

Abelian and non-Abelian geometric phases in adiabatic open quantum systemsM. S. Sarandy¹ and D. A. Lidar²¹*Instituto de Física de São Carlos, Universidade de São Paulo, São Carlos, SP, 13560-970, Brazil*²*Departments of Chemistry, Electrical Engineering-Systems, and Physics, University of Southern California, Los Angeles, California 90089, USA*

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We introduce a self-consistent framework for the analysis of both Abelian and non-Abelian geometric phases associated with open quantum systems, undergoing cyclic adiabatic evolution. We derive a general expression for geometric phases, based on an adiabatic approximation developed within an inherently open-systems approach. This expression provides a natural generalization of the analogous one for closed quantum systems, and we prove that it satisfies all the properties one might expect of a good definition of a geometric phase, including gauge invariance. A striking consequence is the emergence of a finite time interval for the observation of geometric phases. The formalism is illustrated via the canonical example of a spin-1/2 particle in a time-dependent magnetic field. Remarkably, the geometric phase in this case is immune to dephasing and spontaneous emission in the renormalized Hamiltonian eigenstate basis. This result positively impacts holonomic quantum computing.

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I. INTRODUCTION

More than 20 years ago, Berry first observed that quantum systems may retain a memory of their motion in Hilbert space through the acquisition of geometric phases [1]. Remarkably, these phase factors depend only on the geometry of the path traversed by the system during its evolution. Soon after Berry's discovery, geometric phases became a subject of intense theoretical and experimental studies [2]. In recent years, renewed interest has arisen in the study of geometric phases in connection with quantum-information processing [3,4]. Indeed, geometric or holonomic quantum computation (QC) may be useful in achieving fault tolerance [5], since the geometric character of the phase provides protection against certain classes of errors [6–8]. However, a comprehensive investigation in this direction requires a generalization of the concept of geometric phases to the domain of *open* quantum systems, i.e., quantum systems that may decohere due to their interaction with an external environment.

Geometric phases in open systems, and more recently their applications in holonomic QC, have been considered in a number of works since the late 1980s. The first approach to the subject used the Schrödinger equation with non-Hermitian Hamiltonians [9,10]. This is a phenomenological, nonrigorous approach (e.g., it cannot guarantee completely positivity). A consistent non-Hermitian Hamiltonian description of an open system in general requires the theory of stochastic Schrödinger equations [11]. Nevertheless, this approach indicated for the first time that complex Abelian geometric phases should appear for systems undergoing cyclic evolution. In Refs. [12–17], geometric phases acquired by the density operator were analyzed for various explicit models within a master equation approach, but no general theory was formulated for open-system geometric phases. In Refs. [8,18], the quantum jumps method was employed to provide a definition of geometric phases in Markovian open systems (related difficulties with stochastic unravelings have been pointed out in Ref. [19]). In another approach, the den-

sity operator, expressed in its eigenbasis, was lifted to a purified state [20,21]. In Ref. [22], a formalism in terms of mean values of distributions was presented. An interferometric approach for evaluating geometric phases for mixed states evolving unitarily was introduced in Ref. [23] and extended to nonunitary evolution in Refs. [24,25]. This interferometric approach can also be considered from a purification point of view [23,25]. This multitude of different proposals has revealed various interesting facets of the problem. Nevertheless, the concept of adiabatic geometric phases in open systems remains unresolved in general, since most of the previous treatments did not employ an adiabatic approximation genuinely developed for open systems. Note that the applicability of the closed-systems adiabatic approximation [27] to open-systems problems is not *a priori* clear and should be justified on a case-by-case basis. Moreover, almost all of the previous works on open-systems geometric phases were concerned with the Abelian (Berry phase) case. Exceptions are the very recent Refs. [8,28,29], which discuss both nonadiabatic and adiabatic dynamics, but employ the standard adiabatic theorem for closed systems in the latter case.

In this work, we introduce a self-consistent open-systems framework, based on a recent generalization of the adiabatic approximation [30], which allows for a general definition and evaluation of both Abelian and non-Abelian geometric phases in open systems undergoing cyclic adiabatic evolution. As we shall show, this approach yields new insights and lends itself to a simple and elegant generalization of the concept of geometric phases. An important feature emerging from this picture is the appearance of a distinguished time scale for the observation of adiabatic geometric phases in open systems. We illustrate our results by considering the canonical example of a spin-1/2 in a magnetic field. In this example, we find a remarkable robustness of the geometric phase against both dephasing and spontaneous emission in the instantaneous renormalized Hamiltonian eigenstate basis. This result should have a positive impact on the robustness against external disturbances of holonomic QC.

We note that an alternative theory for adiabaticity in open systems was recently developed by Thunström, Åberg, and Sjöqvist, for systems coupled weakly to their environment. This theory was then employed to study a non-Abelian geometric phase gate in holonomic QC [26]. The main difference between the approach of Ref. [26] and our approach to adiabaticity in open systems is that Thunström *et al.* emphasize the decoupling of eigenspaces of the system Hamiltonian, while we focus on the entire superoperator. Central conclusions, such as the breakdown of adiabaticity in open systems, are shared by the two approaches. A full comparison is beyond the scope of the present work.

The structure of this paper is as follows. In the next section, we review the coherence vector approach to solving quantum master equations, the Jordan form of the superoperator, and the notion of adiabaticity in open quantum systems. In Sec. III, we derive the geometric phase in open systems, in both the Abelian (Berry phase) and non-Abelian cases. We prove that this geometric phase satisfies the expected properties, such as a proper closed system limit and gauge invariance. In Sec. IV, we focus on applications, namely we show that our theory predicts that there is a distinguished time scale for open-system geometric phases (related to the breakdown of adiabaticity), and then we consider the example of a spin-1/2 coupled to a slowly varying magnetic field, in the presence of dephasing and spontaneous emission. We conclude in Sec. V.

II. MASTER EQUATIONS AND THE ADIABATIC REGIME OF OPEN QUANTUM SYSTEMS

Open quantum systems typically do not undergo unitary dynamics, i.e., they are not governed by the Schrödinger equation, or even by its non-Hermitian generalization [9,10]. Instead, quite generally we may consider open quantum systems evolving under a convolutionless master equation [31]

$$\partial\rho/\partial t = \mathcal{L}[\vec{R}(t)]\rho(t), \quad (1)$$

where \mathcal{L} is a superoperator that depends on time only through a set of parameters $\vec{R}(t) \equiv \vec{R}$. The Lindblad equation [32] is an important example of this class of master equations,

$$\partial\rho/\partial t = -i[H, \rho] + \frac{1}{2} \sum_i ([\Gamma_i, \rho \Gamma_i^\dagger] + [\Gamma_i \rho, \Gamma_i^\dagger]), \quad (2)$$

where we have suppressed the explicit dependence of the operators on $\vec{R}(t)$. Here H is the effective Hamiltonian of the open system (it is renormalized, i.e., contains the “Lamb shift”—the unitary contribution of the system-bath interaction [33,34]), the Γ_i are operators describing the system-bath interaction, and we work in $\hbar=1$ units. In this work, we consider the general class of convolutionless master equations (1), and in a later section we illustrate our formalism with an example using the case of Eq. (2). In this example of a spin-1/2 in a magnetic field, we allow *both* H and Γ_i to depend on $\vec{R}(t)$. In a slight abuse of nomenclature, we will refer to the implicitly time-dependent generator \mathcal{L} [Eq. (1)]

as the Lindblad superoperator and the Γ_i [Eq. (2)] as Lindblad operators. This terminology is usually associated with time-independent generators [32,33], but recent work has clarified the conditions under which Eq. (2) with time-dependent Γ_i can be derived in the usual Davies [35] weak-coupling limit [36]. What is important to note is that the microscopic weak-coupling limit derivation is consistent with the general class of master equations postulated here, wherein the Lindblad operators depend implicitly on time through their explicit dependence on external control fields. More specifically, in the microscopic derivation one shows that the Lindblad operators are the Fourier components of the time-dependent system operator terms in the system-bath interaction Hamiltonian, where the time dependence arises by working in the interaction picture with respect to the *renormalized* system Hamiltonian [36]. This is how the Γ_i 's appearing here and in our example below must be interpreted.

