

## Entangling capacities of noisy two-qubit Hamiltonians

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We study the limits imposed by intrinsic fluctuations in system-control parameters on the ability of two-qubit (exchange) Hamiltonians to generate entanglement, starting from arbitrary initial states. We find three classes for Gaussian and Laplacian fluctuations. For the Ising and  $XYZ$  models there are qualitatively distinct, sharp entanglement-generation transitions, while the class of Heisenberg,  $XY$ , and  $XXZ$  Hamiltonians is capable of generating entanglement for any finite noise level. Our findings imply that exchange Hamiltonians are surprisingly robust in their ability to generate entanglement in the presence of noise, thus potentially reducing the need for quantum error correction.

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### I. INTRODUCTION

Considerable experimental efforts have been devoted in the past few years to the creation of entangled states, with impressive success in systems such as trapped ions, coupled atomic gas samples, polarized photons, and most recently, superconducting qubits [1]. An important motivation comes from quantum-information processing (QIP), where entanglement is believed to play an important role in algorithmic speedup, communication tasks, and cryptographic applications [2]. As the generation of entanglement often involves the manipulation of an interaction Hamiltonian, recent theoretical work has focused on the entanglement capabilities of such Hamiltonians. In particular, questions concerning optimality [3,4], equivalence classes [5], and entangling power/capacity [6], have been raised and answered, under the assumption of *noiseless* controls. Here we take the first step toward addressing what happens when this assumption is relaxed. In particular, we wish to find out the answer to the following question: *What are the limits imposed on entanglement generation via two-body Hamiltonians by fluctuations in system-control parameters?* [7]. We note that, as is well known, quantum error correction [2] offers a solution to both decoherence and the type of control errors we consider here; however, this solution involves a high cost in extra qubits and logic gates. In view of the central importance of entanglement in QIP, it is of significant interest to find out the limits imposed on entanglement generation via interaction Hamiltonians and state preparation, without any error correction.

### II. THE MODEL

Almost all quantum computing proposals are governed by interaction Hamiltonians that are used to enact two-qubit operations. The most general two-qubit (“exchange”) Hamiltonian has the form

$$H = \sum_{i < j} \sum_{\alpha, \beta = x, y, z} J_{\alpha\beta}^{ij} S_{\alpha}^i S_{\beta}^j, \quad (1)$$

where  $S_{\alpha} \equiv \frac{1}{2}\sigma_{\alpha}$  are the angular momentum operators ( $\sigma_{\alpha}$  are the Pauli matrices) and  $i, j$  are qubit indices. By applying local unitary operations it is always possible to transform  $H$

into a canonical, diagonal form [3]. Hence we will restrict our attention to the case  $J_{\alpha\beta}^{ij} = J_{\alpha}^{ij}\delta_{\alpha\beta}$  from now on. The various models are then classified as follows:  $XYZ$ :  $J_x \neq J_y \neq J_z$ ,  $XXZ$ :  $J_x = J_y \neq J_z$ ,  $XY$ :  $J_x = J_y, J_z = 0$ , Heisenberg:  $J_x = J_y = J_z$ , Ising:  $J_x = J_y = 0$ .

Two qubits can be entangled by first preparing a product state and then running the interaction for a desired amount of time to generate, e.g., a controlled-NOT (CNOT) or controlled-PHASE (CPHASE gate) [2]. Tunability of the coupling constants in  $H$  need not always be possible, even though it is a common assumption of QIP proposals. This leads to two qualitatively distinct scenarios we consider in this work: (a) Tunable interactions—where the interaction can be switched on and off (e.g., an exchange interaction mediated by a tunable tunneling barrier [8]), and (b) nontunable interactions—where the interaction is always on (e.g., a Coulomb interaction [9]), thus requiring, e.g., external single-qubit operations to refocus the interactions and enable controlled entanglement generation. Recent work has addressed the problem of universal quantum computation with nontunable couplings [10], and even unknown parameters [11]. In the laboratory, however, the execution of every single and two-qubit operation will generally be noisy due to system and experimental imperfections, over which we have limited control. We consider phenomenological noise models to describe noisy single and two-qubit operations, wherein certain control parameters vary stochastically. Specifically, we have considered two models: (a) Gaussian and (b) Laplacian parameter fluctuations. The Gaussian model has universal applicability in the case of noise induced by many weakly coupled random sources (by the central limit theorem). It has been extensively used and discussed in stochastic quantum mechanics (e.g., [12]). We consider the Laplacian model mainly to test the robustness of our results. Another important model is  $1/f$  noise due to bistable random fluctuators, which will be considered in a future publication.

