PHYSICAL REVIEW A **69**, 030302(R) (2004)

Dynamical decoupling using slow pulses: Efficient suppression of 1/*f* noise

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(Received 23 May 2003; revised manuscript received 24 November 2003; published 9 March 2004)

The application of dynamical decoupling pulses to a single qubit interacting with a linear harmonic oscillator bath with $1/f$ (more generally, strong low-frequency) spectral density is studied, and compared to the Ohmic case. Decoupling pulses that are slower than the fastest bath time scale are shown to drastically reduce the decoherence rate in the $1/f$ case. Contrary to conclusions drawn from previous studies, this shows that dynamical decoupling pulses do not always have to be ultrafast. Our results explain the recent experiment in which dephasing due to $1/f$ charge noise affecting a charge qubit in a small superconducting electrode was successfully suppressed using spin-echo-type gate-voltage pulses.

DOI: 10.1103/PhysRevA.69.030302 PACS number(s): 03.67.Pp, 03.65.Yz, 03.67.Lx, 05.40.Ca

The most serious problem in the physical implementation of quantum information processing is that of maintaining quantum coherence. Decoherence due to interaction with the environment can spoil the advantage of quantum algorithms [1]. One of the proposed remedies is the method of "dynamical decoupling" or "bang-bang" (BB) pulses, in which strong and sufficiently fast pulses are applied to the system. In this manner one can either eliminate or symmetrize the system-bath Hamiltonian so that system and bath are effectively decoupled [2–11]. The BB method was proposed in [2], where a quantitative analysis was first performed for pure dephasing in the linear spin-boson model: $H_{SB}=g\sigma_z\otimes B$, where σ_z is the Pauli z matrix and B is a Hermitian boson operator. The analysis was recently extended to the nonlinear spin-boson model, with similar conclusions about performance [3]. Decoupling has also been applied to the suppression of spontaneous emission [4] and magnetic state decoherence induced by collisions in a vapor [5]. Since the decoupling pulses are *strong* one ignores the evolution under H_{SB} while the pulses are on, and since the pulses are *fast* one ignores the evolution of the bath under its free Hamiltonian H_B during the pulse cycle. The latter assumption is usually stated as

$$\Delta t \ll 1/\Lambda_{uv}, \tag{1}$$

where Δt is the pulse interval length and Λ_{uv} is the high-frequency cutoff of the bath spectral density $I(\omega)$ [2] [see Eq. (2) below]. It can be shown that the overall system-bath coupling strength g is then renormalized by a factor $\Delta t\Lambda_{uv}$ after a cycle of decoupling pulses [6], and that the bath-induced error rate is reduced by a factor proportional to $(\Delta t\Lambda_{uv})^2$ [7]. A temperature $T>0$ sets an additional, thermal decoherence time scale that must be beaten in order for the decoupling method to work [2,8].

The conclusion (1) is extremely stringent, as the time scale Δt may be too small to be practically attainable. Moreover, as we show below, and has been argued before on the basis of the inverse quantum Zeno effect [10], decoherence may be enhanced, rather than suppressed, if Eq. (1) is not satisfied. Equation (1) is based on studies in which the bath was modeled as a system of harmonic oscillators, with a spectral density of the form $I(\omega)\propto\omega^{\nu}e^{-\omega/\Lambda_{uv}}$, with $\nu=1$ (Ohmic case) [2], or using a flat spectral density with a finite cutoff Λ_{uv} [8], or without reference to a specific spectral density but emphasizing features of its high-frequency components [3,7]. However, a ubiquitous class of baths does not fall into this category, and we show here that then the condition (1) is *overly restrictive*. This is the case for so-called $1/f$ noise, or more generally $1/f^{\alpha}$ $(\alpha>0)$. In these cases the bath spectral density decays as a power law, bounded between infrared (ir, lower) and ultraviolet (uv, upper) cutoffs Λ_{ir} and Λ_{uv}, respectively. In quantum computer implementations this is often attributable to (but certainly not limited to) charge fluctuations in electrodes providing control voltages. The need for such electrodes is widespread in quantum computer proposals, e.g., trapped ions (where observed $1/f$ noise was reported in [12]), quantum dots [13], doped silicon [14], electrons on helium [15], and superconducting qubits [16]. In the last case, in a recent experiment involving a charge qubit in a small superconducting electrode (Cooper-pair box), a spin-echo-type version of the BB was successfully used to suppress low-frequency energy-level fluctuations (causing dephasing) due to $1/f$ charge noise [16]. Here we explain the origin of such a result and discuss its general applicability.