In the superoperator formalism, the density matrix for a quantum state in a D -dimensional Hilbert space is represented by a D^2 -dimensional “coherence vector” $|\rho\rangle\rangle = (\rho_1, \rho_2, \dots, \rho_{D^2})^t$ and the Lindblad superoperator \mathcal{L} becomes a $(D^2 \times D^2)$ -dimensional supermatrix [33], so that the master equation (1) can be written as a linear vector equation in D^2 -dimensional Hilbert-Schmidt space, in the form

$$\partial|\rho\rangle\rangle/\partial t = \mathcal{L}[\vec{R}(t)]|\rho\rangle\rangle. \quad (3)$$

Such a representation can be generated, e.g., by introducing a basis of Hermitian, trace-orthogonal, and traceless operators [e.g., the D -dimensional irreducible representation of the generators of $\mathfrak{su}(D)$], whence the ρ_i are the expansion coefficients of ρ in this basis [33], with ρ_1 the coefficient of I (the identity matrix). In this case, the condition $\text{Tr}\rho^2 \leq 1$ corresponds to $\| |\rho\rangle\rangle \| \leq 1$, $\rho = \rho^\dagger$ to $\rho_i = \rho_i^*$, and positive semidefiniteness of ρ is expressed in terms of inequalities satisfied by certain Casimir invariants [e.g., of $\mathfrak{su}(D)$] [37,38]. A simple and well-known example of this procedure is the representation of the density operator of a two-level system (qubit) on the Bloch sphere, via $\rho = (I_2 + \vec{v} \cdot \vec{\sigma})/2$, where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the vector of Pauli matrices [generators of $\mathfrak{su}(2)$] and I_2 is the 2×2 identity matrix.

The master equation generates a nonunitary evolution since \mathcal{L} is non-Hermitian. In fact, \mathcal{L} need not even be a normal operator ($\mathcal{L}^\dagger \mathcal{L} \neq \mathcal{L} \mathcal{L}^\dagger$). Therefore, \mathcal{L} is generally not diagonalizable, i.e., it does not possess a complete set of linearly independent eigenvectors. Equivalently, it cannot be put into diagonal form via a similarity transformation. However, one can always apply a similarity transformation to \mathcal{L} which puts it into the (block-diagonal) Jordan canonical form [39], namely $\mathcal{L}_J = S^{-1} \mathcal{L} S$. The Jordan form \mathcal{L}_J of a $D^2 \times D^2$ matrix \mathcal{L} is a direct sum of blocks of the form $\mathcal{L}_J = \bigoplus_{\alpha=1}^m J_\alpha$ (α enumerates Jordan blocks), where $m \leq D^2$ is the number of linearly independent eigenvectors of \mathcal{L} , $\sum_{\alpha=1}^m n_\alpha = D^2$, where $n_\alpha \equiv \dim J_\alpha$ is the dimension of the α th Jordan block, and $J_\alpha = \lambda_\alpha I_{n_\alpha} + K_\alpha$, where λ_α is the α th (generally complex-valued) Lindblad-Jordan (LJ) eigenvalue of \mathcal{L} (obtained as roots of the characteristic polynomial), I_{n_α} is the $n_\alpha \times n_\alpha$ dimensional identity matrix, and K_α is a nilpotent matrix with elements $(K_\alpha)_{ij} = \delta_{i,j-1}$ (1's above the main diagonal), where δ

is the Kronecker symbol. Instantaneous right $\{|\mathcal{D}_\beta^{(j)}[\vec{R}(t)]\rangle\rangle$ and left $\{\langle\langle\mathcal{E}_\alpha^{(i)}[\vec{R}(t)]|\rangle\rangle$ bi-orthonormal bases in Hilbert-Schmidt space can always be systematically constructed such that they obey the orthonormality condition $\langle\langle\mathcal{E}_\alpha^{(i)}|\mathcal{D}_\beta^{(j)}\rangle\rangle = \delta_{\alpha\beta}\delta^{ij}$ [30]. Here superscripts enumerate basis states inside a given Jordan block ($i, j \in \{0, \dots, n_\alpha - 1\}$). When \mathcal{L} is diagonalizable, $\{|\mathcal{D}_\beta^{(j)}[\vec{R}(t)]\rangle\rangle$ and $\{\langle\langle\mathcal{E}_\alpha^{(i)}[\vec{R}(t)]|\rangle\rangle$ are simply the bases of right and left eigenvectors of \mathcal{L} , respectively. If \mathcal{L} is not diagonalizable, these right and left bases can be constructed by suitably completing the set of right and left eigenvectors of \mathcal{L} (see Ref. [30] and also Appendix A for a detailed discussion of the left and right basis vectors, including their completeness relation). Based on the above considerations, we gave, in Ref. [30], a definition of adiabaticity in open quantum systems:

An open quantum system is said to undergo adiabatic dynamics when its Hilbert-Schmidt space can be decomposed into decoupled LJ-eigenspaces with distinct, time-continuous, and noncrossing instantaneous eigenvalues of \mathcal{L} .

Note that the key to establishing the concept of adiabaticity in open systems is to replace the idea of decoupling of the Hamiltonian eigenstates by that of decoupling of the Jordan blocks of the Lindblad superoperator (each Jordan block is associated with an independent eigenstate of \mathcal{L}). The definition of adiabaticity given above implies a condition on the total evolution time T which generalizes the well-known closed-systems condition, $T \gg \max_{0 \leq s \leq 1} |\langle k(s) | dH(s)/ds | m(s) \rangle| / g_{mk}(s)^2$, where $s = t/T$ is the normalized time, H is the time-dependent Hamiltonian, $|k\rangle$ and $|m\rangle$ are eigenstates of H , and g_{mk} is the energy gap between these two states. For further details, we refer the reader to Refs. [30,42]; the condition in the case of one-dimensional Jordan blocks is given in Eq. (32) below. The theory developed in Ref. [30] applies also in the more general case of explicitly time-dependent generators $\mathcal{L}(t)$, but since we focus here on geometric phases, we shall only consider implicit time dependence as in $\mathcal{L}[\vec{R}(t)]$.

III. GEOMETRIC PHASES FOR OPEN SYSTEMS IN CYCLIC ADIABATIC EVOLUTION

In order to define geometric phases in open systems, we expand the coherence vector in the instantaneous right vector basis $\{|\mathcal{D}_\beta^{(j)}[\vec{R}(t)]\rangle\rangle$ as

$$|\rho(t)\rangle\rangle = \sum_{\beta=1}^m \sum_{j=0}^{n_\beta-1} p_\beta^{(j)}(t) e^{i \int_0^t \lambda_\beta^{(j)} dt'} |\mathcal{D}_\beta^{(j)}[\vec{R}(t)]\rangle\rangle, \quad (4)$$

where we have explicitly factored out the dynamical phase $\exp[i \int_0^t \lambda_\beta^{(j)} dt']$. The coefficients $\{p_\beta^{(j)}(t)\}$ play the role of “geometric” (nondynamical) amplitudes. We assume that the open system is in the adiabatic regime, i.e., *Jordan blocks associated to distinct eigenvalues evolve in a decoupled manner* (recall the definition of open systems adiabaticity given above). Then, Eqs. (3) and (4), together yield

$$\dot{p}_\alpha^{(i)} = p_\alpha^{(i+1)} - \sum_{\beta | \lambda_\beta = \lambda_\alpha} \sum_{j=0}^{n_\beta-1} p_\beta^{(j)} \langle\langle \mathcal{E}_\alpha^{(i)} | \dot{\mathcal{D}}_\beta^{(j)} \rangle\rangle. \quad (5)$$

A condition on the total evolution time T , which allows one to neglect the coupling between Jordan blocks used in deriving Eq. (5), was given in Ref. [30]. It is given by $T \gg \max_\alpha T_\alpha^c$, where T_α^c is given for the case of one-dimensional Jordan blocks in Eq. (32) below. Note that, due to the restriction $\lambda_\beta = \lambda_\alpha$, the dynamical phase has disappeared. For closed systems, Abelian geometric phases are associated with nondegenerate levels of the Hamiltonian, while non-Abelian phases appear in the case of degeneracy. In the latter case, a subspace of the Hilbert space acquires a geometric phase which is given by a matrix rather than a scalar. Here, for open systems, one-dimensional Jordan blocks can be associated either with Abelian or non-Abelian geometric phases (depending on the possibility of degeneracy) while multidimensional Jordan blocks are naturally tied to a non-Abelian phase.

