}=\rho$ we have

$$P_k\mathcal{E}(\rho)P_k=\frac{1}{d^2}\sum_{l=0}^{d-1}\omega^{(i_m-k)l}\sum_{l'=0}^{d-1}\omega^{(k-i_n)l'}\sum_{m,n=0}^{d^2-1}\chi_{mn}E_m\rho E_n^\dagger.$$

Now, using the discrete Fourier transform properties $\sum_{l=0}^{d-1}\omega^{(i_m-k)l}=d\delta_{i_m,k}$ and $\sum_{l'=0}^{d-1}\omega^{(k-i_n)l'}=d\delta_{i_n,k}$, we obtain

$$P_k\mathcal{E}(\rho)P_k=\sum_m\chi_{mm}E_m^A\rho E_m^{A\dagger}+\sum_{m<n}(\chi_{mn}E_m^A\rho E_n^{A\dagger}+\chi_{mn}^*E_n^A\rho E_m^{A\dagger}).\tag{5}$$

Here, the summation runs over all E_m^A and E_n^B that belong to the same W_k^i; see Lemma 2. That is, the summation is over all unitary operator basis elements E_m^A and E_n^B that anticommute with E_i^A with a particular eigenvalue ω^k. Since the number of elements in each W_k is d, the state of the two-qudit system after the projective measurement comprises $d+2[d(d-1)/2]=d^2$ terms. The probability of getting the outcome ω^k is

$$\mathrm{Tr}[P_k\mathcal{E}(\rho)]=\sum_m\chi_{mm}+2\sum_{m<n}\mathrm{Re}[\chi_{mn}\,\mathrm{Tr}(E_n^{A\dagger}E_m^A\rho)].\tag{6}$$

Therefore, the normalized post-measurement states are $\rho_k=P_k\mathcal{E}(\rho)P_k/\mathrm{Tr}[P_k\mathcal{E}(\rho)]$. These d equations provide us with information about off-diagonal elements of the superoperator if and only if $\mathrm{Tr}[(E_n^A)^\dagger E_m^A\rho]\neq0$. Later we will derive some general properties of the state ρ such that this condition can be satisfied.

Next we measure the expectation value of any other unitary operator basis element $T_{rs}=E_r^AE_s^B$ on the output state, such that $E_r^A\neq I$, $E_s^B\neq I$, $T_{rs}\in N(S)$, and $T_{rs}\neq(S_{ij})^a$, where $0\leq a<d$. Let us write the spectral decomposition of T_{rs} as $T_{rs}=\sum_{k'}\omega^{k'}P_{k'}$. The joint probability distribution of the commuting Hermitian operators P_k and $P_{k'}$ on the output state $\mathcal{E}(\rho)$ is $\mathrm{Tr}[P_{k'}P_k\mathcal{E}(\rho)]$. The average of these joint probability distributions of P_k and $P_{k'}$ over different values of k' becomes $\sum_{k'}\omega^{k'}\mathrm{Tr}[P_{k'}P_k\mathcal{E}(\rho)]=\mathrm{Tr}[T_{rs}P_k\mathcal{E}(\rho)]=\mathrm{Tr}(T_{rs}\rho_k)$, which can be explicitly written as

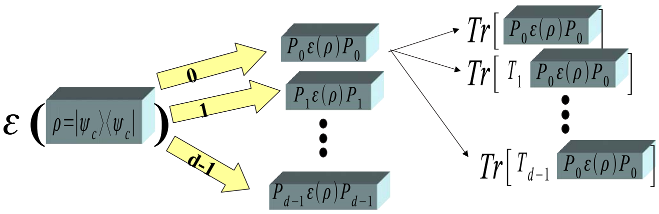

FIG. 4. (Color online) A diagram of the error-detection measurement for estimating quantum dynamical coherence: we measure the sole stabilizer generator at the output state, by applying projection operators corresponding to its different eigenvalues P_k. We also measure $d-1$ commuting operators that belong to the normalizer group. Finally, we calculate the probability of each stabilizer outcome, and joint probability distributions of the normalizers and the stabilizer outcomes. Optimally, we can obtain d^2 linearly independent equations by appropriate selection of the normalizer operators as it is shown in the next section.

$$\begin{aligned}\mathrm{Tr}(T_{rs}\rho_k) = \sum_m \chi_{mm}\,\mathrm{Tr}(E_m^{A\dagger}E_r^AE_s^BE_m^A\rho) \\ + \sum_{m<n}[\chi_{mn}\,\mathrm{Tr}(E_n^{A\dagger}E_r^AE_s^BE_m^A\rho) \\ + \chi_{mn}^*\,\mathrm{Tr}(E_m^{A\dagger}E_r^AE_s^BE_n^A\rho)].\end{aligned}$$

Using $E_r^AE_m^A=\omega^{rm}E_m^AE_r^A$ and $E_r^AE_n^A=\omega^{rn}E_n^AE_r^A$ this becomes

$$\begin{aligned}\mathrm{Tr}(T_{rs}\rho_k) = \frac{1}{\mathrm{Tr}[P_k\mathcal{E}(\rho)]}\Bigg(\sum_m \omega^{rm}\chi_{mm}\,\mathrm{Tr}(T_{rs}\rho) \\ + \sum_{m<n}[\omega^{rm}\chi_{mn}\,\mathrm{Tr}(E_n^{A\dagger}E_m^AT_{rs}\rho) \\ + \omega^{rn}\chi_{mn}^*\,\mathrm{Tr}(E_m^{A\dagger}E_n^AT_{rs}\rho)]\Bigg). \qquad (7)\end{aligned}$$

Therefore, we have an additional set of d equations to identify the off-diagonal elements of the superoperator, provided that $\mathrm{Tr}(E_n^{A\dagger}E_m^AT_{rs}\rho)\neq 0$. Suppose we now measure another unitary operator $T_{r's'}=E_{r'}^AE_{s'}^B$ that commutes with S_{ij}, i.e., $T_{r's'}\in N(S)$, and also commutes with T_{rs}, and satisfies the relations $T_{r's'}\neq T_{rs}^bS_{ij}^a$ (where $0\leq a,b<d$), $E_r^A\neq I$ and $E_s^B\neq I$. Such a measurement results in d equations for $\mathrm{Tr}(T_{r's'}\rho_k)$, similar to those for $\mathrm{Tr}(T_{rs}\rho_k)$. However, for these equations to be useful for characterization of the dynamics, one must show that they are all linearly independent. In the next section, we find the maximum number of independent and commutating unitary operators T_{rs} such that their expectation values on the output state, $\mathrm{Tr}(T_{rs}\rho_k)$, result in linearly independent equations to be $d-1$, see Fig. 4. That is, we find an optimal Abelian set of unitary operators such that the joint probability distribution functions of their eigenvalues and stabilizer eigenvalues at the output state are linearly independent.

VI. LINEAR INDEPENDENCE AND OPTIMALITY OF MEASUREMENTS

Before presenting the proof of linear independence of the functions $\mathrm{Tr}(T_{rs}\rho_k)$ and of the optimality of the DCQD algorithm, we need to introduce the following lemmas and definitions.

Lemma 3. If a stabilizer group, S, has a single generator, the order of its normalizer group, $N(S)$, is d^3.

Proof. Let us consider the sole stabilizer generator $S_{12}=E_1^AE_2^B$, and a typical normalizer element $T_{1'2'}=E_{1'}^AE_{2'}^B$, where $E_1^A=X^{q_1}Z^{p_1}$, $E_2^B=X^{q_2}Z^{p_2}$, $E_{1'}^A=X^{q_1'}Z^{p_1'}$, and $E_{2'}^B=X^{q_2'}Z^{p_2'}$. Since S_{12} and $T_{1'2'}$ commute, we have $S_{12}T_{1'2'}=\omega^{\sum_{i=1}^2 p_iq_{i'}'-q_ip_{i'}'}T_{1'2'}S_{12}$, where $\sum_{i=1}^2 p_iq_{i'}'-q_ip_{i'}'\equiv 0(\mathrm{mod}\ d)$. We note that for any particular code with a single stabilizer generator, all q_1, p_1, q_2, and p_2 are fixed. Now, by Lemma 1, for given values of q_1', p_1', and q_2' there is only one value for p_2' that satisfies the above equation. However, each q_1', p_1', and q_2' can have d different values. Therefore, there are d^3 different normalizer elements $T_{1'2'}$.

Lemma 4. Each Abelian subgroup of a normalizer, which includes the stabilizer group $\{S_{ij}^a\}$ as a proper subgroup, has order d^2.

Proof. Suppose T_{rs} is an element of $N(S)$, i.e., it commutes with S_{ij}. Moreover, all unitary operators of the form $T_{rs}^bS_{ij}^a$, where $0\leq a,b<d$, also commute. Therefore, any Abelian subgroup of the normalizer, $A\subset N(S)$, which includes $\{S_{ij}^a\}$ as a proper subgroup, is at least an order of d^2. Now let $T_{r's'}$ be any other normalizer element, i.e., $T_{r's'}\neq T_{rs}^bS_{ij}^a$ with $0\leq a$, $b<d$, which belongs to the same Abelian subgroup A. In this case, any operator of the form $T_{r's'}^{b'}T_{rs}^bS_{ij}^a$ would also belong to A. Then all elements of the normalizer should commute or $A=N(S)$, which is unacceptable. Thus, either $T_{r's'}=T_{rs}^bS_{ij}^a$ or $T_{r's'}\notin A$, i.e., the order of the Abelian subgroup A is at most d^2.