On the time scale $t>1/\Lambda_{uv}$, the details of the system-bath interaction and internal bath dynamics become important. Since for $1/f$ noise most of the bath spectral density is concentrated in the low, rather than the high end of the frequency range, it turns out that in this case BB decoupling with slow pulses $(\Delta t>1/\Lambda_{uv})$ depends more sensitively on the lower than on the upper cutoff. In particular, we show that the suppression of dephasing is more effective when the noise originates in a bath with $1/f$ spectrum than in the Ohmic case, owing to the abundance of ir modes in the former. In the following we present the results of our analysis contrasting the BB results for $1/f$ and Ohmic baths.

Decoupling for spin-boson model. We consider the linear spin-boson model including periodic decoupling pulses. We first briefly review and somewhat simplify the results derived in [2]. We use $k_B=\hbar=1$ units. The Hamiltonian is

$$H=H_S+H_B+H_{SB}+H_P$$
$$=\frac{\epsilon}{2}\sigma_z+\sum_k \omega_k b_k^\dagger b_k+\sum_k \sigma_z(g_k^* b_k+g_k b_k^\dagger)+H_P,$$

where the first (second) term governs the free system (bath) evolution; the third term is the (linear) system-bath interaction in which b_k is the kth-mode boson annihilation operator and g_k is a coupling constant; and the last term is the fully controllable Hamiltonian generating the decoupling pulses:

$$H_P(t)=\sum_{n=1}^{N} V_n(t)e^{i\epsilon t\sigma_z/2}\sigma_x e^{-i\epsilon t\sigma_z/2},$$

where the pulse amplitude $V_n(t)=V$ for $t_n\leqslant t\leqslant t_n+\tau$ and 0 otherwise, lasting for a duration $\tau\ll\Delta t$, with $t_n=n\Delta t$ being the time at which the nth pulse is applied. The properties of the bath are captured by its spectral density

$$I(\omega)=\sum_k \delta(\omega-\omega_k)|g_k|^2. \tag{2}$$

The reduced system density matrix is obtained from the total density matrix by tracing over the bath degrees of freedom

$$\rho_S(t)=\mathrm{Tr}_B[\rho(t)]=\mathrm{Tr}_B[U(t)\rho_S(0)\otimes\rho_B(0)U^\dagger(t)],$$

where we have assumed a factorized initial condition between the system and thermal bath, and $U(t)$ is the time evolution generated by H: $U(t)=\mathcal{T}\exp[-i\int_0^t ds H(s)]$ ($\mathcal{T}$ denotes time ordering). We are interested in how decoupling improves the system coherence, defined as $\rho_{01}(t)=\langle 0|\rho_S(t)|1\rangle$, where $|0\rangle,|1\rangle$ are σ_z eigenstates. In the interaction picture with respect to H_S and H_B the result in the absence of decoupling pulses (free evolution) is $\rho_{01}^I(t)=e^{-\Gamma_0(t)}\rho_{01}^I(0)$, where

$$\Gamma_0(t)=\int_{\Lambda_{ir}}^{\Lambda_{uv}} d\omega \coth\left(\frac{\beta\omega}{2}\right)\frac{1-\cos\omega t}{\omega^2}I(\omega), \tag{3}$$

$\beta=1/(k_B T)$. In the Schrödinger picture there are oscillations at the natural frequency ϵ, i.e., $\rho_{01}(t)=e^{-i\epsilon t}\rho_{01}^I(t)$.

Similarly in the presence of the decoupling pulses, at $t_{2N}=2N\Delta t$, $\rho_{01}^I(t_{2N})=e^{-i\epsilon t_{2N}}e^{-\Gamma_P(N,\Delta t)}\rho_{01}^I(0)$, where we can show from Eqs. (46) and (47) of [2] that

$$\Gamma_P(N,\Delta t)=4\int_{\Lambda_{ir}}^{\Lambda_{uv}} d\omega \coth\left(\frac{\beta\omega}{2}\right)\frac{1-\cos\omega t_{2N}}{\omega^2}\times I(\omega)\tan^2\left(\frac{\omega\Delta t}{2}\right). \tag{4}$$