A. The Abelian case: Generalized Berry phase

Consider the simple case of a nondegenerate one-dimensional Jordan block (a block that is a 1×1 submatrix containing an eigenvalue of \mathcal{L}). In this case, the absence of degeneracy implies in Eq. (5) that $\lambda_\beta = \lambda_\alpha \Rightarrow \alpha = \beta$ (nondegenerate blocks). Moreover, since the blocks are assumed to be one-dimensional we have $n_\alpha = 1$, which allows us to remove the upper indices in Eq. (5), resulting in $\dot{p}_\alpha = -p_\alpha \langle\langle \mathcal{E}_\alpha | \dot{\mathcal{D}}_\alpha \rangle\rangle$. The solution of this equation is $p_\alpha(t) = p_\alpha(0) \exp[i \gamma_\alpha(t)]$, with $\gamma_\alpha(t) = i \int_0^t \langle\langle \mathcal{E}_\alpha(t') | \dot{\mathcal{D}}_\alpha(t') \rangle\rangle dt'$. In order to establish the geometric character of $\gamma_\alpha(t)$, we now recall that \mathcal{L} depends on time implicitly through the parameters $\vec{R}(t)$. Then, for a cyclic evolution in parameter space along a closed curve C , we obtain that the Abelian geometric phase associated with the Jordan block α is given by

$$\gamma_\alpha(C) = i \oint_C \langle\langle \mathcal{E}_\alpha(\vec{R}) | \vec{\nabla} | \mathcal{D}_\alpha(\vec{R}) \rangle\rangle \cdot d\vec{R}. \quad (6)$$

This elegant generalized expression for the geometric phase, which bears similarity to the original Berry formula [1], is our first main result. As expected for open systems, $\gamma_\alpha(C)$ is complex, since $\langle\langle \mathcal{E}_\alpha |$ and $| \mathcal{D}_\alpha \rangle\rangle$ are not related by transpose conjugation. Thus, the geometric phase may have real and imaginary contributions, the latter affecting the visibility of the phase.

In Refs. [9,10], Garrison and Wright and Dattoli *et al.* found an expression for the open-systems Berry phase that resembles our Eq. (6). Their result is

$$\tilde{\gamma}_\alpha(C) = i \oint_C \langle \theta_\alpha(\vec{R}) | \vec{\nabla} | \psi_\alpha(\vec{R}) \rangle \cdot d\vec{R}. \quad (7)$$

Here $\{|\psi_\alpha\rangle\rangle$ and $\{|\theta_\alpha\rangle\rangle$ are a bi-orthonormal set of eigenvectors of a non-Hermitian Hamiltonian H and its Hermitian conjugate H^\dagger , respectively. There are some important methodological and technical differences between this and our

result. First, here, instead of working with a phenomenological non-Hermitian Hamiltonian, we started from the outset with a fully consistent master equation approach, where the left $\{\langle\langle\mathcal{E}_\alpha|\}\}$ and right $\{|\mathcal{D}_\alpha\rangle\rangle\}$ basis vectors are associated with the dynamical superoperator \mathcal{L} , rather than with the non-Hermitian Hamiltonian. Second, as a result in our case, the basis vectors span the D^2 -dimensional Hilbert-Schmidt space, whereas in the non-Hermitian Hamiltonian case the geometric phase expression involves vectors in the usual D -dimensional Hilbert space. As we note below, this implies that in our case γ_α is a relative, not absolute, geometric phase, and hence there is in general no connection between the expressions (6) and (7). Third, unlike Refs. [9,10], where adiabaticity is imported from the theory of closed systems, we work within a consistent theory of adiabaticity for open systems, as formulated in Ref. [30].

The expression for $\gamma_\alpha(C)$ exhibits a number of important properties expected from a good definition of a geometric phase, as follows

Geometric character. $\gamma_\alpha(C)$ is geometric, i.e., it depends only on the path traversed in parameter space.

Gauge invariance. $\gamma_\alpha(C)$ is gauge invariant, i.e., we cannot modify (or eliminate) the geometric phase by redefining $\langle\langle\mathcal{E}_\alpha|$ or $|\mathcal{D}_\alpha\rangle\rangle$ by multiplying one of them by a complex factor $\chi(\vec{R})\exp[i\nu(\vec{R})]$. Indeed, let us define $|\mathcal{D}'_\alpha\rangle\rangle = \chi \exp(i\nu) |\mathcal{D}_\alpha\rangle\rangle$ [$\chi(\vec{R}) \neq 0 \forall \vec{R}$]. Redefinition of right-vectors automatically implies redefinition of left-vectors due to the normalization constraint $\langle\langle\mathcal{E}_\alpha|\mathcal{D}_\alpha\rangle\rangle = 1$, so that $\langle\langle\mathcal{E}'_\alpha| = \langle\langle\mathcal{E}_\alpha|X^{-1}\exp(-i\nu)$. Therefore, $\langle\langle\mathcal{E}'_\alpha|\vec{\nabla}|\mathcal{D}'_\alpha\rangle\rangle = \langle\langle\mathcal{E}_\alpha|\vec{\nabla}|\mathcal{D}_\alpha\rangle\rangle + (\vec{\nabla}\chi)/\chi + i\vec{\nabla}\nu$. Gauge invariance then follows from the computation of γ'_α using Eq. (6), with Stokes's theorem leading to $\gamma'_\alpha(C) = \gamma_\alpha(C)$. Below, we provide a detailed proof of gauge invariance in the non-Abelian case, which includes the latter as a special case.

Closed-system limit. if the interaction with the bath vanishes, $\gamma_\alpha(C)$ reduces to the usual difference of geometric phases acquired by the density operator in the closed case. In order to prove this, consider the expansion of the vectors $|\mathcal{D}_\alpha\rangle\rangle$ and $\langle\langle\mathcal{E}_\alpha|$ in a basis $\{I_D, \Lambda_i | i=1, \dots, D^2-1\}$, where I_D is the $D \times D$ identity matrix and Λ_i are traceless Hermitian matrices, with $\text{Tr}(\Lambda_i \Lambda_j) = \delta_{ij}$. Then, by using the normalization condition $\langle\langle\mathcal{E}_\alpha|\mathcal{D}_\alpha\rangle\rangle = 1$ and the matrix inner product $\langle\langle u|v\rangle\rangle = (1/D)\text{Tr}(u^\dagger v)$ [39], we obtain in the closed-case limit $|\mathcal{D}_\alpha\rangle\rangle \rightarrow \sqrt{D}|\psi_m\rangle\langle\psi_n|$ (equivalently, $\langle\langle\mathcal{E}_\alpha| \rightarrow \sqrt{D}|\psi_n\rangle\langle\psi_m|$), with $\{|\psi_m\rangle\}$ denoting a set of normalized eigenstates of the Hamiltonian operator. Therefore, Eq. (6) yields $\gamma_\alpha \rightarrow \gamma_m^{\text{closed}} - \gamma_n^{\text{closed}}$, which is exactly the difference of phases acquired by the density matrix in closed systems. Note that only phase *differences* are experimentally observable, so that the fact that our expression for the geometric phase involves phase differences, rather than an absolute phase, is natural. As mentioned above, this is an important aspect in which our expressions differ from the ones derived using non-Hermitian Hamiltonians [9,10].