Lemma 5. There are $d+1$ Abelian subgroups, A, in the normalizer $N(S)$.

Proof. Suppose that the number of Abelian subgroups which includes the stabilizer group as a proper subgroup is n. Using Lemmas 3 and 4, we have $d^3=nd^2-(n-1)d$, where the term $(n-1)d$ has been subtracted from the total number of elements of the normalizer due to the fact that the elements of the stabilizer group are common to all Abelian subgroups. Solving this equation for n, we find that $n=\frac{d^2-1}{d-1}=d+1$.

Lemma 6. The basis of eigenvectors defined by $d+1$ Abelian subgroups of $N(S)$ are mutually unbiased.

Proof. It has been shown [30] that if a set of d^2-1 traceless and mutually orthogonal $d\times d$ unitary matrices can be partitioned into $d+1$ subsets of equal size, such that the $d-1$ unitary operators in each subset commute, then the basis of eigenvectors corresponding to these subsets are mutually unbiased. We note that, based on Lemmas 3, 4, and 5, and in the code space (i.e., up to multiplication by the stabilizer elements $\{S_{ij}^a\}$), the normalizer $N(S)$ has d^2-1 nontrivial elements, and each Abelian subgroup A, has $d-1$ nontrivial commuting operators. Thus, the bases of eigenvectors defined by $d+1$ Abelian subgroups of $N(S)$ are mutually unbiased.

Lemma 7. Let C be a cyclic subgroup of A, i.e., $C\subset A\subset N(S)$. Then, for any fixed $T\in A$, the number of distinct left (right) cosets, $TC(CT)$, in each A is d.

Proof. We note that the order of any cyclic subgroup

$C \subset A$, such as T_{rs}^{b} with $0 \leq b < d$, is d. Therefore, by Lemma 4, the number of distinct cosets in each A is $\frac{d^2}{d} = d$.

Definition 2. We denote the cosets of an (invariant) cyclic subgroup, C_a, of an Abelian subgroup of the normalizer, A_v, by A_v/C_a, where $v = 1, 2, \ldots, d+1$. We also represent generic members of A_v/C_a as $T_{rs}^{b} S_{ij}^{a}$, where $0 \leq a, b < d$. The members of a specific coset A_v/C_{a_0} are denoted as $T_{rs}^{b} S_{ij}^{a_0}$, where a_0 represents a fixed power of stabilizer generator S_{ij}, that labels a particular coset A_v/C_{a_0}, and b $(0 \leq b < d)$ labels different members of that particular coset.

Lemma 8. The elements of a coset, $T_{rs}^{b} S_{ij}^{a_0}$ (where $T_{rs} = E_r^A E_s^B$, $S_{ij} = E_i^A E_j^B$, and $0 \leq b < d$) anticommute with E_i^A with different eigenvalues ω^k. That is, there are no two different members of a coset, A_v/C_{a_0}, that anticommute with E_i^A with the same eigenvalue.

Proof. First we note that for each $T_{rs}^{b} = (E_r^A)^b (E_s^B)^b$, the unitary operators acting only on the principal subsystem, $(E_r^A)^b$, must satisfy either (a) $(E_r^A)^b = E_i^A$ or (b) $(E_r^A)^b \neq E_i^A$. In the case (a), and due to $[T_{rs}, S_{ij}] = 0$, we should also have $(E_s^B)^b = E_j^B$, which results in $T_{rs}^{b} = S_{ij}$; i.e., T_{rs}^{b} is a stabilizer and not a normalizer. This is unacceptable. In the case (b), in particular for $b = 1$, we have $E_r^A E_i^A = \omega^{r_i} E_i^A E_r^A$. Therefore, for arbitrary b we have $(E_r^A)^b E_i^A = \omega^{b r_i} E_i^A (E_r^A)^b$. Since $0 \leq b < d$, we conclude that $\omega^{b r_i} \neq \omega^{b' r_i}$ for any two different values of b and b'.

As a consequence of this lemma, different $(E_r^A)^b$, for $0 \leq b < d$, belong to different W_k^i's.

Lemma 9. For any fixed unitary operator $E_r^A \in W_k^i$, where $k \neq 0$, and any other two independent operators E_m^A and E_n^A that belong to the same W_k^i, we always have $\omega^{r_m} \neq \omega^{r_n}$, where $E_r^A E_m^A = \omega^{r_m} E_m^A E_r^A$ and $E_r^A E_n^A = \omega^{r_n} E_n^A E_r^A$.

Proof. We need to prove for operators $E_r^A, E_m^A, E_n^A \in W_k^i$ (where $k \neq 0$), that we always have $E_m^A \neq E_n^A \Rightarrow \omega^{r_m} \neq \omega^{r_n}$. Let us prove the converse, $\omega^{r_m} = \omega^{r_n} \Rightarrow E_m^A = E_n^A$. We define $E_i^A = X^{q_i} Z^{p_i}$, $E_r^A = X^{q_r} Z^{p_r}$, $E_m^A = X^{q_m} Z^{p_m}$, $E_n^A = X^{q_n} Z^{p_n}$. Based on the definition of subsets W_k^i with $k \neq 0$, we have $p_i q_m - q_i p_m \equiv p_i q_n - q_i p_n = k (\mathrm{mod}\ d) = k + td$ (I), where t is an integer number. We need to show if $p_r q_m - q_r p_m \equiv p_r q_n - q_r p_n = k' (\mathrm{mod}\ d) = k' + t'd$ (II), then $E_m^A = E_n^A$.

We divide the equations (I) by $q_i q_m$ or $q_i q_n$ to get $\frac{p_i}{q_i} = \frac{k+td}{q_i q_m} + \frac{p_m}{q_m} = \frac{k+td}{q_i q_n} + \frac{p_n}{q_n}$ (I$'$). We also divide the equations (II) by $q_r q_m$ or $q_r q_n$ to get $\frac{p_r}{q_r} = \frac{k'+t'd}{q_r q_m} + \frac{p_m}{q_m} = \frac{k'+t'd}{q_r q_n} + \frac{p_n}{q_n}$ (II$'$). By subtracting the equation (II$'$) from (I$'$) we get

$$q_n \left(\frac{k+td}{q_i} - \frac{k'+t'd}{q_r} \right) = q_m \left(\frac{k+td}{q_i} - \frac{k'+t'd}{q_r} \right). \tag{8}$$

Similarly, we can obtain the equation

$$p_n \left(k + tdp_i - \frac{k'+t'd}{p_r} \right) = p_m \left(\frac{k+td}{p_i} - \frac{k'+t'd}{p_r} \right). \tag{9}$$

Note that the expressions within the parentheses in both equations (8) or (9) cannot be simultaneously zero, because it will result in $p_i q_r - q_i p_r = 0$, which is unacceptable for $k \neq 0$. Therefore, the expression within the parentheses in at least one of the equations (8) or (9) is nonzero. This results in $q_n = q_m$ and/or $p_n = p_m$. Consequently, considering the equation (8), we have $E_m^A = E_n^A$.

A. Linear independence of the joint distribution functions

Theorem 1. The expectation values of normalizer elements on a post-measurement state, ρ_k, are linearly independent if these elements are the $d-1$ nontrivial members of a coset A_v/C_{a_0}. That is, for two independent operators T_{rs}, $T_{r's'} \in A_v/C_{a_0}$, we have $\mathrm{Tr}(T_{rs}\rho_k) \neq c\, \mathrm{Tr}(T_{r's'}\rho_k)$, where c is an arbitrary complex number.

Proof. We know that the elements of a coset can be written as $T_{rs}^{b} S_{ij}^{a_0} = (E_r^A E_s^B)^b S_{ij}^{a_0}$, where $b = 1, 2, \ldots, d-1$. We also proved that $(E_r^A)^b$ belongs to different W_k^i $(k \neq 0)$ for different values of b (see Lemma 8). Therefore, according to Lemma 9 and regardless of the outcome of k (after measuring the stabilizer S_{ij}), there exists one member in the coset A_v/C_{a_0} that has different eigenvalues ω^{r_m} with all (independent) members $E_m^A \in W_k^i$. The expectation value of $T_{rs}^{b} S_{ij}^{a_0}$ is

$$\begin{aligned} \mathrm{Tr}(T_{rs}^{b} S_{ij}^{a_0} \rho_k) = {} & \sum_m \chi_{mm}\, \mathrm{Tr}(E_m^{A\dagger} T_{rs}^{b} S_{ij}^{a_0} E_m^A \rho), \\ & + \sum_{m<n} [\chi_{mn}\, \mathrm{Tr}(E_n^{A\dagger} T_{rs}^{b} S_{ij}^{a_0} E_m^A \rho) \\ & + \chi_{mn}^{*}\, \mathrm{Tr}(E_m^{A\dagger} T_{rs}^{b} S_{ij}^{a_0} E_n^A \rho)], \end{aligned} \tag{10}$$