The $\tan^2(\omega\Delta t/2)$ term (which was not found in [2]) is the suppression factor arising from the decoupling procedure. In effect, the bath spectral density in the presence of decoupling pulses has been transformed from $I(\omega)$ to the effective spectral density $I'(\omega)=I(\omega)\tan^2(\omega\Delta t/2)$. This is a crucial point: owing to the $\tan^2$ factor the low-frequency modes in $I'(\omega)$ are strongly suppressed. In other words, *the BB pulses act as a high-pass filter for environmental noise*. For this reason the BB method is particularly effective for *general* strong low-frequency noise, in particular, but certainly not limited to, the $1/f$ case. Note further that the singularity of $\tan^2$ at $\omega\Delta t=(2l+1)\pi$ for an integer l is canceled by the vanishing of $1-\cos\omega t_{2N}$ at the same points, so Γ_P remains finite. Nevertheless, and as already pointed out in [2], the value $\omega\Delta t=\pi$ is special: In the limit $N\gg 1$ the integrand of Eq. (4) is highly oscillatory for $\omega\Delta t>\pi$, and grows to $16N^2$ at $\omega\Delta t=\pi$. Thus, decoherence suppression is effective when

$$\Lambda_{uv}\Delta t<\pi. \tag{5}$$

This is an upper bound on Δt that is independent of the specific form of $I(\omega)$. Note further that decoupling *enhances* decoherence from all modes with $(4l+1)\pi/2<\omega\Delta t<(4l+3)\pi/2$, since for these values $\tan^2(\omega\Delta t/2)>1$. However, this effect may be quenched if the weight of these modes is sufficiently low; this is indeed what happens in the $1/f$ case.

Results for $1/f$ and Ohmic spectral densities. Let us now assume that the spectral density has the following form:

$$I(\omega)=\gamma\omega^\nu, \quad \nu=\pm 1, \tag{6}$$

with uv cutoff Λ_{uv} and ir cutoff Λ_{ir}. Thus we are comparing $1/f$ noise (the case $\nu=-1$) to an Ohmic bath (the case $\nu=1$, considered in [2]).

To explain the effect of pulses qualitatively, we approximate $\tan^2 x$ by $x^2(1-2x/\pi)^{-1}$, which allows us to obtain an explicit form for Γ_P for $0\leqslant\Lambda_{uv}\Delta t<\pi/2$. We further expand $\coth x\approx 1+2\exp(-2x)$ $(x>1)$. Then, the contribution to Γ_P for $1/f$ noise at low temperature is the sum of the zero temperature part

$$\Gamma_P^{(T=0)}(N,\Delta t)=\gamma(\Delta t)^2\left[\ln\left\{\frac{\Lambda_{uv}}{\Lambda_0}\right\}-\ln\left\{\frac{\pi-\Lambda_{uv}\Delta t}{\pi-\Lambda_{ir}\Delta t}\right\}-\mathrm{Ci}(\Lambda_{uv}t_{2N})+\mathrm{Ci}(\Lambda_0 t_{2N})+O(\Delta t)\right], \tag{7}$$

and the low-temperature correction

$$\Gamma_P^{(T>0)}(N,\Delta t)=\frac{\gamma(\Delta t)^2}{2}\left[\ln(1+T^2t_{2N}^2)+\frac{2\Delta t T}{\pi}\times\left\{1-\frac{1}{1+T^2t_{2N}^2}\right\}+O(T^2)\right], \tag{8}$$

where Ci is the cosine integral. In Eq. (8), the limits $\Lambda_{ir}\to 0$ and $\Lambda_{uv}\to\infty$ are taken. All terms are finite in these limits. The first and second terms in $\Gamma_P^{(T=0)}(N,\Delta t)$ (independent of t_{2N}) determine the asymptotic value $\Gamma_P^{(T=0)}(\infty,\Delta t)$; the remainder is a damped oscillatory part, given by the difference of two cosine integrals, that vanishes at long times. The second logarithmic term diverges as the pulse interval approaches the inverse uv cutoff frequency time scale of the bath leading to decoherence enhancement from the $\tan^2$ term

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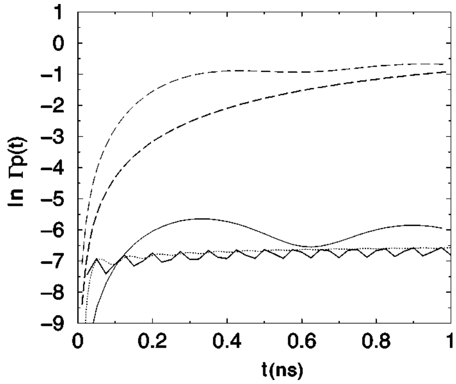