B. The non-Abelian case: Generalized holonomic connection

Let us now generalize these considerations to *degenerate* one-dimensional Jordan blocks, whence the geometric phase

becomes non-Abelian. From Eq. (5) we obtain

$$\dot{p}_\alpha = - \sum_{\beta|\lambda_\beta=\lambda_\alpha} p_\beta \langle\langle\mathcal{E}_\alpha|\dot{\mathcal{D}}_\beta\rangle\rangle. \quad (8)$$

Each decoupled subspace is associated with a different value of λ_α , and is spanned by the set $\{|\mathcal{D}_\beta(\vec{R})\rangle\rangle | \lambda_\beta = \lambda_\alpha\}$. Then, enumerating this list for each decoupled subspace (denoted by λ_α as $|\mathcal{D}_{\lambda_\alpha}^{(1)}\rangle\rangle, |\mathcal{D}_{\lambda_\alpha}^{(2)}\rangle\rangle, \dots, |\mathcal{D}_{\lambda_\alpha}^{(G)}\rangle\rangle$ (the left basis $\{\langle\langle\mathcal{E}_\alpha(\vec{R})|\}\}$ is similarly enumerated), with G the degeneracy (dimension of the decoupled subspace), we have that $\dot{p}_{\lambda_\alpha}^{(i)} = -\sum_{j=1}^G p_{\lambda_\alpha}^{(j)} \langle\langle\mathcal{E}_{\lambda_\alpha}^{(i)}|\dot{\mathcal{D}}_{\lambda_\alpha}^{(j)}\rangle\rangle$. Writing this equation in a vector notation, we obtain

$$\dot{\mathbf{P}}_{\lambda_\alpha} = -(\vec{A}_{\lambda_\alpha} \cdot \dot{\vec{R}})\mathbf{P}_{\lambda_\alpha}, \quad (9)$$

where $\mathbf{P}_{\lambda_\alpha} = (p_{\lambda_\alpha}^{(1)}, \dots, p_{\lambda_\alpha}^{(G)})^t$ is a vector in Hilbert-Schmidt space (superscript t denotes transposition) and

$$\vec{A}_{\lambda_\alpha} = \begin{pmatrix} \langle\langle\mathcal{E}_{\lambda_\alpha}^{(1)}|\vec{\nabla}|\mathcal{D}_{\lambda_\alpha}^{(1)}\rangle\rangle & \dots & \langle\langle\mathcal{E}_{\lambda_\alpha}^{(1)}|\vec{\nabla}|\mathcal{D}_{\lambda_\alpha}^{(G)}\rangle\rangle \\ \vdots & \ddots & \vdots \\ \langle\langle\mathcal{E}_{\lambda_\alpha}^{(G)}|\vec{\nabla}|\mathcal{D}_{\lambda_\alpha}^{(1)}\rangle\rangle & \dots & \langle\langle\mathcal{E}_{\lambda_\alpha}^{(G)}|\vec{\nabla}|\mathcal{D}_{\lambda_\alpha}^{(G)}\rangle\rangle \end{pmatrix}. \quad (10)$$

Note that each element of \vec{A} is a vector in parameter space (we use boldface and arrow superscripts to denote vectors in Hilbert-Schmidt space and parameter space, respectively). The non-Abelian geometric phase in a cyclic evolution associated with a degenerate level λ_α is determined by the solution of Eq. (9), which is formally provided by $\mathbf{P}_{\lambda_\alpha}(C) = \mathcal{U} \mathbf{P}_{\lambda_\alpha}(0)$, where

$$\mathcal{U} = \mathcal{P} e^{-\oint_C \vec{A}_{\lambda_\alpha} \cdot d\vec{R}} \quad (11)$$

is the corresponding Wilson loop, and \mathcal{P} denotes path-ordering.

Equations (10) and (11) and (11) constitute our second main result. They are the generalization of the concept of non-Abelian geometric phases to the open-systems case. In particular, the matrix \vec{A}_{λ_α} [Eq. (10)] naturally generalizes the Wilczek-Zee gauge potential [40], also known as the holonomic connection.

A non-Abelian geometric phase will also appear in the case of multidimensional Jordan blocks. However, in this case, it is not possible to obtain a general analytical solution due to the presence of the term $p_\alpha^{(i+1)}$ in Eq. (5). One should then solve Eq. (5) on a case by case basis for all pairs (α, i) . This yields a set of coupled differential equations in a ladder structure.

The geometric character of the non-Abelian geometric phase is evident from the expression (11) for the Wilson loop operator (it depends only on the path and not on its parametrization). The closed-system limit is obtained in a manner exactly analogous to the proof above for the Abelian case. What is left, therefore, in order to demonstrate that our expressions for the generalized non-Abelian geometric phase

have the desired properties, is a proof of gauge invariance of the eigenvalues of the Wilson loop. We consider this issue next.

C. Gauge invariance of the non-Abelian geometric phase in open systems

Let us show that the geometric phase derived in Eq. (11) is invariant under gauge transformations. First, we can rewrite Eq. (9) in the form

$$\dot{p}_{\lambda_\alpha}^{(i)} = - \sum_{j=1}^G A_{\lambda_\alpha}^{(ij)} p_{\lambda_\alpha}^{(j)}, \quad (12)$$

where

$$A_{\lambda_\alpha}^{(ij)} \equiv \langle\langle \mathcal{E}_{\lambda_\alpha}^{(i)} | \dot{\mathcal{D}}_{\lambda_\alpha}^{(j)} \rangle\rangle = \sum_{\mu} (A_{\lambda_\alpha}^{(ij)})_{\mu} \frac{\partial x_{\mu}(t)}{\partial t} = (A_{\lambda_\alpha}^{\tilde{i}j}) \cdot \dot{\vec{R}}, \quad (13)$$

where we have introduced the explicit dependence on $\vec{R}(t) = (x_1(t), \dots, x_n(t))$, and

$$(A_{\lambda_\alpha}^{(ij)})_{\mu} \equiv \langle\langle \mathcal{E}_{\lambda_\alpha}^{(i)} | \partial_{\mu} | \mathcal{D}_{\lambda_\alpha}^{(j)} \rangle\rangle. \quad (14)$$

Let us now apply a local gauge transformation $\Omega(\vec{R}(t))$ on the right eigenvectors,

$$| \mathcal{D}_{\lambda_\alpha}^{(a)} \rangle\rangle = \sum_d \Omega_{ad} | \mathcal{D}_{\lambda_\alpha}^{(d)} \rangle\rangle, \quad (15)$$

where Ω is an arbitrary complex matrix. Here Ω is taken as a function of $\vec{R}(t)$ instead of t because the basis vectors depend on t only through the parameters $\vec{R}(t)$. The normalization constraint $\langle\langle \mathcal{E}_{\lambda_\alpha}' | \mathcal{D}_{\lambda_\alpha}' \rangle\rangle = 1$ implies

$$\langle\langle \mathcal{E}_{\lambda_\alpha}^{(b)} | = \sum_c \langle\langle \mathcal{E}_{\lambda_\alpha}^{(c)} | \Omega_{cb}^{-1}. \quad (16)$$

The gauge transformation of $A_{\lambda_\alpha}^{(ba)} \equiv \langle\langle \mathcal{E}_{\lambda_\alpha}^{(b)} | \mathcal{D}_{\lambda_\alpha}^{(a)} \rangle\rangle$, is then given by

$$\begin{aligned} A_{\lambda_\alpha}^{(ba)'} &= \langle\langle \mathcal{E}_{\lambda_\alpha}^{(b)'} | \mathcal{D}_{\lambda_\alpha}^{(a)'} \rangle\rangle = \sum_{cd} \Omega_{ad} (A_{\lambda_\alpha}^t)_{dc} \Omega_{cb}^{-1} + \dot{\Omega}_{ad} \delta_{cd} \Omega_{cb}^{-1} \\ &= (\Omega A_{\lambda_\alpha}^t \Omega^{-1})_{ab} + (\dot{\Omega} \Omega^{-1})_{ab}, \end{aligned} \quad (17)$$

where superscript t denotes transposition. Therefore,

$$A_{\lambda_\alpha}^{\prime t} = \Omega A_{\lambda_\alpha}^t \Omega^{-1} + \dot{\Omega} \Omega^{-1}, \quad (18)$$

which proves that $A_{\lambda_\alpha}^t$ transforms as a gauge potential.