$$\begin{aligned} \mathrm{Tr}(T_{rs}^{b} \rho_k) = {} & \sum_m \omega^{b r_m} \chi_{mm}\, \mathrm{Tr}(T_{rs}^{b} \rho) \\ & + \sum_{m<n} [\omega^{b r_m} \chi_{mn}\, \mathrm{Tr}(E_n^{A\dagger} E_m^A T_{rs}^{b} \rho) \\ & + \omega^{b r_n} \chi_{mn}^{*}\, \mathrm{Tr}(E_m^{A\dagger} E_n^A T_{rs}^{b} \rho)], \end{aligned} \tag{11}$$

where $\omega^{r_m} \neq \omega^{r_n} \neq \cdots$ for all elements $E_m^A, E_n^A, \cdots$ that belong to a specific W_k^i. Therefore, for two independent members of a coset denoted by b and b' (i.e., $b \neq b'$), we have $(\omega^{b' r_m}, \omega^{b' r_n}, \ldots) \neq c(\omega^{b r_m}, \omega^{b r_n}, \ldots)$ for all values of $0 \leq b, b' < d$, and any complex number c. We also note that we have $\mathrm{Tr}(E_n^{A\dagger} E_m^A T_{rs}^{b} \rho) \neq c\, \mathrm{Tr}(E_n^{A\dagger} E_m^A T_{rs}^{b'} \rho)$, since $T_{rs}^{b'-b}$ is a normalizer, not a stabilizer element, and its action on the state cannot be expressed as a global phase. Thus, for any two independent members of a coset A_v/C_{a_0}, we always have $\mathrm{Tr}(T_{rs}^{b'} \rho_k) \neq c\, \mathrm{Tr}(T_{rs}^{b} \rho_k)$.

In summary, after the action of the unknown dynamical process, we measure the eigenvalues of the stabilizer generator, $E_i^A E_j^B$, that has d eigenvalues for $k = 0, 1, \ldots, d-1$ and provides d linearly independent equations for the real and imaginary parts of χ_{mn}. This is due to the fact that the outcomes corresponding to different eigenvalues of a unitary operator are independent. We also measure expectation values of all the $d-1$ independent and commuting normalizer operators $T_{rs}^{b} S_{ij}^{a_0} \in A_v/C_{a_0}$, on the post-measurement state ρ_k, which provides $(d-1)$ linearly independent equations for each outcome k of the stabilizer measurements. Overall, we obtain $d + d(d-1) = d^2$ linearly independent equations for characterization of the real and imaginary parts of χ_{mn} by a single ensemble measurement. In the following, we show that the above algorithm is optimal. That is, within the d^2 Hilbert space of principal system and ancilla, there does not exist any other possible strategy that can provide more than d^2 linearly independent equations by a single measurement on the output state $\mathcal{E}(\rho)$.

B. Optimality

Theorem 2. The maximum number of commuting normalizer elements that can be measured simultaneously to provide linear independent equations for the joint distribution functions $\mathrm{Tr}(T_{rs}^b S_{ij}^a \rho_k)$ is $d-1$.

Proof. Any Abelian subgroup of the normalizer has order d^2 (see Lemma 4). Therefore, the desired normalizer operators should all belong to a particular A_v and are limited to d^2 members. We already showed that the outcomes of measurements for $d-1$ elements of a coset A_v/C_a, represented by $T_{rs}^b S_{ij}^a$ (with $b \neq 0$), are independent (see Theorem 1). Now we show that measuring any other operator, $T_{rs}^b S_{ij}^{a'}$, from any other coset $A_v/C_{a'}$, results in linearly dependent equations for the functions $w=\mathrm{tr}(T_{rs}^b S_{ij}^a \rho_k)$ and $w'=\mathrm{tr}(T_{rs}^b S_{ij}^{a'} \rho_k)$ as the following:

$$\begin{aligned} w &= \mathrm{Tr}(T_{rs}^b S_{ij}^a \rho_k) \\ &= \sum_m \chi_{mm} \mathrm{Tr}(E_m^{A\dagger} T_{rs}^b S_{ij}^a E_m^A \rho) + \sum_{m<n} [\chi_{mn} \mathrm{Tr}(E_n^{A\dagger} T_{rs}^b S_{ij}^a E_m^A \rho) \\ &\quad + \chi_{mn}^* \mathrm{Tr}(E_m^{A\dagger} T_{rs}^b S_{ij}^a E_n^A \rho)], \end{aligned}$$

$$\begin{aligned} w' &= \mathrm{Tr}(T_{rs}^b S_{ij}^{a'} \rho_k) \\ &= \sum_m \chi_{mm} \mathrm{Tr}(E_m^{A\dagger} T_{rs}^b S_{ij}^{a'} E_m^A \rho) + \sum_{m<n} [\chi_{mn} \mathrm{Tr}(E_n^{A\dagger} T_{rs}^b S_{ij}^{a'} E_m^A \rho) \\ &\quad + \chi_{mn}^* \mathrm{tr}(E_m^{A\dagger} T_{rs}^b S_{ij}^{a'} E_n^A \rho)]. \end{aligned}$$

Using the commutation relations $T_{rs}^b S_{ij}^a E_m^A = \omega^{br_m + ai_m} E_m^A T_{rs}^b S_{ij}^a$, we obtain

$$\begin{aligned} w = \sum_m \omega^{br_m + ai_m} \chi_{mm} \mathrm{Tr}(T_{rs}^b \rho) &+ \sum_{m<n} [\omega^{br_m + ai_m} \chi_{mn} \mathrm{Tr}(E_n^{A\dagger} E_m^A T_{rs}^b \rho) \\ &+ \omega^{br_n + ai_n} \chi_{mn}^* \mathrm{Tr}(E_m^{A\dagger} E_n^A T_{rs}^b \rho)] \end{aligned}$$

$$\begin{aligned} w' = \sum_m \omega^{br_m + a' i_m} \chi_{mm} \mathrm{tr}(T_{rs}^b \rho) &+ \sum_{m<n} [\omega^{br_m + a' i_m} \chi_{mn} \mathrm{Tr}(E_n^{A\dagger} E_m^A T_{rs}^b \rho) \\ &+ \omega^{br_n + a' i_n} \chi_{mn}^* \mathrm{Tr}(E_m^{A\dagger} E_n^A T_{rs}^b \rho)], \end{aligned}$$

where we also used the fact that both S_{ij}^a and $S_{ij}^{a'}$ are stabilizer elements. Since all of the operators E_m^A belong to the same W_k^i, we have $i_m = i_n = k$, and obtain

$$\begin{aligned} w = \omega^{ak} \Bigg(\sum_m \omega^{br_m} \chi_{mm} \mathrm{Tr}(T_{rs}^b \rho) &+ \sum_{m<n} [\omega^{br_m} \chi_{mn} \mathrm{Tr}(E_n^{A\dagger} E_m^A T_{rs}^b \rho) \\ &+ \omega^{br_n} \chi_{mn}^* \mathrm{Tr}(E_m^{A\dagger} E_n^A T_{rs}^b \rho)] \Bigg), \end{aligned}$$

$$\begin{aligned} w' = \omega^{a'k} \Bigg(\sum_m \omega^{br_m} \chi_{mm} \mathrm{Tr}(T_{rs}^b \rho) &+ \sum_{m<n} [\omega^{br_m} \chi_{mn} \mathrm{Tr}(E_n^{A\dagger} E_m^A T_{rs}^b \rho) \\ &+ \omega^{br_n} \chi_{mn}^* \mathrm{Tr}(E_m^{A\dagger} E_n^A T_{rs}^b \rho)] \Bigg). \end{aligned}$$

Thus, we have $w' = \omega^{(a'-a)k} w$, and consequently the measurements of operators from other cosets $A_v/C_{a'}$ do not provide any new information about χ_{mn} beyond the corresponding measurements from the coset A_v/C_a.

For another proof of the optimality, based on fundamental limitation of transferring information between two parties given by the Holevo bound see Ref. [26]. In principle, one can construct a set of *non-Abelian* normalizer measurements, from different A_v, where $v=1,2,\ldots,d+1$, to obtain information about the off-diagonal elements χ_{mn}. However, determining the eigenvalues of a set of noncommuting operators cannot be done via a single measurement. Moreover, as mentioned above, by measuring the stabilizer and $d-1$ commuting normalizer elements, one can in principle transfer $\log_2 d^2$ bits of classical information between two parties, which is the maximum allowed by the Holevo bound [31]. Therefore, other strategies involving non-Abelian, or a mixture of Abelian and non-Abelian normalizer measurements, cannot improve the above scheme. It should be noted that there are several possible alternative sets of Abelian normalizers that are equivalent for this task. We address this issue in the next lemma.

Lemma 10. The number of alternative sets of Abelian normalizer measurements that can provide optimal information about quantum dynamics, in one ensemble measurement, is d^2.

Proof. We have $d+1$ Abelian normalizers A_v (see Lemma 5). However, there are d of them that contain unitary operators that act nontrivially on both qudit systems A and B, i.e., $T_{rs}^b = (E_r^A E_s^B)^b$, where $E_r^A \neq I$, $E_s^B \neq I$. Moreover, in each A_v we have d cosets (see Lemma 5) that can be used for optimal characterization of χ_{mn}. Overall, we have d^2 possible sets of Abelian normalizers that are equivalent for our purpose.

In the next section, we develop the algorithm further to obtain complete information about the off-diagonal elements of the superoperator by repeating the above scheme for different input states.