### III. GAUSSIAN FLUCTUATIONS

Given a Hamiltonian  $K(t, J)$  (where  $J$  is a parameter or set of parameters), a unitary transformation  $U$  is generated by evolving under  $K$  for some time  $\tau$ :  $U(\phi) = \mathcal{T} \exp[-i \int_0^{\tau} K(t, J) dt]$ , where  $\mathcal{T}$  denotes time ordering. In

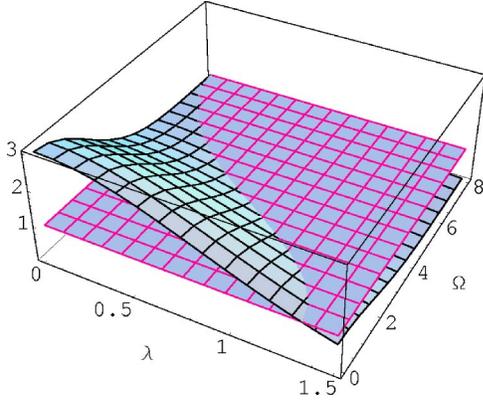


FIG. 1. Condition for entanglement as a function of interaction error  $\Omega$  and preparation error  $\lambda$  in the Ising model. Plotted is the PPT criterion [Eq. (3)]. The transition line between entanglement and separability is clearly visible, with the region above the horizontal plane corresponding to entanglement.

our analysis below we only deal with piecewise constant Hamiltonians, then the angle(s)  $\phi = \pi J$ . An initial state  $\rho$  transforms as  $\rho \rightarrow \rho(\phi) = U(\phi)\rho U(\phi)^\dagger$ . We now assume that  $\phi$  is Gaussian distributed with mean  $\bar{\phi}$  (the desired angle) and standard deviation (SD)  $\lambda$ :  $\phi \sim N(\bar{\phi}, \lambda)$ . This may be the outcome of Gaussian noise in  $J$ ,  $\tau$ , or both. Thus, under a noisy control the actual transformation is

$$\rho \rightarrow \rho_{\text{noisy}}(\lambda, \bar{\phi}) = \frac{1}{\sqrt{2\pi\lambda}} \int_{-\infty}^{\infty} e^{-[(\phi - \bar{\phi})^2/2\lambda^2]} \rho(\phi) d\phi. \quad (2)$$

Below we take  $K$  to be either an exchange Hamiltonian with noisy coupling constants  $J_{\alpha}^{ij}$  (in which case we assume for simplicity an equal SD, denoted as  $\Omega$ ), or a noisy single-qubit Hamiltonian needed to refocus an always-on exchange Hamiltonian (and denote the SD as  $\Lambda$ ).

*Case (i): Tunable Ising interaction.* In the Ising model, the exchange Hamiltonian takes the form  $H_{ZZ} = \frac{1}{4} J \sigma_z^1 \sigma_z^2 = J S_z^1 S_z^2$ . Consider the preparation of a maximally entangled state, starting from the initial state  $\rho_i = |00\rangle\langle 00|$ , in the presence of noisy interactions [13]. Without noise, application of the Hadamard transform  $U_H(\pi) = \exp(-i\pi S_y)$  on both qubits, followed by

$$U_{ZZ}(\pi) = \exp\left(-i \int_0^{\pi} \frac{1}{4} J \sigma_z^1 \sigma_z^2 dt\right) = e^{-i(\pi/4) \sigma_z^1 \sigma_z^2},$$

prepares the maximally entangled pure state  $|\xi\rangle = \frac{1}{2}(|00\rangle + i|01\rangle + i|10\rangle + |11\rangle)$ . In the presence of noise, the action of both  $U_H$  and  $U_{ZZ}$  must be averaged over a distribution of angles, as in Eq. (2). The noisy Hadamard transform is a rotation about the  $y$  axis with average angle  $\pi$  and SD  $\lambda$ , resulting in the mixed state

$$\rho_0(\lambda) = (1/\sqrt{2\pi\lambda}) \int e^{-[(\theta - \pi)^2/2\lambda^2]} [U_H(\theta)\rho_i U_H(\theta)^\dagger] d\theta,$$

which can be easily evaluated. The mixedness of  $\rho_0$  is measured by its von Neumann entropy:  $M(\rho) = -\text{Tr}[\rho \log_2 \rho]$ , as a function of the noise parameter  $\lambda$ .  $M(\rho) = 0$  for a pure state,