FIG. 1. Temporal behavior of the logarithm of the decoherence factors at $T=0$. The initial coherence $\rho_{01}^{I}(0)=1$. Parameters are $\gamma=0.05$, $\Lambda_{uv}=10$ for Ohmic and $\gamma=0.25$, $\Lambda_{uv}=80$ for $1/f$, $\Lambda_{ir}=1$, $\Delta t=0.025$ for both. Thick solid (dashed) line: $1/f$ case with (without) decoupling pulses. Thin solid (dashed) line: Ohmic case with (without) decoupling pulses. Equation (3) was used for the case without decoupling pulses, while Eq. (4) was used for the case with decoupling pulses at each $t=t_{2N}$. The dotted line is our analytical result in Eq. (7).

in Eq. (4). These behaviors are reflected in the exact solutions displayed in Fig. 1. The leading order finite temperature correction $\Gamma_P^{(T>0)}(N,\Delta t)$ can be separated into two terms. The first term characterizes the asymptotic power law decay and the second term gives the initial damping and the asymptotic relaxation to the t_{2N}-independent constant.

In Fig. 1 the logarithm of the decoherence factors $\Gamma_0(t)$ (free evolution) and $\Gamma_P(t)$ (pulsed evolution) for the $1/f$ and Ohmic cases are shown. The smaller Γ, the more coherent is the evolution. The apparent oscillations with a frequency given by Λ_{uv} are caused by the use of a sudden cutoff. Given the parameters used in Fig. 1, the standard time scale condition $\Delta t \ll 1/\Lambda_{uv}$ is *not* satisfied in the $1/f$ case, while it is $(\Delta t \Lambda_{uv}=0.25)$ in the Ohmic case. The most striking feature apparent in Fig. 1 is the highly efficient suppression of decoherence in the case of $1/f$ noise, in spite of the seemingly unfavorable pulse interval length. In addition, it can be shown that decoherence due to the $1/f$ bath is accelerated when the ir cutoff is decreased, and is more sensitive to the ir cutoff than the Ohmic case. This is a direct consequence of the fact that most of the modes in a $1/f$ spectrum are concentrated around Λ_{ir}. For $1/f$ baths we therefore expect slow and strong decoherence on a long time scale, that may be efficiently suppressed by relatively *slow* and strong pulses. A similar conclusion is applicable to baths with $1/f^{\alpha}$ spectral density. More generally, our results imply that the effectiveness of decoupling pulses is determined by how well the modes are concentrated in the low-frequency regime compared to the suppression of these modes by the $\tan^2$ factor of Eq. (4). Thus for baths with general spectral density in Eq. (6), the smaller ν $(=-\alpha)$ is, the more successfully the decoupling works.

For our pure dephasing case at finite temperature, there is the thermal time scale $t_\beta \equiv T^{-1}$ at which thermal fluctuations start affecting the system's coherence. In particular, for $T \gg \Lambda_{uv}$, decoherence is governed by the thermal fluctuations.

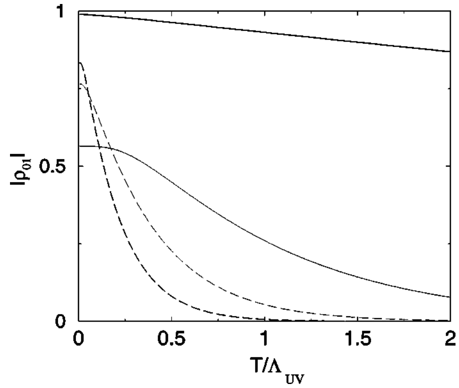

FIG. 2. The temperature dependence of coherence at $t=4$. $\gamma=0.1$ for Ohmic case and $\gamma=0.5$ for $1/f$ case, $\Lambda_{uv}=20$, $\Lambda_{ir}=1$, $\Delta t=0.125$. Legend as in Fig. 1.