VII. REPEATING THE ALGORITHM FOR OTHER STABILIZER STATES

We have shown that by performing one ensemble measurement one can obtain d^2 linearly independent equations for χ_{mn}. However, a complete characterization of quantum dynamics requires obtaining d^4-d^2 independent real parameters of the superoperator (or d^4 for nontrace preserving maps). We next show how one can obtain complete information by appropriately rotating the input state and repeating the above algorithm for a complete set of rotations.

Lemma 11. The number of independent eigenkets for the error operator basis $\{E_j\}$, where $j=1,2,\ldots, d^2-1$, is $d+1$. These eigenkets are mutually unbiased.

Proof. We have d^2-1 unitary operators, E_i. We note that the operators E_i^a for all values of $1\leq a\leq d-1$ commute and have a common eigenket. Therefore, overall we have $(d^2-1)/(d-1)=d+1$ independent eigenkets. Moreover, it has been shown [30] that if a set of d^2-1 traceless and mutually orthogonal $d\times d$ unitary matrices can be partitioned into $d+1$ subsets of equal size, such that the $d-1$ unitary operators in each subset commute, then the basis of eigenvectors defined by these subsets are mutually unbiased.

Let us construct a set of $d+1$ stabilizer operators $E_i^A E_j^B$, such that the following conditions hold: (a) $E_i^A, E_j^B \neq I$, (b) $(E_i^A)^a \neq E_{i'}^A$ for $i\neq i'$ and $1\leq a\leq d-1$. Then, by preparing the eigenstates of these $d+1$ independent stabilizer operators, one at a time, and measuring the eigenvalues of S_{ij} and its corresponding $d-1$ normalizer operators $T_{rs}^b S_{ij}^a \in A_v/C_a$, one can obtain $(d+1)d^2$ *linearly independent* equations to characterize the superoperator's off-diagonal elements. The linear independence of these equations can be understood by noting that the eigenstates of all operators E_i^A of the $d+1$ stabilizer operator S_{ij} are mutually unbiased (i.e., the measurements in these mutual unbiased bases are maximally noncommuting). For example, the bases $\{|0\rangle,|1\rangle\}$, $\{(|+\rangle_X,|-\rangle_X\}$ and $\{|+\rangle_Y, |-\rangle_Y\}$ (the eigenstates of the Pauli operators Z, X, and Y) are *mutually unbiased*, i.e., the inner products of each pair of elements in these bases have the same magnitude. Then measurements in these bases are maximally noncommuting [32].

To obtain complete information about the quantum dynamical coherence, we again prepare the eigenkets of the above $d+1$ stabilizer operators $E_i^A E_j^B$, but after the stabilizer measurement we calculate the expectation values of the operators $T_{r's'}^b S_{ij}^a$ belonging to other Abelian subgroups $A_{v'}/C_a$ of the normalizer, i.e., $A_{v'}\neq A_v$. According to Lemma 6 the bases of different Abelian subgroups of the normalizer are mutually unbiased, therefore, the expectation values of $T_{r's'}^b S_{ij}^a$ and $T_{rs}^b S_{ij}^a$ from different Abelian subgroups $A_{v'}$ and A_v are independent. In order to make the stabilizer measurements also independent we choose a different superposition of logical basis in the preparation of $d+1$ possible stabilizer state in each run. Therefore in each of these measurements we can obtain at most d^2 linearly independent equations. By repeating these measurements for $d-1$ different A_v over all $d+1$ possible input stabilizer states, we obtain $(d+1)(d-1)d^2=d^4-d^2$ linearly independent equations, which suffice to fully characterize all independent parameters of the superoperator's off-diagonal elements. In the next section, we address the general properties of these $d+1$ stabilizer states.

VIII. GENERAL CONSTRAINTS ON THE STABILIZER STATES

The restrictions on the stabilizer states ρ can be expressed as follows:

Condition 1. The state $\rho=|\phi_{ij}\rangle\langle\phi_{ij}|$ is a nonseparable pure state in the Hilbert space of the two-qudit system $\mathcal{H}$. That is, $|\psi_{ij}\rangle_{AB}\neq|\phi\rangle_A\otimes|\varphi\rangle_B$.

Condition 2. The state $|\phi_{ij}\rangle$ is a stabilizer state with a sole stabilizer generator $S_{ij}=E_i^A E_j^B$. That is, it satisfies $S_{ij}^a|\phi_{ij}\rangle=\omega^{ak}|\phi_{ij}\rangle$, where $k\in\{0,1,\ldots,d-1\}$ denotes a fixed eigenvalue of S_{ij}, and $a=1,\ldots,d-1$ labels $d-1$ nontrivial members of the stabilizer group.

The second condition specifies the stabilizer subspace, V_S, that the state ρ lives in, which is the subspace fixed by all the elements of the stabilizer group with fixed eigenvalues k. More specifically, an arbitrary state in the entire Hilbert space $\mathcal{H}$ can be written as $|\phi\rangle=\sum_{u,u'=0}^{d-1}\alpha_{uu'}|u\rangle_A|u'\rangle_B$ where $\{|u\rangle\}$ and $\{|u'\rangle\}$ are bases for the Hilbert spaces of the qudits A and B, such that $X^q|u\rangle=|u+q\rangle$ and $Z^p|u\rangle=\omega^{pu}|u\rangle$. However, we can expand $|\phi\rangle$ in another basis as $|\phi\rangle=\sum_{v,v'=0}^{d-1}\beta_{vv'}|v\rangle_A|v'\rangle_B$, such that $X^q|v\rangle=\omega^{qv}|v\rangle$ and $Z^p|v\rangle=|v+p\rangle$. Let us consider a stabilizer state fixed under the action of a unitary operator $E_i^A E_j^B=(X^A)^q(X^B)^{q'}(Z^A)^p(Z^B)^{p'}$ with eigenvalue ω^k. Regardless of the basis chosen to expand $|\phi_{ij}\rangle$, we should always have $S_{ij}|\phi_{ij}\rangle=\omega^k|\phi_{ij}\rangle$. Consequently, we have the constraints $pu\oplus p'u'=k$, for the stabilizer subspace V_S spanned by the $\{|u\rangle\otimes|u'\rangle\}$basis, and $q(v\oplus p)\oplus q'(v'\oplus p')=k$, if V_S is spanned by $\{|v\rangle\otimes|v'\rangle\}$ basis, where $\oplus$ is addition mod(d). From these relations, and also using the fact that the bases $\{|v\rangle\}$ and $\{|u\rangle\}$ are related by a unitary transformation, one can find the general properties of V_S for a given stabilizer generator $E_i^A E_j^B$ and a given k.

We have already shown that the stabilizer states ρ should also satisfy the set of conditions $\mathrm{Tr}[E_n^{A\dagger}E_m^A\rho]\neq 0$ and $\mathrm{Tr}(E_n^{A\dagger}E_m^A T_{rs}^b\rho)\neq 0$ for all operators E_m^A belonging to the same W_k^i, where T_{rs}^b $(0<b\leq d-1)$ are the members of a particular coset A_v/C_a of an Abelian subgroup, A_v, of the normalizer $N(S)$. These relations can be expressed more compactly as follows.

Condition 3. For stabilizer state $\rho=|\phi_{ij}\rangle\langle\phi_{ij}|\equiv|\phi_c\rangle\langle\phi_c|$ and for all $E_m^A\in W_k^i$, we have

$$\langle\phi_c|E_n^{A\dagger}E_m^A T_{rs}^b|\phi_c\rangle\neq 0, \tag{12}$$

where here $0\leq b\leq d-1$.

Before developing the implications of the above formula for the stabilizer states we give the following definition and lemma.

Definition 3. Let $\{|l\rangle_L\}$ be the logical basis of the code space that is fixed by the stabilizer generator $E_i^A E_j^B$. The stabilizer state in that basis can be written as $|\phi_c\rangle=\sum_{l=0}^{d-1}\alpha_l|l\rangle_L$, and all the normalizer operators, T_{rs}, can be generated from tensor products of logical operations $\bar{X}$ and $\bar{Z}$ defined as $\bar{Z}|l\rangle_L=\omega^l|l\rangle_L$ and $\bar{X}|l\rangle_L=|l+1\rangle$. For example, $|l\rangle_L=|k\rangle|k\rangle$, $\bar{Z}=Z\otimes I$, and $\bar{X}=X\otimes X$, where $X|k\rangle=|k+1\rangle$ and $Z|k\rangle=\omega^k|k\rangle$.

Lemma 12. For a stabilizer generator $E_i^A E_j^B$ and all unitary operators $E_m^A\in W_k^i$, we always have $E_n^{A\dagger}E_m^A=\omega^c\bar{Z}^a$, where $\bar{Z}$ is the logical Z operation acting on the code space and a and c are integers.