$M(\rho) = 2$  for a maximally mixed state. In the present case the entropy rises rapidly from zero (at  $\lambda = 0$ ) and reaches its maximum of 2 for  $\lambda \approx 2$ . Next we apply the noisy version of the  $U_{ZZ}$  gate, with angle  $J\tau \sim N(\pi, \Omega)$ . The resulting density matrix  $\rho(\lambda, \Omega)$  is easily computed but is not particularly illuminating; instead, we present the result of using the partial transposition test for entanglement [14]: a  $2 \otimes 2$  state is entangled if and only if it has negative partial transpose (NPT). One then arrives at the following condition for inseparability for the density matrix  $\rho(\lambda, \Omega)$ :

$$e^{-\lambda^2} + 2e^{-(1/2)[\lambda^2 + \Omega^2/4]} > 1. \quad (3)$$

Figure 1 illustrates this condition. Observe that there is a significant region of entanglement in parameter space, with a tradeoff between the tolerated level of noise in state preparation and interaction. Interestingly, except along the cut  $\lambda = 0$ , the transition from entangled to separable is sharp. Solving the inequality (3), we find that the condition for entanglement is

$$\lambda \leq \{-2 \log[(e^{-(\Omega^2/4)} + 1)^{1/2} - e^{-(\Omega^2/8)}]\}^{1/2}. \quad (4)$$

The finite range of the state preparation parameter  $\lambda$  indicates that the purity of the initial state is crucial. However, with a high quality interaction a significantly mixed initial state can be tolerated: From the above follows that if the interaction is perfect ( $\Omega = 0$ ) then  $\lambda_{\text{max}} = [-2 \ln(\sqrt{2} - 1)]^{1/2} = 1.327$ , meaning that even an initial mixed state with entropy 96% as high as the maximally mixed state would still enable entanglement generation. Conversely, if the initial state preparation does not involve any noise ( $\lambda = 0$ ), then the interaction, no matter how noisy, will be able to produce some entanglement. Finally, note that for  $2 \otimes 2$  systems of the type we are considering here, if a state is entangled then it is distillable as well, i.e., one can extract pure states from such noisy states using local operations and classical communication [15]. In particular, Eq. (4) therefore guarantees that a state is useful for teleportation and all other QIP primitives.

*Case (ii): Untunable Ising interaction.* Now we do not assume the ability to switch the interaction off (as, e.g., in NMR). It is therefore necessary to refocus the interaction using single-qubit operations [16]. This is done by pulsing an external magnetic field along the  $x$  axis (we choose the first qubit for this operation). We assume that such pulses can be made very fast and strong compared to the interaction [16]. Formally, let  $X, Y, Z$  be operators satisfying  $su(2)$  commutation relations:  $[X, Y] = iZ$  and cyclic permutations (e.g., the angular momentum operators  $S_{\alpha}$ ). Then it follows from the Baker-Campbell-Hausdorff formula that upon ‘‘conjugation by  $\varphi$ :’’  $C_Z^{\varphi} X \equiv \exp(-i\varphi Z) X \exp(i\varphi Z) = X \cos \varphi + Y \sin \varphi$ . Note further that  $U e^A U^\dagger = e^{U A U^\dagger}$  for unitary  $U$  and arbitrary  $A$ . Thus,  $C_Z^{\varphi} \circ \exp(i\theta X) = \exp(-i\varphi Z) \exp(i\theta X) \exp(i\varphi Z) = \exp[i\theta(X \cos \varphi + Y \sin \varphi)]$ . One application is ‘‘time-reversal,’’ which results from conjugation by  $\pi$ ,

$$C_Z^{\pi} \circ e^{i\theta X} = e^{-iZ\pi} e^{i\theta X} e^{iZ\pi} = e^{-i\theta X}. \quad (5)$$