Now let us show that the Wilson loop has gauge-invariant eigenvalues (a similar proof for the closed case can be found in Ref. [41]). As a first step, consider the Wilson operator for an open curve,

$$\begin{aligned} \mathcal{U} &= \mathcal{P} \exp \left(- \int_{\vec{R}(0)}^{\vec{R}(t)} \vec{A} \cdot d\vec{R} \right) \\ &= \mathcal{P} \exp \left(- \int_0^t dt' A_{\mu}(x) \frac{dx^{\mu}}{dt'} \right) \\ &= 1 - \int_0^t dt_1 A_{\mu_1}(t_1) \frac{dx^{\mu_1}}{dt_1} \\ &\quad + \int_0^t dt_1 A_{\mu_1}(t_1) \frac{dx^{\mu_1}}{dt_1} \int_0^{t_1} dt_2 A_{\mu_2}(t_2) \frac{dx^{\mu_2}}{dt_2} + \dots, \end{aligned} \quad (19)$$

where repeated indices are summed over, and we suppress the subscript λ_α for notational simplicity. The transposed Wilson operator then yields

$$\begin{aligned} \mathcal{U}^t &= 1 - \int_0^t dt_1 A_{\mu_1}^t(t_1) \frac{dx^{\mu_1}}{dt_1} \\ &\quad + \int_0^t dt_1 \int_0^{t_1} dt_2 A_{\mu_2}^t(t_2) \frac{dx^{\mu_2}}{dt_2} A_{\mu_1}^t(t_1) \frac{dx^{\mu_1}}{dt_1} + \dots \end{aligned} \quad (20)$$

Note the inversion of the order of the operators due to the transposition. Therefore, the transposed Wilson operator $W \equiv \mathcal{U}^t$ obeys the differential equation

$$\frac{dW}{dt} + W A_{\mu}^t \frac{dx^{\mu}}{dt} = 0. \quad (21)$$

We can determine the gauge transformation of W by imposing gauge invariance of Eq. (21).

After a gauge transformation, Eq. (21) reads

$$\frac{dW'}{dt} + W' A_{\mu}^{\prime t} \frac{dx^{\mu}}{dt} = 0, \quad (22)$$

where primes indicate gauge-transformed operators. Note that A_{μ}^t , which is given by

$$(A_{\mu}^t)_{ab} = \langle\langle \mathcal{E}_{\lambda_\alpha}^{(b)} | \partial_{\mu} | \mathcal{D}_{\lambda_\alpha}^{(a)} \rangle\rangle, \quad (23)$$

transforms, according to Eq. (18), under a gauge transformation as

$$A_{\mu}^{\prime t} = \Omega A_{\mu}^t \Omega^{-1} + (\partial_{\mu} \Omega) \Omega^{-1}. \quad (24)$$

Then, using Eq. (24), we obtain

$$\left(\frac{d(W' \Omega)}{dt} + (W' \Omega) A_{\mu}^t \frac{dx^{\mu}}{dt} \right) \Omega^{-1} = 0. \quad (25)$$

Since Ω is arbitrary, it follows from Eq. (25) that gauge invariance of the equation of motion implies

$$W \rightarrow W' = \tilde{\Omega} W \Omega^{-1}, \quad (26)$$

where $\Omega^{-1} = \Omega^{-1}(x^{\mu}(t))$ and $\tilde{\Omega}$ is independent of $x^{\mu}(t)$.

The gauge transformation of a product of paths allows us to further restrict $\tilde{\Omega}$: we can show that $\tilde{\Omega} = \Omega^{-1}(x_0)$, where here and below $x_i \equiv x^{\mu}(t_i)$ [$x^{\mu}(t_0)$ is the initial position $\vec{R}(0)$]. To see this, consider an open curve $\Gamma = \Gamma_1 + \Gamma_2$, where Γ_1 is a

continuous curve in the interval $[x_0, x_a]$ and Γ_2 is a continuous curve in the interval $[x_a, x_b]$. The transposed Wilson operators $W(\Gamma_1)$ and $W(\Gamma_2)$ associated with these curves are, according to Eq. (20), given by

$$\begin{aligned} W(\Gamma_1) &= 1 - \int_0^{t_a} dt_1 A_{\mu_1}^t(t_1) \frac{dx^{\mu_1}}{dt_1} \\ &\quad + \int_0^{t_a} dt_1 \int_0^{t_1} dt_2 A_{\mu_2}^t(t_2) \frac{dx^{\mu_2}}{dt_2} A_{\mu_1}^t(t_1) \frac{dx^{\mu_1}}{dt_1} + \dots, \\ W(\Gamma_2) &= 1 - \int_{t_a}^{t_b} dt_1 A_{\mu_1}^t(t_1) \frac{dx^{\mu_1}}{dt_1} \\ &\quad + \int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_1} dt_2 A_{\mu_2}^t(t_2) \frac{dx^{\mu_2}}{dt_2} A_{\mu_1}^t(t_1) \frac{dx^{\mu_1}}{dt_1} + \dots. \end{aligned} \quad (27)$$

Then, under gauge transformation, we have from Eq. (26)

$$\begin{aligned} W(\Gamma_1) &\rightarrow \tilde{\Omega}_1 W(\Gamma_1) \Omega(x_a)^{-1}, \\ W(\Gamma_2) &\rightarrow \tilde{\Omega}_2 W(\Gamma_2) \Omega(x_b)^{-1}, \\ W(\Gamma) &\rightarrow \tilde{\Omega} W(\Gamma) \Omega(x_b)^{-1}. \end{aligned} \quad (28)$$

Then, by applying a gauge transformation on $W(\Gamma) = W(\Gamma_1)W(\Gamma_2)$, we obtain

$$\tilde{\Omega} W(\Gamma) \Omega(x_b)^{-1} = \tilde{\Omega}_1 W(\Gamma_1) \Omega(x_a)^{-1} \tilde{\Omega}_2 W(\Gamma_2) \Omega(x_b)^{-1}. \quad (29)$$

This implies (i) $\tilde{\Omega}_2 = \Omega(x_b)$, i.e., $\tilde{\Omega}$ depends only on the initial position, and (ii) $\tilde{\Omega}_1 = \tilde{\Omega}$ ($\tilde{\Omega}$ must be independent of the path index). We can therefore write $\tilde{\Omega} = \Omega(x_0)$. Thus, for a general open curve Γ in the interval $[x_0, x_a]$, the transposed Wilson operator transforms as $W \rightarrow W' = \Omega(x_0)W\Omega^{-1}(x_a)$. For a closed curve C , we then get $W \rightarrow W' = \Omega(x_0)W\Omega^{-1}(x_0)$, or, expressed in terms of the Wilson loop itself

$$\mathcal{U}(C) \rightarrow \mathcal{U}'(C) = (\Omega^t)^{-1}(x_0) \mathcal{U}(C) \Omega^t(x_0). \quad (30)$$

Thus, the Wilson loop transforms as a similarity transformation, and consequently its eigenvalues are gauge invariant, as desired. In particular, this implies that the gauge-transformed amplitude vector $\mathbf{P}'_{\lambda_\alpha} \equiv (\Omega^t)^{-1}(x_0) \mathbf{P}_{\lambda_\alpha}$ obeys the same geometric phase transformation rule as $\mathbf{P}_{\lambda_\alpha}$,

$$\begin{aligned} \mathcal{U}' \mathbf{P}'_{\lambda_\alpha}(0) &= (\Omega^t)^{-1}(x_0) \mathcal{U} \Omega^t(x_0) (\Omega^t)^{-1}(x_0) \mathbf{P}_{\lambda_\alpha}(0) \\ &= (\Omega^t)^{-1}(x_0) \mathbf{P}_{\lambda_\alpha}(C) = \mathbf{P}'_{\lambda_\alpha}(C). \end{aligned} \quad (31)$$