Proof. Let us consider $E_i^A=X^{q_i}Z^{p_i}$, and two generic operators E_n^A and E_m^A that belong to W_k^i, $E_m^A=X^{q_m}Z^{p_m}$ and $E_n^A=X^{q_n}Z^{p_n}$. From the definition of W_k^i (see Definition 1) we have $p_iq_m-q_ip_m=p_iq_n-q_ip_n=k(\mathrm{mod}\ d)=k+td$. We can solve these two equations to get $q_m-q_n=q_i(p_mq_n-q_mp_n)/(k+td)$ and $p_m-p_n=p_i(p_mq_n-q_mp_n)/(k+td)$. We also define $p_mq_n-q_mp_n=k'+t'd$. Therefore, we obtain $q_m-q_n=q_ia$ and $p_m-p_n=p_ia$, where we have introduced

$$a=(k'+t'd)/(k+td). \tag{13}$$

Moreover, we have $E_n^{A\dagger}=X^{(t''d-q_n)}Z^{(t''d-p_n)}$ for some other integer t''. Then we get

$$\begin{aligned}E_n^{A\dagger}E_m^A &= \omega^c X^{(t''d+q_m-q_n)}Z^{(t''d+p_m-p_n)}\\ &= \omega^c X^{(q_m-q_n)}Z^{(p_m-q_n)}\\ &= \omega^c (X^{q_i}Z^{p_i})^a,\end{aligned}$$

where $c=(t''d-p_n)(t''d+q_m-q_n)$. However, $X^{q_i}Z^{p_i}\otimes I$ acts as logical $\bar{Z}$ on the code subspace, which is the eigenstate of $E_i^A E_j^B$. Thus, we obtain $E_n^{A\dagger}E_m^A=\omega^c\bar{Z}^a$.

Based on the above lemma, for the case of $b=0$ we obtain

$$\langle\phi_c|E_n^{A\dagger}E_m^A|\phi_c\rangle=\omega^c\langle\phi_c|\bar{Z}^a|\phi_c\rangle=\omega^c\sum_{l=0}^{d-1}\omega^{al}|\alpha_l|^2.$$

Therefore, our constraint in this case becomes $\sum_{k=0}^{d-1}\omega^{al}|\alpha_l|^2\neq 0$, which is not satisfied if the stabilizer state is maximally entangled. For $b\neq 0$, we note that T_{rs}^b are in fact the normalizers. By considering the general form of the normalizer elements as $T_{rs}^b=(\bar{X}^q\bar{Z}^p)^b$, where $q,\ p\in\{0,1,\ldots,d-1\}$, we obtain

$$\begin{aligned}\langle\phi_c|E_n^{A\dagger}E_m^A T_{rs}^b|\phi_c\rangle &= \omega^c\langle\phi_c|\bar{Z}^a(\bar{X}^q\bar{Z}^p)^b|\phi_c\rangle\\ &= \omega^c\sum_{k=0}^{d-1}\omega^{a(l+bq)}\omega^{bpl}\alpha_l^*\alpha_{l+bq}\\ &= \omega^{(c+abq)}\sum_{l=0}^{d-1}\omega^{(a+bp)l}\alpha_l^*\alpha_{l+bq}.\end{aligned}$$

Overall, the constraints on the stabilizer state, due to condition (iii), can be summarized as

$$\sum_{l=0}^{d-1}\omega^{(a+bp)l}\alpha_l^*\alpha_{l+bq}\neq 0. \tag{14}$$

This inequality should hold for all $b\in\{0,1,\ldots,d-1\}$, and all a defined by Eq. (13), however, for a particular coset A_v/C_a the values of q and p are fixed. One important property of the stabilizer code, implied by the above formula with $b=0$, is that it should always be a *nonmaximally entangled state*. In the next section, by utilizing the quantum Hamming bound, we show that the minimum number of physical qudits, n, needed for encoding the required stabilizer state is in fact two.

IX. MINIMUM NUMBER OF REQUIRED PHYSICAL QUDITS

In order to characterize off-diagonal elements of a superoperator we must use degenerate stabilizer codes, in order to preserve the coherence between operator basis elements. Degenerate stabilizer codes do not have a classical analog [1]. Due to this fact, the classical techniques used to prove bounds for nondegenerate error-correcting codes cannot be applied to degenerate codes. In general, it is yet unknown if there are degenerate codes that exceed the quantum Hamming bound [1]. However, due to the simplicity of the stabilizer codes used in the DCQD algorithm and their symmetry, it is possible to generalize the quantum Hamming bound for them. Let us consider a stabilizer code that is used for encoding k logical qudits into n physical qudits such that we can correct any subset of t or fewer errors on any $n_e\leq n$ of the physical qudits. Suppose that $0\leq j\leq t$ errors occur. Therefore, there are $\binom{n_e}{j}$ possible locations, and in each location there are (d^2-1) different operator basis elements that can act as errors. The total possible number of errors is $\sum_{j=0}^{t}\binom{n_e}{j}(d^2-1)^j$. If the stabilizer code is nondegenerate each of these errors should correspond to an orthogonal d^k-dimensional subspace; but if the code is uniformly g-fold degenerate (i.e., with respect to all possible errors), then each set of g errors can be fit into an orthogonal d^k-dimensional subspace. All these subspaces must be fit into the entire d^n-dimensional Hilbert space. This leads to the following inequality:

$$\sum_{j=0}^{t}\binom{n_e}{j}\frac{(d^2-1)^j d^k}{g}\leq d^n. \tag{15}$$

We are always interested in finding the errors on one physical qudit. Therefore, we have $n_e=1$, $j\in\{0,1\}$ and $\binom{cn_e}{j}=1$, and Eq. (15) becomes $\sum_{j=0}^{1}\frac{(d^2-1)^j d^k}{g}\leq d^n$. In order to characterize diagonal elements, we use a nondegenerate stabilizer code with $n=2$, $k=0$, and $g=1$, and we have $\sum_{j=0}^{1}(d^2-1)^j=d^2$. For off-diagonal elements, we use a degenerate stabilizer code with $n=2$, $k=1$, and $g=d$, and we have $\sum_{j=0}^{1}\frac{(d^2-1)^j d}{d}=d^2$. Therefore, in both cases the upper-bound of the quantum Hamming bound is satisfied by our codes. Note that if instead we use $n=k$, i.e., if we encode n logical qudits into n separable physical qubits, we get $\sum_{j=0}^{1}\frac{(d^2-1)^j}{g}\leq 1$. This can only be satisfied if $g=d^2$, in which case we cannot obtain any information about the errors. The above argument justifies condition (i) of the stabilizer state being nonseparable. Specifically, it explains why alternative encodings such as $n=k=2$ and $n=k=1$ are excluded from our discussions. However, if we encode zero logical qubits into one physical qubit, i.e., $n=1,k=0$, then, by using a d-fold degenerate code, we can obtain $\sum_{j=0}^{1}\frac{(d^2-1)^j}{d}=d$ which satisfies the quantum Hamming bound and could be useful for characterizing off-diagonal elements. For this to be true, the code $|\phi_c\rangle$ should also satisfy the set of conditions $\langle\phi_c|E_n^{A\dagger}E_m^A|\phi_c\rangle\neq 0$ and $\langle\phi_c|E_n^{A\dagger}E_m^A T_{rs}^b|\phi_c\rangle\neq 0$. Due to the d-fold degeneracy of the code, the condition $\langle\phi_c|E_n^{A\dagger}E_m^A|\phi_c\rangle\neq 0$ is automatically satisfied. However, the condition $\langle\phi_c|E_n^{A\dagger}E_m^A T_{rs}^b|\phi_c\rangle\neq 0$ can never be satisfied, since the code space is one-dimensional, i.e., $d^k=1$, and the normalizer operators cannot be defined. That is, there does not exist any nontrivial unitary operator T_{rs}^b that can perform logical operations on the one-dimensional code space.

We have demonstrated how we can characterize quantum dynamics using the most general form of the relevant stabilizer states and generators. In the next section, we choose a standard form of stabilizers, in order to simplify the algorithm and to derive a standard form of the normalizer.

X. STANDARD FORM OF STABILIZER AND NORMALIZER OPERATORS

Let us choose the set $\{|0\rangle, |1\rangle, \ldots, |k-1\rangle\}$ as a standard basis, such that $Z|k\rangle = \omega^k |k\rangle$ and $X|k\rangle = |k+1\rangle$. In order to characterize the quantum dynamical population, we choose the standard stabilizer generators to be $(X^A X^B)^q$ and $[Z^A (Z^B)^{d-1}]^p$. Therefore, the maximally entangled input states can be written as $|\varphi_c\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} |k\rangle_A |k\rangle_B$. In order to characterize the quantum dynamical coherence we choose the sole stabilizer operator as $[E_i^A (E_i^B)^{d-1}]^a$, which has an eigenket of the form $|\varphi_c\rangle = \sum_{i=0}^{d-1} \alpha_i |i\rangle_A |i\rangle_B$, where $E_i|i\rangle = \omega^i |i\rangle$ and $|i\rangle$ represents one of $d+1$ mutually unbiased basis states in the Hilbert space of one qudit. The normalizer elements can be written as $T_{qp}^b = (\overline{X}^q \overline{Z}^p)^b \in A_{v_0}/C_{a_0}$, for all $0 < b \leq d-1$, where $\overline{X} = \widetilde{E}_i \otimes \widetilde{E}_i$, $\overline{Z} = E_i \otimes I$, $\widetilde{E}_i |i\rangle = |i+1\rangle$ and $E_i|i\rangle = \omega^i |i\rangle$; and A_{v_0}/C_{a_0} represents a fixed coset of a particular Abelian subgroup, A_{v_0}, of the normalizer $N(S)$. For example, for a stabilizer generator of the form $[E_i^A (E_i^B)^{d-1}]^a = [Z^A (Z^B)^{d-1}]^p$ we prepare its eigenket $|\varphi_c\rangle = \sum_{k=0}^{d-1} \alpha_k |k\rangle_A |k\rangle_B$, and the normalizers become $T_{qp}^b = (\overline{X}^q \overline{Z}^p)^b$, where $\overline{X} = X \otimes X$ and $\overline{Z} = Z \otimes I$. Using this notations for stabilizer and the normalizer operators, we provide an overall outline for the DCQD algorithm in the next section.