We then have  $\rho(\tau_2) = U_R(\pi)\rho_0 U_R(\pi)^\dagger$ , where

$$\begin{aligned}
U_R(\pi) &= \left[ C_{S_x^1}^\pi \circ \exp\left(-i \int_{\tau_1}^{\tau_2} JS_z^1 S_z^2 dt\right) \right] \\
&\quad \times \exp\left(-i \int_0^{\tau_1} JS_z^1 S_z^2 dt\right) \\
&= e^{-i(\pi/4)\sigma_z^1 \sigma_z^2},
\end{aligned}$$

which in conjunction with our Hadamard-state preparation, yields the desired CPHASE gate. The parameters must satisfy the condition  $J(2\tau_1 - \tau_2) = \pi$ . In the noisy scenario, the recoupling step is implemented with a rotation around the  $x$  axis by an angle  $\theta \sim N(\pi, \Lambda)$  (where  $\Lambda^2 = \Lambda_1^2 + \Lambda_2^2$ , with  $\Lambda_i$  the SD's of the independent Gaussian-random variables  $J\tau_i$ ,  $i = 1, 2$ ,  $J$  fixed); the only change is then  $U_R(\pi) \mapsto U_R(\theta)$ , and the final state is given by  $\rho(\tau_2, \lambda, \Lambda) = (1/\sqrt{2\pi\Lambda}) \int e^{-(\theta - \pi)^2/2\Lambda^2} [U_R(\theta)\rho_0 U_R(\theta)^\dagger] d\theta$ , which can be analytically evaluated. The inseparability condition, obtained using the partial transposition criterion, yields the same form as the tunable case [Eq. (3)], provided we replace  $\Omega/2$  by  $\Lambda$ . Note that this is by no means an *a priori* obvious substitution:  $\Omega$  is the two-body interaction strength error, whereas  $\Lambda$  is the error in the single-body coupling parameter.

*Case (iii): Tunable XYZ Hamiltonians.* We now consider the XYZ model  $H_{XYZ} = \sum_{\alpha=x,y,z} J_\alpha S_\alpha^1 S_\alpha^2$ , of which, in the noiseless case, the XXZ, XY, and Heisenberg models are special cases. As in the Ising case we assume the initial state is  $\rho_i = |00\rangle\langle 00|$  [17]. In the noiseless scenario, we first apply  $U_X(\pi) = \exp(-i\pi S_x)$  on the second qubit, yielding  $\rho_0 = |01\rangle\langle 01|$ . This is followed by  $U_{XYZ}(\theta_x, \theta_y, \phi) = \exp(-i \int_0^\tau H_{XYZ} dt)$ , where  $\theta_{x,y} = J_{x,y}\tau$ ,  $\phi = J_z\tau$ . Letting  $\theta_x + \theta_y = \pi/4$  and leaving  $\phi$  arbitrary, this prepares the maximally entangled pure state  $|\xi\rangle = (1/\sqrt{2})(|i01\rangle + |10\rangle)$ . Thus in the noiseless case, and for all the exchange models considered here, there is no dependence on  $J_z$ . In the noisy scenario, we first apply  $U_X(\omega) = \exp(-i\omega S_x)$ , where  $\omega \sim N(\pi, \lambda)$ . Then,  $\rho_0(\lambda)$  is a mixture of the states  $|00\rangle, |01\rangle$  with respective weights  $[1 \mp \exp(-\lambda^2/2)]/2$ . Note that now, in the worst case scenario ( $\lambda \rightarrow \infty$ )  $\rho_0$  can only be 50% as mixed as the totally mixed state, whereas in the Ising model above  $\rho_0$  could be fully mixed (or 96% mixed if we insist on entanglement generation).