IV. APPLICATIONS

A. A distinguished time scale for open-system geometric phases

As a first application of these general considerations, we now show the existence of a distinguished time scale for the

observation of open-system geometric phases. To this end, it is convenient to express the variables in terms of the dimensionless time $s = t/T$, where T denotes the total evolution time. Then, adiabatic dynamics in the interval $0 \leq s \leq 1$ occurs if and only if the following time condition is satisfied: $T \gg \max_\alpha \{T_\alpha^c\}$, where T_α^c denotes the *crossover time* for the Jordan block J_α [30]. For the particular case of one-dimensional blocks, we have [30,42]

$$\begin{aligned} T_\alpha^c &= \max_{0 \leq s \leq 1} \left| \sum_{\beta \neq \alpha} \left[Q_{\beta\alpha}(0) - Q_{\beta\alpha}(s) e^{T\Omega_{\beta\alpha}(s)} \right. \right. \\ &\quad \left. \left. + \int_0^s ds' e^{T\Omega_{\beta\alpha}(s')} dQ_{\beta\alpha}(s')/ds' \right] \right|, \end{aligned} \quad (32)$$

where

$$\begin{aligned} \Omega_{\beta\alpha}(s) &= \int_0^s \omega_{\beta\alpha}(s') ds', \\ \omega_{\beta\alpha}(s) &= \gamma_\beta(s) - \gamma_\alpha(s) \end{aligned} \quad (33)$$

is the gap between Jordan eigenvalues,

$$V_{\beta\alpha}(s) = p_\beta(s) \langle \langle \mathcal{E}_\alpha(s) | \frac{d\mathcal{L}(s)}{ds} | \mathcal{D}_\beta(s) \rangle \rangle \quad (34)$$

are the matrix elements of the time derivative of the Lindblad superoperator, and

$$Q_{\beta\alpha}(s) \equiv V_{\beta\alpha}(s) / \omega_{\beta\alpha}^2(s). \quad (35)$$

Note that a quantity analogous to $Q_{\beta\alpha}$, namely the time derivative of the Hamiltonian divided by the square of the spectral gap, appears in the standard condition for adiabaticity in closed systems [30]. In the expression for $V_{\beta\alpha}(s)$, upper indices in $p_\beta^{(j)}(s)$ and in the basis vectors $\{ | \mathcal{D}_\beta^{(j)}(s) \rangle \}$ and $\{ \langle \mathcal{E}_\alpha^{(i)}(t) | \}$ were removed because the Jordan blocks are one-dimensional. The crossover time T_α^c provides a decoupling time scale for each Jordan block: *provided $T \gg T_\alpha^c$, the Jordan block J_α is adiabatically decoupled from all other blocks associated to a different eigenvalue.* The general expression for T_α^c in the case of multidimensional Jordan blocks, as well as a more detailed discussion of its meaning, are given in Refs. [30,42].

Now, the important observation is that the decoupling time scale is *finite* due to the presence of complex exponentials in T_α^c [30], which have real and imaginary parts in the case of open systems. Therefore, since the geometric phases are defined in the adiabatic regime, they will only be observable during the finite time in which the Jordan blocks are decoupled. This fact implies the existence of a distinguished, finite time scale for geometric phases in open systems. Such a time scale was noted in Ref. [16] in the context of a specific example, namely the case of a spin-1/2 particle in a magnetic field. Finite adiabaticity time -scales have been revealed as a general property of open systems [26,30,43], a fact that has also been observed in adiabatic QC, both theoretically [42] and experimentally [44]. Physically, the reason for this phenomenon is the broadening of the system energy levels due to the presence of a dense spectrum of bath energy

levels, until the broadened system energies overlap. When this happens, different eigenspaces may no longer be decoupled (provided there are no selection rules preventing the coupling), and the adiabatic approximation breaks down. In the case of static Hamiltonians, this is known as quantum diffusion [45].

B. Spin-1/2 in a time-dependent magnetic field under decoherence

As an illustration of the general theory presented above, let us consider the canonical example of a spin-1/2 in a time-dependent magnetic field, originally considered by Berry in the context of closed quantum systems [1]. The renormalized Hamiltonian of the system is given by $H(\vec{B}) = -\mu\vec{S}\cdot\vec{B}$, where $\vec{S}=(1/2)(\sigma^x, \sigma^y, \sigma^z)$ is the spin operator, with σ^i ($i=x, y, z$) denoting the Pauli matrices, $\vec{B}(t) = (B_x(t), B_y(t), B_z(t))$ is a time-dependent magnetic field (including the Lamb shift correction [33,34]), and μ is a constant. A standard evaluation of the geometric phase in this case yields $\gamma_{\pm}^{\text{closed}}(C) = \pm\Omega(C)/2$, where $\gamma_{\pm}^{\text{closed}}(C)$ are the geometric phases associated with the energy levels $E_{\pm} = \pm(\mu/2)B$, with $B = |\vec{B}(t)|$, and $\Omega(C)$ being the solid angle subtended by the closed curve C traversed by the magnetic field in parameter space.

In the weak-coupling regime, it is common to consider decoherence in the eigenbasis of the renormalized system Hamiltonian [36,46,47]. Let us now analyze the effects of decoherence in this basis, assuming that open-systems dynamics is described by the master equation (2). We consider two important sources of decoherence, namely dephasing and spontaneous emission in the eigenenergy basis. The Lindblad operators modeling these processes are given, respectively, by $\Gamma_z = \beta_z W(\vec{B})\sigma_z W^\dagger(\vec{B})$ and $\Gamma_- = \beta_- W(\vec{B})\sigma_- W^\dagger(\vec{B})$ ($\sigma_-|1\rangle = 2|0\rangle, \sigma_-|0\rangle = 0$), where β_z and β_- are the error probabilities per unit time and $W(\vec{B})$ is the unitary matrix that diagonalizes $H(\vec{B})$. The Lindblad superoperator is then given by $\mathcal{L}(\vec{B}) = \mathcal{H}(\vec{B}) + \mathcal{R}(\vec{B})$, where $\mathcal{H}(\vec{B})$ is the Hamiltonian superoperator [obtained from $H(\vec{B})$] and $\mathcal{R}(\vec{B})$ is the superoperator containing the decoherence contribution (obtained from Γ_z and Γ_-). In this case, explicit calculation reveals that (i) \mathcal{H} and \mathcal{R} are diagonalizable, (ii) $[\mathcal{H}, \mathcal{R}] = 0$. Hence \mathcal{L} , \mathcal{H} , and \mathcal{R} have a common eigenstate basis, and in particular it follows that \mathcal{L} has only one-dimensional Jordan blocks, whence it is diagonalizable. Thus, bearing in mind that the eigenstate basis for $\mathcal{H}(\vec{B})$ is independent of β_z and β_- , it follows that the eigenstate basis for $\mathcal{L}(\vec{B})$ is also independent of β_z and β_- . This implies that *the adiabatic geometric phases, which can be computed here from Eq. (6), are robust against dephasing (Γ_z) and spontaneous emission (Γ_-)*. In this case, integration in parameter space, which is the relevant space for adiabatic geometric phases, is not affected by decoherence. Nevertheless, the adiabaticity crossover time T_α^c does depend on β_z and β_- through the eigenvalues of \mathcal{L} .

The robustness against dephasing is in agreement with Ref. [18], but obtained here in a totally different framework. A microscopic derivation of the geometric phase for a spin-1/2 in a magnetic field was developed in Ref. [17], with no robustness against dephasing detected. However, note that the robustness of the geometric phase obviously depends on the basis in which the environment acts. As is clear from Eq. (4) of Ref. [17], Whitney *et al.* consider dephasing in a *fixed* (time-independent) basis, where robustness is absent. In this basis, our approach is in agreement with this lack of robustness. However, in the example we discuss here, we consider decoherence in the *instantaneous* eigenenergy basis. In the weak-coupling regime, one should consider decoherence in the eigenbasis of the system Hamiltonian (see, e.g., Refs. [46,47]). This follows from the fundamental Davies derivation of the quantum Markovian master equation [35], which was recently reviewed and generalized in a context relevant to ours in Ref. [36], and which shows that in the Markovian limit time-dependent system Hamiltonians are always coupled to the Lindblad operators. This difference in basis explains the apparent disparity between Ref. [17] and our result. Then, in the instantaneous eigenbasis, we obtain robustness against both dephasing and spontaneous emission for adiabatic evolution as a simple consequence of the commutation relation between the Hamiltonian superoperator \mathcal{H} and the corresponding decoherence superoperators \mathcal{R} . In fact, the commutation between \mathcal{H} and \mathcal{R} provides a general sufficient condition for robustness of adiabatic geometric phases against \mathcal{R} . Note also that the nonadiabatic geometric phase is usually affected by corrections due to the system-bath interaction, in particular in the case of spontaneous emission [18]. Remarkably, by imposing adiabaticity on the open system, robustness of the geometric phase against this decoherence process is obtained.