XI. ALGORITHM: DIRECT CHARACTERIZATION OF QUANTUM DYNAMICS

The DCQD algorithm for the case of a qudit system is summarized as follows (see also Figs. 5 and 6.):

Inputs. (1) An ensemble of two-qudit systems, A and B, prepared in the state $|0\rangle_A \otimes |0\rangle_B$. (2) An arbitrary unknown CP quantum dynamical map $\mathcal{E}$, whose action can be expressed by $\mathcal{E}(\rho) = \sum_{m,n=0}^{d^2-1} \chi_{mn} E_m^A \rho E_n^{A\dagger}$, where ρ denotes the state of the primary system and the ancilla.

Output. $\mathcal{E}$, given by a set of measurement outcomes in the procedures (a) and (b) below.

Procedure (*a*). Characterization of quantum dynamical population (diagonal elements χ_{mm} of χ), see Fig. 5.

(1) Prepare $|\varphi_0\rangle = |0\rangle_A \otimes |0\rangle_B$, a pure initial state.

(2) Transform it to $|\varphi_c\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} |k\rangle_A |k\rangle_B$, a maximally entangled state of the two qudits. This state has the stabilizer operators $E_i^A E_j^B = (X^A X^B)^q$ and $E_{i'}^A E_{j'}^B = [Z^A (Z^B)^{d-1}]^p$ for $0 < p, q \leq d-1$.

(3) Apply the unknown quantum dynamical map to the qudit A: $\mathcal{E}(\rho) = \sum_{m,n=0}^{d^2-1} \chi_{mn} E_m^A \rho E_n^{A\dagger}$, where $\rho = |\phi_c\rangle\langle\phi_c|$.

(4) Perform a projective measurement $P_k P_{k'} : \mathcal{E}(\rho) \mapsto P_k P_{k'} \mathcal{E}(\rho) P_k P_{k'}$, where $P_k = \frac{1}{d} \sum_{l=0}^{d-1} \omega^{-lk} (E_i^A E_j^B)^l$, and $P_{k'} = \frac{1}{d} \sum_{l'=0}^{d-1} \omega^{-l'k'} (E_{i'}^A E_{j'}^B)^{l'}$, and calculate the joint probability distributions of the outcomes k and k',

$$\mathrm{Tr}[P_k P_{k'} \mathcal{E}(\rho)] = \chi_{mm}.$$

Number of ensemble measurements for procedure (a), 1.

Procedure (*b*). Characterization of quantum dynamical coherence (off-diagonal elements χ_{mn} of χ), see Fig. 6.

(1) Prepare $|\varphi_0\rangle = |0\rangle_A \otimes |0\rangle_B$, a pure initial state.

(2) Transform it to $|\varphi_c\rangle = \sum_{i=0}^{d-1} \alpha_i |i\rangle_A |i\rangle_B$, a nonmaximally entangled state of the two qudits. This state has stabilizer operators $[E_i^A (E_i^B)^{d-1}]^a$.

(3) Apply the unknown quantum dynamical map to the qudit A: $\mathcal{E}(\rho) = \sum_{m,n=0}^{d^2-1} \chi_{mn} E_m^A \rho E_n^{A\dagger}$, where $\rho = |\phi_c\rangle\langle\phi_c|$.

(4) Perform a projective measurement

$$P_k : \mathcal{E}(\rho) \mapsto \rho_k = P_k \mathcal{E}(\rho) P_k = \sum_m \chi_{mm} E_m^A \rho E_m^{A\dagger} + \sum_{m<n} (\chi_{mn} E_m^A \rho E_n^{A\dagger} + \chi_{mn}^* E_n^A \rho E_m^{A\dagger}),$$

where $P_k = \frac{1}{d} \sum_{l=0}^{d-1} \omega^{-lk} (E_i^A E_j^B)^l$ and $E_m^A = X^{q_m} Z^{p_m} \in W_k^i$, and calculate the probability of outcome k:

$$\mathrm{Tr}[P_k \mathcal{E}(\rho)] = \sum_m \chi_{mm} + 2 \sum_{m<n} \mathrm{Re}[\chi_{mn} \mathrm{Tr}(E_n^{A\dagger} E_m^A \rho)].$$

(5) Measure the expectation values of the normalizer operators $T_{qp}^b = (\overline{X}^q \overline{Z}^p)^b \in A_{v_0}/C_{a_0}$, for all $0 < b \leq d-1$, where $\overline{X} = \widetilde{E}_i \otimes \widetilde{E}_i$, $\overline{Z} = E_i \otimes I$, $E_i|i\rangle = \omega^i |i\rangle$, $\widetilde{E}_i |i\rangle = |i+1\rangle$, where A_{v_0}/C_{a_0} represents a fixed coset of a particular Abelian subgroup, A_{v_0}, of the normalizer $N(S)$,

$$\mathrm{Tr}(T_{qp}^b \rho_k) = \sum_m \omega^{pq_m - qp_m} \chi_{mm} \mathrm{Tr}(T_{rs}^b \rho) + \sum_{m<n} [\omega^{pq_m - qp_m} \chi_{mn} \mathrm{Tr}(E_n^{A\dagger} E_m^A T_{rs}^b \rho) + \omega^{pq_n - qp_n} \chi_{mn}^* \mathrm{Tr}(E_m^{A\dagger} E_n^A T_{rs}^b \rho)].$$

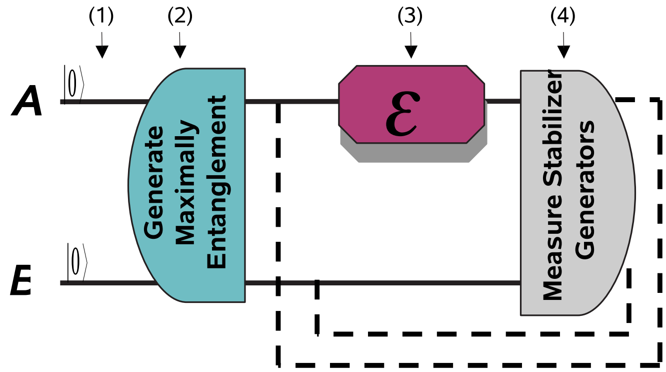

FIG. 5. (Color online) Procedure (a): Measuring the quantum dynamical population (diagonal elements χ_{mm}). The arrows indicate direction of time. (1) Prepare $|\varphi_0\rangle = |0\rangle_A \otimes |0\rangle_B$, a pure initial state. (2) Transform it to $|\varphi_C\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} |k\rangle_A |k\rangle_B$, a maximally entangled state of the two qudits. This state has the stabilizer operators $S = X^A X^B$ and $S' = Z^A (Z^B)^{d-1}$. (3) Apply the unknown quantum dynamical map to the qudit A, $\mathcal{E}(\rho)$, where $\rho = |\phi_C\rangle\langle\phi_C|$. (4) Perform a projective measurement $P_k P_{k'}$, for $k, k' = 0, \ldots, d-1$, corresponding to eigenvalues of the stabilizer operators S and S'. Then calculate the joint probability distributions of the outcomes k and k': $\mathrm{Tr}[P_k P_{k'} \mathcal{E}(\rho)] = \chi_{mm}$. The elements χ_{mm} represent the population of error operators that anticommute with the stabilizer generators S and S' with eigenvalues ω^k and $\omega^{k'}$, respectively. The number of ensemble measurements for procedure (a) is one.

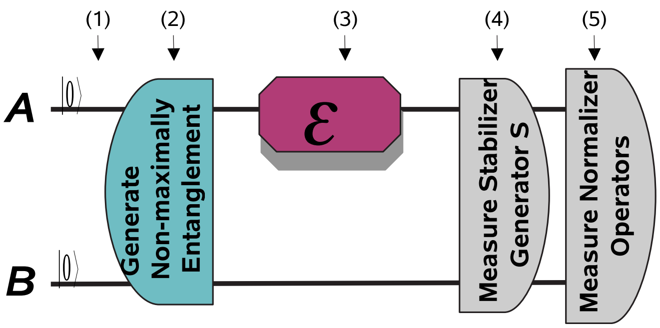

FIG. 6. (Color online) Procedure (b): Measuring the quantum dynamical coherence (off-diagonal elements χ_{mn}). (1) Prepare $|\varphi_0\rangle = |0\rangle_A \otimes |0\rangle_B$, a pure initial state. (2) Transform it to $|\varphi_C\rangle = \sum_{i=0}^{d-1} \alpha_i |i\rangle_A |i\rangle_B$, a nonmaximally entangled state of the two qudits. This state has a sole stabilizer operator of the form $S = E_i^A (E_i^B)^{d-1}$. (3) Apply the unknown quantum dynamical map to the qudit A, $\mathcal{E}(\rho)$, where $\rho = |\phi_C\rangle\langle\phi_C|$. (4) Perform a projective measurement P_k, and calculate the probability of the outcome k, $\mathrm{tr}[P_k \mathcal{E}(\rho)]$. (5) Measure the expectation values of all normalizer operators T_{rs} that simultaneously commute with the stabilizer generator S. There are only $d-1$ such operators T_{rs} that are independent of each other, within a multiplication by a stabilizer generator; and they belong to an Abelian subgroup of the normalizer group. (6) Repeat the steps (1)–(5) $d+1$ times, by preparing the eigenkets of the other stabilizer operator $E_i^A (E_i^B)^{d-1}$ for all $i \in \{1, 2, \ldots, d+1\}$, such that states $|i\rangle_A$ in step (2) belong to a mutually unbiased basis [32]. (7) Repeat the step (6) up to $d-1$ times, each time choosing normalizer elements T_{rs} from a different Abelian subgroup of the normalizer, such that these measurements become maximally noncommuting, i.e., their eigenstates form a set of mutually unbiased bases. The number of ensemble measurements for Procedure (b) is $(d+1)(d-1)$.