The unitarily transformed density matrix is given by  $\rho_0(\lambda) \mapsto U_{XYZ}(\theta_x, \theta_y, \phi)\rho_0(\lambda)U_{XYZ}^\dagger(\theta_x, \theta_y, \phi) = \rho(\lambda, \theta^+, \theta^-)$ , where  $\theta^\pm = \theta_x \pm \theta_y$ , whose explicit form can be found without much difficulty, and again is independent of  $\phi$  (i.e.,  $J_z$ ), now even when the initial state is noisy. We next integrate over  $\theta^+ \sim N(\pi/4, \Omega)$  and  $\theta^- \sim N(\bar{\theta}, \Omega)$ , where  $\Omega = (\Omega_x^2 + \Omega_y^2)^{1/2}$ , and where  $\Omega_i$  are the SD's of  $\theta_i$ . Upon applying the partial transposition criterion we find the condition for inseparability

$$|z| > [1/b^2 - (1+a)^2/(1-a)^2]^{1/2}, \quad (6)$$

where  $a = e^{-1/2\lambda^2}$ ,  $b = e^{-2\Omega^2}$ ,  $z = \cos(2\bar{\theta})$ . This condition, plotted in Fig. 2, depends on the noise parameters  $\lambda, \Omega$  (as in the Ising case), but also periodically on the (mean) coupling constants through the distance between exchange models  $\bar{\theta}$

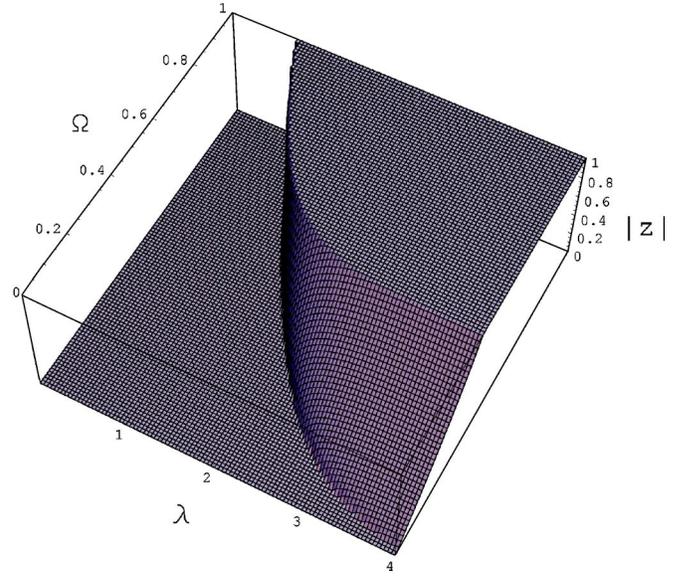


FIG. 2. Plot of the inseparability condition Eq. (6). Entanglement in the noisy tunable XYZ model is the region where  $|z|$  is above the surface shown and  $< 1$ .

$= (\overline{J_x - J_y})\tau$ . From  $0 \leq |z|, a, b \leq 1$ , we have the following sufficient conditions for [inseparability right-hand side (RHS) of Eq. (6)  $< 0$ ]:  $b > (1-a)/(1+a)$  (curve at  $|z|=0$  in Fig. 2), and separability [RHS of Eq. (6)  $> 1$ ]:  $b < (1-a)/\sqrt{2(1+a^2)}$  (curve at  $|z|=1$  in Fig. 2). Thus we find that that the XYZ model with  $J_x \neq J_y$ , similarly to the Ising model, exhibits a sharp entanglement/separability transition as a function of preparation and interaction noise. However, in contrast to the Ising model, in general there is also a dependence on a third parameter, the (noisy) interaction distance  $J_x - J_y$ , so the XYZ model belongs to a distinct class.

*Case (iv): Tunable XY, XXZ, and Heisenberg Hamiltonians.* Note that because of the integration over the distribution of  $J_x - J_y$  above we cannot specialize to the XY, XXZ, Heisenberg models ( $J_x = J_y \equiv J$  case). Repeating the calculations above now with  $J \sim N(\pi/4, \Omega)$  we find that the final state  $\rho(\lambda, \Omega)$  is a mixture of the states  $|00\rangle$  and  $|\Psi^\pm\rangle = (1/\sqrt{2})(|01\rangle \pm |10\rangle)$ , with respective weights  $\frac{1}{2}(1 - e^{-\lambda^2/2})$  and  $\frac{1}{4}[1 + e^{-(1/2)\lambda^2}](1 \mp e^{-\Omega^2/2})$ . Such states are always entangled as long as the proportions of the states  $|\Psi^\pm\rangle$  are different, which clearly holds in our case for  $\lambda, \Omega < \infty$ . Hence, the final state in the tunable XY, XXZ, and Heisenberg models is always entangled for all practical purposes. This, then, is a third class.