We stress that, to the best of our knowledge, the approach presented here for dealing with geometric phases is the first to predict robustness against both dephasing and spontaneous emission for a spin-1/2. We expect that such a robustness will serve as a useful protection mechanism in holonomic QC (see, e.g., Ref. [8] for difficulties in the correction of spontaneous emission). The robustness is reminiscent of the emergence of a decoherence-free subspace (DFS) [48], but unlike the symmetry-driven appearance of the latter, here the robustness is due to a, fundamentally different, adiabatic mechanism. Related observations were made, using very different methods, in Ref. [49], for a system in a DFS, coupled to a cyclically evolving reservoir.

As for closed systems, B plays an important role in setting the time interval for the observation of the geometric phase. The reason is that the Zeeman effect further splits the system energy levels, thus postponing the breakdown of adiabaticity due to overlap caused by environmentally induced broadening. This behavior is illustrated in Fig. 1 for the case of dephasing, where we take $\beta_z = 0.1$ (in units such that $\mu = 1$) and use the following spherically symmetric configuration for the magnetic field: $B_x(s) = B \cos(2\pi s) \sin \theta$, $B_y(s) = B \sin(2\pi s) \sin \theta$, and $B_z = B \cos \theta$, with θ denoting the azimuthal angle, set at $\pi/3$. The initial state of the system is chosen to be an equal superposition of energy eigenstates. Due to the commutation relation $[\mathcal{H}, \mathcal{R}] = 0$, we find that the

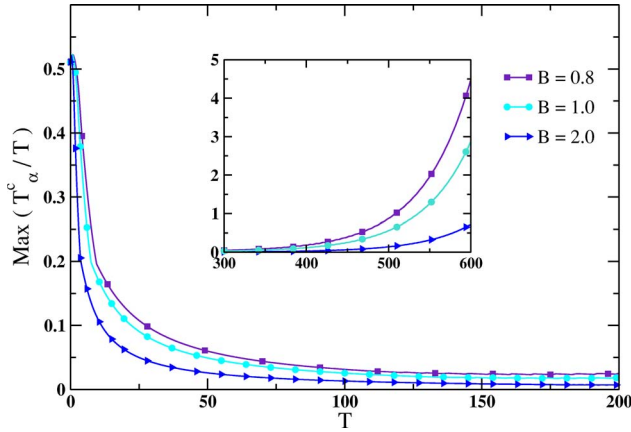


FIG. 1. (Color online) Maximum value of the ratio T_α^c/T of the crossover time to the total evolution time, taken over all the Jordan blocks, as a function of T for a spin-1/2 particle in a magnetic field undergoing dephasing. Parameter values are given in the text. The adiabatic interval requires $T_\alpha^c \ll T$. Observe that the stronger the magnetic field (at fixed decoherence rate), the better the adiabatic approximation, and hence the longer we can observe the geometric phase.

four Jordan blocks are associated with the set of Hilbert-Schmidt space vectors $\{|\psi_m\rangle\langle\psi_n|\}$ ($m, n=1, 2$), where $\{|\psi_m\rangle\}$ are the normalized eigenstates of the system Hamiltonian. Then, direct computation of γ_α from Eq. (6) (for $s=1$) yields $\gamma_\alpha = \pm 2\pi \cos \theta$ for the vectors $|\psi_m\rangle\langle\psi_n|$ with $m \neq n$. Up to an unimportant 2π factor, these are exactly the differences of geometric phases $\pm 2\pi(1 - \cos \theta)$ appearing in the density operator for these states in the closed case. As suggested from the analytical treatment above, similar results hold for spontaneous emission.

It should be noticed, however, that robustness is, naturally, not universal; e.g., it does not hold for the bit-flip channel $\Gamma_x(\vec{B}) = \beta_x W(\vec{B}) \sigma_x W(\vec{B})^\dagger$, since Γ_x does not commute with the Hamiltonian superoperator. This result is illustrated in Fig. 2, where it is shown that the real part of the geometric phase under bit-flip is slightly affected by the decoherence process. This is in contrast with the robustness of dephasing and spontaneous emission. The corresponding imaginary part of the geometric phase (not shown) has negligible variation, which means that the visibility of the phase is not significantly affected by the bit-flip channel for the decoherence strengths considered in the plot.

V. CONCLUSIONS

We have introduced a general framework for geometric phases in open systems undergoing cyclic adiabatic evolution. Expressions which naturally generalize the familiar closed systems' (Abelian) Berry phase and (non-Abelian) Wilczek-Zee gauge potential and Wilson loop were derived, and their gauge invariance proven. An important feature of our approach is the existence of a distinguished time scale for the observation of the adiabatic geometric phase. This property imposes time constraints on realistic schemes for holonomic QC based on adiabatic phases. Remarkably, ro-

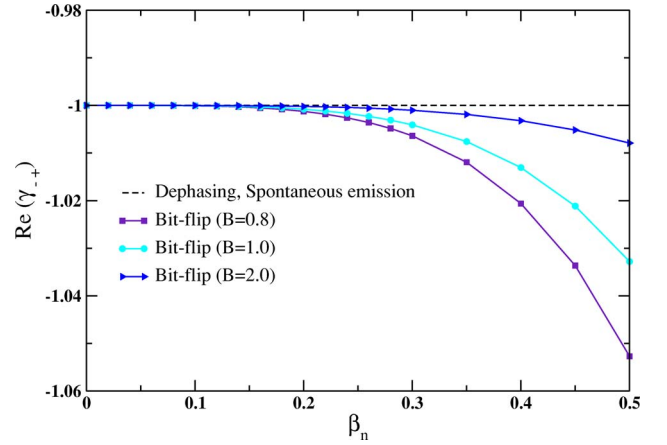


FIG. 2. (Color online) Geometric phase (in units of π) of a spin-1/2 in a magnetic field, as a function of the decoherence strengths β_n (n is chosen as z , $-$, or x depending on the decoherence process, as indicated in the legend). The magnetic field is applied in a spherically symmetric configuration whose parameters are chosen as in Fig. 1. The geometric phase $\gamma_{\rightarrow}(C)$ plotted is for the Jordan block associated—in the limit of vanishing decoherence—with the closed-system geometric phase difference $\gamma_-^{\text{closed}}(C) - \gamma_+^{\text{closed}}(C)$, which equals $-2\pi[1 - \cos(\theta = \pi/3)] = -\pi$.

bustness against dephasing and spontaneous emission was found for the geometric phase acquired by a spin-1/2 in a magnetic field. This robustness is, however, not universal; e.g., it does not hold for the bit-flip channel, since the latter does not commute with the Hamiltonian superoperator. The development of methods for overcoming decoherence affecting geometric phases is therefore of significant interest [8,50].

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APPENDIX A

We define the right $\{|\mathcal{D}_\beta^{(j)}[\vec{R}(t)]\rangle\rangle\}$ and left $\{\langle\langle \mathcal{E}_\alpha^{(i)}[\vec{R}(t)] | \}$ basis vectors associated with $\mathcal{L}[\vec{R}(t)]$ and prove their bi-orthogonality and completeness. Instantaneous right and left eigenstates of a general time-dependent superoperator $\mathcal{L}(t)$ are defined by

$$\mathcal{L}(t)|\mathcal{P}_\alpha(t)\rangle\rangle = \lambda_\alpha(t)|\mathcal{P}_\alpha(t)\rangle\rangle, \quad (\text{A1})$$

$$\langle\langle \mathcal{Q}_\alpha(t) | \mathcal{L}(t) = \langle\langle \mathcal{Q}_\alpha(t) | \lambda_\alpha(t), \quad (\text{A2})$$

where possible degeneracies correspond to $\lambda_\alpha = \lambda_\beta$, with $\alpha \neq \beta$. In other words, we reserve a different index α for each independent eigenvector since each eigenvector is in a distinct Jordan block. It follows from Eqs. (A1) and (A2) that, on the one hand,

$$\langle\langle \mathcal{Q}_\beta(t) | \mathcal{L}(t) | \mathcal{P}_\alpha(t) \rangle\rangle = \lambda_\alpha(t) \langle\langle \mathcal{Q}_\beta(t) | \mathcal{P}_\alpha(t) \rangle\rangle, \quad (\text{A3})$$

while on the other hand this equals

$$\langle\langle \mathcal{Q}_\beta(t) | \mathcal{L}(t) | \mathcal{P}_\alpha(t) \rangle\rangle = \langle\langle \mathcal{Q}_\beta(t) | \mathcal{P}_\alpha(t) \rangle\rangle \lambda_\beta(t). \quad (\text{A4})$$

Therefore, for $\lambda_\alpha \neq \lambda_\beta$, we have $\langle\langle \mathcal{Q}_\alpha(t) | \mathcal{P}_\beta(t) \rangle\rangle = 0$.