(6) Repeat the steps (1)–(5) $d+1$ times, by preparing the eigenkets of other stabilizer operator $[E_i^A (E_i^B)^{d-1}]^a$ for all $i \in \{1, 2, \ldots, d+1\}$, such that states $|i\rangle_A |i\rangle_B$ in the step (2) belong to a mutually unbiased basis.

(7) Repeat the step (6) up to $d-1$ times, each time choosing normalizer elements T_{qp}^b from a different Abelian subgroup A_v / C_a, such that these measurements become maximally noncommuting.

Number of ensemble measurements for procedure (b), $(d+1)(d-1)$.

Overall number of ensemble measurements, d^2.

In the case of performing nondemolition measurements, the output state in Figs. 5 and 6 becomes a maximally entangled state $|\varphi_E\rangle = \sum_{i=0}^{d-1} |i\rangle_A |i\rangle_B$ which is the common eigenket of the stabilizer generator and its commuting normalizer operators. For procedure (a), this state can be directly used for other measurements. This is indicated by the dashed lines in Fig. 5. For procedure (b), the state $|\varphi_E\rangle$ can be unitarily transformed to another member of the same input stabilizer code, $|\varphi_C\rangle = \sum_{i=0}^{d-1} \alpha_i |i\rangle_A |i\rangle_B$, before another measurement. Therefore, all the required ensemble measurements, for measuring the expectation values of the stabilizer and normalizer operators, can be performed in a temporal sequence on the same pair of qudits, by utilizing nondestructive measurements.

In the preceding sections, we have explicitly shown how the DCQD algorithm can be developed for qudit systems when d is prime. In the Appendix , we demonstrate that the DCQD algorithm can be generalized to other N-dimensional quantum systems with N being a power of a prime.

XII. SUMMARY

For convenience, we provide a summary of the DCQD algorithm. The DCQD algorithm for a qudit, with d being a prime, was developed by utilizing the concept of an error operator basis. An arbitrary operator acting on a qudit can be expanded over an orthonormal and unitary operator basis $\{E_0, E_1, \ldots, E_{d^2-1}\}$, where $E_0 = I$ and $\mathrm{tr}(E_i^\dagger E_j) = d\delta_{ij}$. Any element E_i can be generated from tensor products of X and Z, where $X|k\rangle = |k+1\rangle$ and $Z|k\rangle = \omega^k |k\rangle$, such that the relation $XZ = \omega^{-1} ZX$ is satisfied [28]. Here ω is a dth root of unity and X and Z are the generalizations of Pauli operators to higher dimension.

Characterization of dynamical population. A measurement scheme for determining the quantum dynamical population, χ_{mm}, in a single experimental configuration. Let us prepare a maximally entangled state of the two qudits $|\varphi_C\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} |k\rangle_A |k\rangle_B$. This state is stabilized under the action of stabilizer operators $S = X^A X^B$ and $S' = Z^A (Z^B)^{d-1}$, and it is referred to as a *stabilizer state* [1,28]. After applying the quantum map to the qudit A, $\mathcal{E}(\rho)$, where $\rho = |\phi_C\rangle\langle\phi_C|$, we can perform a projective measurement $P_k P_{k'} \mathcal{E}(\rho) P_k P_{k'}$, where $P_k = \frac{1}{d} \sum_{l=0}^{d-1} \omega^{-lk} S^l, P_{k'} = \frac{1}{d} \sum_{l'=0}^{d-1} \omega^{-l'k'} S' l'$, and $\omega = e^{i2\pi/d}$. Then, we calculate the joint probability distributions of the outcomes k and k': $\mathrm{Tr}[P_k P_{k'} \mathcal{E}(\rho)] = \chi_{mm}$, where the elements χ_{mm} represent the population of error operators that anticommute with stabilizer generators S and S' with eigenvalues ω^k and $\omega^{k'}$, respectively. Therefore, with a *single* experimental configuration we can identify all diagonal elements of the superoperator.

Characterization of dynamical coherence. For measuring the quantum dynamical coherence, we create a *nonmaximally* entangled state of the two qudits $|\varphi_C\rangle = \sum_{i=0}^{d-1} \alpha_i |i\rangle_A |i\rangle_B$. This state has the sole stabilizer operator $S = E_i^A (E_i^B)^{d-1}$ (for detailed restrictions on the coefficients α_i see Sec. VIII). After applying the dynamical map to the qudit A, $\mathcal{E}(\rho)$, we perform a projective measurement $\rho_k = P_k \mathcal{E}(\rho) P_k$, and calculate the probability of the outcome k, $\mathrm{Tr}[P_k \mathcal{E}(\rho)] = \sum_m \chi_{mm} + 2\sum_{m<n} \mathrm{Re}[\chi_{mn} \mathrm{Tr}(E_n^{A\dagger} E_m^A \rho)]$; where E_m^A are all the operators in the operator basis, $\{E_j^A\}$, that anticommute with the operator E_i^A with the same eigenvalue ω^k. We also measure the expectation values of all independent operators $T_{rs} = E_r^A E_s^B$ of the Pauli group (where $E_r^A \neq I$; $E_s^B \neq I$) that simultaneously commute with the stabilizer generator S, $\mathrm{Tr}(T_{rs} \rho_k)$. There are only $d-1$ such operators T_{rs} that are independent of each other, within a multiplication by a stabilizer generator; and they belong to an Abelian subgroup of the normalizer group. The normalizer group is the group of unitary operators that preserve the stabilizer group by conjugation, i.e., $TST^\dagger = S$. We repeat this procedure $d+1$ times, by preparing the eigenkets of other stabilizer operator $E_i^A (E_i^B)^{d-1}$ for all $i \in \{1, 2, \ldots, d+1\}$, such that states $|i\rangle_A$ in input states belong to a mutually unbiased basis [32]. Also, we can change the measurement basis $d-1$ times, each time choosing normal-

izer elements T_{rs} from a different Abelian subgroup of the normalizer, such that their eigenstates form a mutually unbiased basis in the code space. Therefore, we can completely characterize quantum dynamical coherence by $(d+1)(d-1)$ different measurements, and the overall number of experimental configuration for a qudit becomes d^2. For N-dimensional quantum systems, with N a power of a prime, the required measurements are simply the tensor product of the corresponding measurements on individual qudits–see the Appendix . For quantum system whose dimension is not a power of a prime, the task can be accomplished by embedding the system in a larger Hilbert space whose dimension is a prime.

XIII. OUTLOOK

An important and promising advantage of DCQD is for use in partial characterization of quantum dynamics, where in general, one cannot afford or does not need to carry out a full characterization of the quantum system under study, or when one has some *a priori* knowledge about the dynamics. Using indirect methods of QPT in those situations is inefficient, because one must apply the whole machinery of the scheme to obtain the relevant information about the system. On the other hand, the DCQD scheme has built-in applicability to the task of partial characterization of quantum dynamics. In general, one can substantially reduce the overall number of measurements, when estimating the coherence elements of the superoperator for only specific subsets of the operator basis and/or subsystems of interest. This fact has been demonstrated in Ref. [26] in a generic fashion, and several examples of partial characterization have also been presented. Specifically, it was shown that DCQD can be applied to (single- and two-qubit) Hamiltonian identification tasks. Moreover, it is demonstrated that the DCQD algorithm enables the simultaneous determination of coarse-grained (semiclassical) physical quantities, such as the longitudinal relaxation time T_1 and the transversal relaxation (or dephasing) time T_2 for a single qubit undergoing a general homogenizing quantum map. The DCQD scheme can also be used for performing generalized quantum dense coding tasks. Other implications and applications of DCQD for partial QPT remain to be investigated and explored.

An alternative representation of the DCQD scheme for higher-dimensional quantum systems, based on generalized Bell-state measurements will be presented in Ref. [33]. The connection of Bell-state measurements to stabilizer and normalizer measurements in DCQD for two-level systems, can be easily observed from Table II of Ref. [3]. Our presentation of the DCQD algorithm assumes ideal (i.e., error-free) quantum state preparation, measurement, and ancilla channels. However, these assumptions can all be relaxed in certain situations, in particular when the imperfections are already known. A discussion of these issues is beyond the scope of this work and will be the subject of a future presentation [33].