*Case (v): Untunable XYZ, XY, XXZ, and Heisenberg Hamiltonians.* Since the interaction is always on, we need to apply external single-qubit refocusing operations. This is done by pulsing an external magnetic field along the  $z$  axis (we choose the first qubit for this operation). After a lengthy calculation we find from the partial transposition criterion, for the XYZ model, the inseparability condition

$$(A + B\mu)^2(1 - \eta)^2(C\mu - D)^2 - \frac{1}{4}\mu^4(1 + \eta)^2 < 0, \quad (7)$$

where  $\mu = e^{-\Lambda^2/2}$ ,  $\eta = e^{-(1/2)\Lambda^2}$ ,  $A = \cos \beta \cos \delta$ ,  $B = \sin \beta \sin \delta$ ,  $C = \sin \beta \cos \delta$ ,  $D = \cos \beta \sin \delta$ ,  $\beta = \Delta\pi/2$ ,  $\delta$

$=\Delta\pi/4$ , and  $\Delta=(J_x-J_y)/(J_x+J_y)$ . The  $XY$ ,  $XXZ$ , and Heisenberg models ( $J_x=J_y$ ) correspond to  $B=C=D=0$ . Interestingly, not only this is achieved when  $J_x=J_y$ , but  $XYZ$  mimicks  $XY$ ,  $XXZ$ , and Heisenberg also when  $\Delta=4n$ , with  $n$  being an integer. It can be shown that the inequality (7) is satisfied as long as  $\Lambda, \lambda < \infty$ . Hence, in all these exchange models the final state is always entangled for all practical purposes. In contrast to the tunable case, the untunable  $XYZ$ ,  $XY$ ,  $XXZ$ , and Heisenberg Hamiltonians all lie in the same class.

#### IV. LAPLACIAN FLUCTUATIONS

As a test of the robustness of our Gaussian-model based conclusions, we have repeated the above analysis when the fluctuations in control parameters obey a Laplace distribution: under a noisy control the actual state transformation is now

$$\rho \rightarrow \rho_{\text{noisy}}(\omega, \bar{\phi}) = \frac{1}{4\omega} \int_{-\infty}^{\infty} e^{-(|\phi-\bar{\phi}|/2\omega)} \rho(\phi) d\phi. \quad (8)$$

While we find quantitative differences compared to the Gaussian model, the qualitative behavior is identical, thus bolstering the universality of the classes obtained. To illustrate this fact let us reconsider the Ising case. Let  $\omega=\lambda$  ( $\Omega$ ) be the SD of the noisy parameter controlling state preparation (interaction) in both the tunable and nontunable cases. We find, repeating the procedure above, that the condition of inseparability can be given for both the tunable and nontunable cases as

$$4\lambda^2(\Omega^2 + 2\lambda^2 + 2\lambda^2\Omega^2) < 1. \quad (9)$$

Recall that, similarly, in the Gaussian case the inseparability condition of the tunable case can be obtained by appropriate

rescaling of the SD. Further, note that as in the Gaussian model, when the initial state is perfect ( $\lambda=0$ ) the state remains inseparable for all practical purposes. By solving inequality (9) the inseparability condition can be written as  $\lambda < \frac{1}{2}\{[(1+(1+\Omega^2)^2)^{1/2}-\Omega^2]/(1+\Omega^2)\}^{1/2}$ , which shows that, again as noticed in the Gaussian case, the purity of the initial state is crucial and actually even more so in the Laplace case. Indeed, if the interaction is perfect, we obtain the inseparability condition  $\lambda < \sqrt[4]{1/8}=0.5946$ , in contrast to the Gaussian case where the threshold value of  $\lambda$  in the no-noise interaction scenario is as high as 1.327.

#### V. CONCLUSIONS

The sharp transition found in the Ising and  $XYZ$  models is reminiscent of the thermal entanglement transition [18], and suggests an interesting avenue for further research. The surprising robustness of entanglement to noise bodes well for quantum-information processing with reduced demands on error correction. Another interesting implication of our work concerns *entanglement verification*: Knowing the underlying two-body interaction Hamiltonian and corresponding level of control, an experimentalist can confidently characterize the degree of entanglement his/her system can generate, without needing to perform a direct, and often difficult, measurement of entanglement [1].

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