The left and right eigenstates can be easily identified when the Lindblad superoperator is in the Jordan form $\mathcal{L}_J(t) = S^{-1}(t) \mathcal{L} S(t)$. Denoting $|\mathcal{P}_\alpha(t)\rangle_J = S^{-1}(t) |\mathcal{P}_\alpha(t)\rangle$, i.e., the right ‘‘Jordan basis’’ (note the J subscript) eigenstate of $\mathcal{L}_J(t)$ associated to a Jordan block J_α , then Eq. (A1) implies that $|\mathcal{P}_\alpha(t)\rangle_J$ is time-independent and, after normalization, is given by

$$|\mathcal{P}_\alpha\rangle_J|_{J_\alpha} = (1, 0, 0, \dots, 0)^t, \quad (\text{A5})$$

where only the vector components associated to the Jordan block J_α are shown, with all the others vanishing. In order to have a complete basis we shall define new states, which will be chosen so that they preserve the block structure of $\mathcal{L}_J(t)$. A suitable set of additional vectors is

$$\begin{aligned} |\mathcal{D}_\alpha^{(1)}\rangle_J|_{J_\alpha} &= (0, 1, 0, \dots, 0)^t, \dots, \\ |\mathcal{D}_\alpha^{(n_\alpha-1)}\rangle_J|_{J_\alpha} &= (0, \dots, 0, 1)^t, \end{aligned} \quad (\text{A6})$$

where again all the components outside J_α are zero. This simple vector structure allows for the derivation of the expression

$$\mathcal{L}_J(t) |\mathcal{D}_\alpha^{(j)}\rangle_J = |\mathcal{D}_\alpha^{(j-1)}\rangle_J + \lambda_\alpha(t) |\mathcal{D}_\alpha^{(j)}\rangle_J, \quad (\text{A7})$$

with $|\mathcal{D}_\alpha^{(0)}\rangle_J = |\mathcal{P}_\alpha\rangle_J$ and $|\mathcal{D}_\alpha^{(-1)}\rangle_J = 0$. The set $\{|\mathcal{D}_\alpha^{(j)}\rangle_J, j=0, \dots, (n_\alpha-1)\}$ can immediately be related to a right vector basis for the original $\mathcal{L}(t)$ by means of the transformation $|\mathcal{D}_\alpha^{(j)}(t)\rangle = S(t) |\mathcal{D}_\alpha^{(j)}\rangle_J$ which, applied to Eq. (A7), yields

$$\mathcal{L}(t) |\mathcal{D}_\alpha^{(j)}(t)\rangle = |\mathcal{D}_\alpha^{(j-1)}(t)\rangle + \lambda_\alpha(t) |\mathcal{D}_\alpha^{(j)}(t)\rangle. \quad (\text{A8})$$

Equation (A8) exhibits an important feature of the set $\{|\mathcal{D}_\beta^{(j)}(t)\rangle\}$, namely, it implies that Jordan blocks are invariant under the action of the Lindblad superoperator, i.e., the index α denoting the Jordan block is preserved under \mathcal{L} .

An analogous procedure can be employed to define the left basis. Denoting by ${}_J\langle\langle \mathcal{Q}_\alpha(t) | = \langle\langle \mathcal{Q}_\alpha(t) | S(t)$ the left eigenstate of $\mathcal{L}_J(t)$ associated to a Jordan block J_α , Eq. (A2) leads to the normalized left vector

$${}_J\langle\langle \mathcal{Q}_\alpha | |_{J_\alpha} = (0, \dots, 0, 1). \quad (\text{A9})$$

The additional left vectors are defined as (note that these are just the transpose of the right vectors in the Jordan basis)

$$\begin{aligned} {}_J\langle\langle \mathcal{E}_\alpha^{(0)} | |_{J_\alpha} &= (1, 0, 0, \dots, 0), \\ &\dots \\ {}_J\langle\langle \mathcal{E}_\alpha^{(n_\alpha-2)} | |_{J_\alpha} &= (0, \dots, 0, 1, 0), \end{aligned} \quad (\text{A10})$$

which imply the following expression for the left basis vector $\langle\langle \mathcal{E}_\alpha^{(i)}(t) | = {}_J\langle\langle \mathcal{E}_\alpha^{(i)} | S^{-1}(t)$ for $\mathcal{L}(t)$:

$$\langle\langle \mathcal{E}_\alpha^{(i)}(t) | \mathcal{L}(t) = \langle\langle \mathcal{E}_\alpha^{(i+1)}(t) | + \langle\langle \mathcal{E}_\alpha^{(i)}(t) | \lambda_\alpha(t), \quad (\text{A11})$$

or, equivalently,

$$\mathcal{L}(t)^\dagger |\mathcal{E}_\alpha^{(i)}\rangle = \lambda_\alpha^*(t) |\mathcal{E}_\alpha^{(i)}\rangle + |\mathcal{E}_\alpha^{(i+1)}\rangle. \quad (\text{A12})$$

Here we have used the notation ${}_J\langle\langle \mathcal{E}_\alpha^{(n_\alpha-1)} | \equiv {}_J\langle\langle \mathcal{Q}_\alpha |$ and ${}_J\langle\langle \mathcal{E}_\alpha^{(n_\alpha)} | \equiv 0$.

We can now derive the orthogonality and completeness relations. First, the left and right basis vectors are orthonormal,

$$\langle\langle \mathcal{E}_\alpha^{(i)}(t) | \mathcal{D}_\beta^{(j)}(t) \rangle\rangle = {}_J\langle\langle \mathcal{E}_\alpha^{(i)} | S^{-1}(t) S(t) | \mathcal{D}_\beta^{(j)} \rangle\rangle_J = \delta_{\alpha\beta} \delta^{ij}. \quad (\text{A13})$$

Second, it is clear that (since it is a standard basis) the Jordan basis is complete in the sense that $\sum_{\alpha,\beta;i,j} |\mathcal{D}_\beta^{(j)}\rangle_J \langle\langle \mathcal{E}_\alpha^{(i)} | = I$. Applying $S(t)$ to the left and $S^{-1}(t)$ to the right of this equation, we therefore find the completeness relation

$$\sum_{\alpha,\beta;i,j} |\mathcal{D}_\beta^{(j)}(t)\rangle \langle\langle \mathcal{E}_\alpha^{(i)}(t) | = I. \quad (\text{A14})$$

As a final point of clarification, note that, even though in the stationary Jordan basis left and right basis states coincide (up to transposition), this is not the case in the time-dependent basis. This difference between left and right vectors is due to the nonunitarity of the similarity matrix S . To see this, note that for a one-dimensional Jordan block, $|\mathcal{D}_\alpha\rangle_J = |\mathcal{E}_\alpha\rangle_J$, so instead of $\langle\langle \mathcal{E}_\alpha(t) | = {}_J\langle\langle \mathcal{E}_\alpha | S^{-1}(t)$ we can write $\langle\langle \mathcal{E}_\alpha(t) | = {}_J\langle\langle \mathcal{D}_\alpha | S^{-1}(t)$. Then $|\mathcal{E}_\alpha(t)\rangle = (S^{-1})^\dagger(t) |\mathcal{D}_\alpha\rangle_J$, which does not equal $|\mathcal{D}_\alpha(t)\rangle = S(t) |\mathcal{D}_\alpha\rangle_J$ since S is not unitary.

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