There are a number of other directions in which the results presented here can be extended. One can combine the DCQD algorithm with the method of maximum likelihood estimation [35], in order to minimize the statistical errors in each experimental configuration invoked in this scheme. Moreover, a new scheme for *continuous* characterization of quantum dynamics can be introduced, by utilizing weak measurements for the required quantum error detections in DCQD [36,37]. Finally, the general techniques developed for direct characterization of quantum dynamics could be further utilized for control of open quantum systems [38].

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APPENDIX: GENERALIZATION TO ARBITRARY OPEN QUANTUM SYSTEMS

Here, we first demonstrate that the overall measurements for a full characterization of the dynamics of an n qudit systems (with d being a prime) become the tensor product of the required measurements on individual qudits. One of the important examples of such systems is a QIP unit with r qubits, thus having a 2^r-dimensional Hilbert space. Let us consider a quantum system consisting of r qudits, $\rho=\rho_1\otimes\rho_2\otimes\cdots\otimes\rho_r$, with a Hilbert space of dimension $N=d^r$. The output state of such a system after a dynamical map becomes $\mathcal{E}(\rho)=\sum_{m,n=0}^{N^2-1}\chi_{mn}E_m\rho E_n^\dagger$ where here $\{E_m\}$ are the unitary operator basis elements of an N-dimensional Hilbert space. These unitary operator basis elements can be written as $E_m=X^{q_{m_1}}Z^{p_{m_1}}\otimes X^{q_{m_2}}Z^{p_{m_2}}\otimes\cdots\otimes X^{q_{m_r}}Z^{p_{m_r}}$ [34]. Therefore, we have

$$\begin{aligned}
\mathcal{E}(\rho) &= \sum_{m,n=0}^{N^2-1}\chi_{mn}(X^{q_{m_1}}Z^{p_{m_1}}\otimes\ \dots\ \otimes X^{q_{m_n}}Z^{p_{m_n}})\rho_1\otimes\ \dots \\
&\quad\otimes\rho_n(X^{q_{n_1}}Z^{p_{n_1}}\otimes\ \dots\ \otimes X^{q_{n_r}}Z^{p_{n_r}})^\dagger \\
&= \sum_{m_1,\dots,m_r,n_1,\dots,n_r=0}^{d^2-1}\chi_{(m_1\dots m_r)(n_1\dots n_r)}(E_{m_1}\rho_1E_{n_1}^\dagger) \\
&\quad\otimes\dots(E_{m_s}\rho_sE_{n_s}^\dagger)\dots\otimes(E_{m_r}\rho_rE_{n_r}^\dagger) \\
&= \sum_{m_1\dots m_r,n_1\dots n_r=0}^{d^2-1}\chi_{(m_1\dots m_r)(n_1\dots n_r)}(E_m\rho E_n^\dagger)_s^{\otimes^r},
\end{aligned}$$

where we have introduced $E_{m_s}=X^{q_{m_s}}Z^{p_{m_s}}$ and $\chi_{mn}=\chi_{(m_1,\dots,m_r)(n_1,\dots,n_r)}$. I.e., $m=(m_1,\dots,m_s,\dots,m_r)$ and $n=(n_1,\dots,n_s,\dots,n_r)$, and the index s represents a generic qudit. Let us first investigate the tensor product structure of the DCQD algorithm for characterization of the diagonal elements of the superoperator. We prepare the eigenstate of the stabilizer operators $(E_i^AE_j^B)_s^{\otimes^r}$ and $(E_{i'}^AE_{j'}^B)_s^{\otimes^r}$. For each qudit, the projection operators corresponding to outcomes ω^k and $\omega^{k'}$ (where $k,k'=0,1,\dots,d-1$), have the form P_k

TABLE I. Required physical resources for the QPT schemes: Standard quantum process tomography (SQPT), ancilla-assisted process tomography using separable joint measurements (AAPT), using mutual unbiased bases measurements (MUB), using generalized measurements (POVM), see Ref. [3], and direct characterization of quantum dynamics (DCQD). The overall number of measurements is reduced quadratically in the DCQD algorithm with respect to the separable methods of QPT. This comes at the expense of requiring entangled input states, and two-qudit measurements of the output states. The nonseparable AAPT schemes require many-body interactions that are not available experimentally [3].

Scheme	$\dim(\mathcal{H})$[a]	N_{inputs}	$N_{\text{exp.}}$[b]	Measurements	Required interactions
SQPT	d^n	$d^2 n$	d^{4n}	One body	Single body
AAPT	d^{2n}	1	d^{4n}	Joint one body	Single body
AAPT (MUB)	d^{2n}	1	$d^{2n}+1$	MUB	Many body
AAPT (POVM)	d^{4n}	1	1	POVM	Many body
DCQD	d^{2n}	$[(d+1)+1]^n$	d^{2n}	Stabilizer/Normalizer	Single and two body

[a] $\mathcal{H}$, the Hilbert space of each experimental configuration.
[b] Overall number of experimental configurations.

$=\frac{1}{d}\sum_{l=0}^{d-1}\omega^{-lk}(E_i^A E_j^B)^l$ and $P_{k'}=\frac{1}{d}\sum_{l'=0}^{d-1}\omega^{-l'k'}(E_{i'}^A E_{j'}^B)^{l'}$. The joint probability distribution of the commuting Hermitian operators $P_{k_1},P_{k_1'},P_{k_2},P_{k_2'},\ldots,P_{k_r},P_{k_r'}$ on the output state $\mathcal{E}(\rho)$ is

$$\mathrm{Tr}[(P_kP_{k'})_s^{\otimes r}\mathcal{E}(\rho)]=\frac{1}{(d^2)^r}\sum_{m_1,\ldots,m_r,n_1,\ldots,n_r=0}^{d^2-1}\chi_{(m_1,\ldots,m_r)(n_1,\ldots,n_r)}\times\left(\sum_{l=0}^{d-1}\sum_{l'=0}^{d-1}\omega^{-lk}\omega^{-l'k'}\mathrm{Tr}[E_n^\dagger(E_i^A)^l(E_{i'}^A)^{l'}\times E_m(E_j^B)^l(E_{j'}^B)^{l'}\rho]\right)_s^{\otimes r}.$$

By introducing $E_iE_m=\omega^{i_m}E_mE_i$ for each qudit and using the relation $[(E_i^AE_j^B)^l(E_{i'}^AE_{j'}^B)^{l'}\rho]_s=\rho_s$ we obtain

$$\mathrm{Tr}[(P_kP_{k'})_s^{\otimes r}\mathcal{E}(\rho)]=\frac{1}{(d^2)^r}\sum_{m_1,\ldots,m_r,n_1,\ldots,n_r=0}^{d^2-1}\chi_{(m_1,\ldots,m_r)(n_1,\ldots,n_r)}\times\left(\sum_{l=0}^{d-1}\sum_{l'=0}^{d-1}\omega^{(i_m-k)l}\omega^{(i_m'-k')l'}\mathrm{Tr}[E_n^\dagger E_m\rho]\right)_s^{\otimes r}$$

Using the QEC condition for nondegenerate codes, $\mathrm{Tr}[E_n^\dagger E_m\rho]_s=(\delta_{mn})_s$, and also using the discrete Fourier transform identities $\sum_{l=0}^{d-1}\omega^{(i_m-k)l}=d\delta_{i_m,k}$ and $\sum_{l'=0}^{d-1}\omega^{(i_m'-k')l'}=d\delta_{i_m',k'}$ for each qudit, we get

$$\mathrm{Tr}[(P_kP_{k'})_s^{\otimes r}\mathcal{E}(\rho)]=\sum_{m_1,\ldots,m_r,n_1,\ldots,n_r=0}^{d^2-1}\chi_{(m_1,\ldots,m_r)(n_1,\ldots,n_r)}\times(\delta_{i_m,k}\delta_{i_m',k'}\delta_{mn})_s^{\otimes r}=\chi_{(m_{01},\ldots,m_{0r})(m_{01},\ldots,m_{0r})},$$

where for each qudit, the index m_0 is defined through the relations $i_{m_0}=k$ and $i'_{m_0}=k'$, etc. I.e., E_{m_0} is the unique error operator that anticommutes with the stabilizer operators of each qudit with a fixed pair of eigenvalues ω^k and $\omega^{k'}$corresponding to experimental outcomes k and k'. Since P_k and $P_{k'}$ operator have d eigenvalues, we have d^2 possible outcomes for each qudit, which overall yields $(d^2)^r$ equations that can be used to characterize all the diagonal elements of the superoperator with a single ensemble measurement and $(2d)^r$ detectors. Similarly, the off-diagonal elements of superoperators can be identified by a tensor product of the operations in the DCQD algorithm for each individual qudit, see Ref. [26]. A comparison of the required physical resources for n qudits is given in Table I.

For a d-dimensional quantum system where d is neither a prime nor a power of a prime, we can always imagine another d'-dimensional quantum system such that d' is prime, and embed the principal qudit as a subspace into that system. For example, the energy levels of a six-level quantum system can be always regarded as the first six energy levels of a virtual seven-level quantum system, such that the matrix elements for coupling to the seventh level are practically zero. Then, by considering the algorithm for characterization of the virtual seven-level system, we can perform only the measurements required to characterize superoperator elements associated with the first six energy levels.

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