In Fig. 2, a finite temperature result is shown. The decoupling pulses enhance the decoherence for the Ohmic bath even at low temperatures, since for the parameters chosen the condition (1) is not satisfied. On the other hand, decoherence suppression in the $1/f$ case is highly effective. At high temperature, it has been argued on the basis of the Ohmic case that decoupling pulses faster than the thermal frequency T are required to suppress decoherence [8]. Once again, the nature of the bath can qualitatively modify this conclusion. Thus decoupling by relatively slow pulses that obey the condition $\Lambda_{uv}\Delta t \sim 1$ can still be effective for decoherence suppression at elevated temperatures. However, as the temperature increases, the effective spectrum shifts toward low frequencies, and at the same time, the influence of the environment increases. Overall, BB decoupling becomes ineffective irrespective of the type of bath. This explains the breakdown of decoherence suppression at $T=1000$ in Fig. 3. Note from the figure that the suppression of decoherence for the $1/f$ bath is more effective than for the Ohmic bath throughout the whole temperature regime.

For pulses that are too slow, BB decoupling accelerates the decoherence [10]. For the Ohmic bath, as the interval

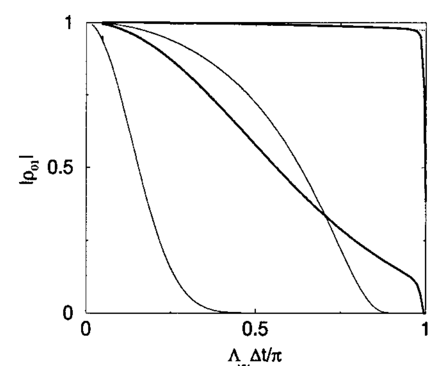

FIG. 3. Coherence as a function of a pulse interval at finite temperature is plotted at $t=2$, calculated from Eq. (4). Parameters are $\gamma=0.5$, $\Lambda_{uv}=100$, $\Lambda_{ir}=0.01$. Thick (thin) curves are $1/f$ (Ohmic) case. $T=10$ for upper lines and $T=1000$ for lower lines. The dotted line (mostly hiding behind the upper thick line) is our approximate analytical result, Eqs. (7) and (8), at $T=10$.

approaches the threshold value (1) from below, there is a crossover from decoherence suppression to decoherence enhancement, as shown in Fig. 3. For the $1/f$ bath, suppression is still effective for longer pulse intervals as long as $\Delta t \Lambda_{uv} < \pi$ is satisfied.

It is of interest to compare our results with the gate-voltage pulse experiment performed in [16] in a Cooper-pair box. The corresponding parameter values in Eq. (4) are $\gamma = 2E_C^2 \alpha / e^2 \hbar^2 \approx 0.1$. Thus the weak coupling (Gaussian) limit we are considering in this work is applicable with the Josephson charging energy $E_C = 122\ \mu$eV and the constant $\alpha = (1.3\times 10^{-3} e)^2$ determined by the noise measurement. To achieve 90% decoherence suppression with $\Lambda_{ir} = 100$ Hz and $\Lambda_{uv} = 10$ GHz at $k_B T = 5\ \mu$eV, the pulse interval $\Delta t \sim 0.25$ ns is required from our analysis based on Eq. (4) with $N = 1$. Although the pulse sequence of [16] differs from ours (theirs is the $\pi/2$-π-$\pi/2$ spin-echo sequence), they play essentially the same role (though their sequence is known to be more robust to systematic errors). Our Δt value roughly agrees with their value, $\Delta t \sim 0.5$ ns, deduced from Fig. 2 in [16]. This agreement nicely illustrates the experimental feasibility of the BB in the case of $1/f$ noise. Note that this agreement is essentially free from paprameter adjustments: (i) The role of temperature in the low T regime, $T = 1-10$, is insignificant. (ii) There is a weak logarithmic dependence on Λ_{uv} in Eq. (7), which does not essentially change the value of Δt (except when Δt is very close to Λ_{uv}^{-1}). The effectiveness of spin-echo-type pulses in relation to superconducting qubits was also recently discussed in [17]. The spin-boson model is appropriate for the study of $1/f$ noise due to a large number of weakly coupled background charges [18] and explains the observed Gaussian decay of coherence [16].

Conclusions. We have shown that the speed requirement of the decoupling method should be stated relative to the *type* of bath spectral density, and not just in terms of its upper cutoff (baths with bimodal spectral distributions provide another example of this [3]). Most significantly, our exact results have demonstrated that BB decoupling can be expected to be highly effective in suppressing decoherence due to the ubiquitous $1/f$ noise, and more generally strong low-frequency noise, without having to satisfy the stringent time constraints that may render the method overly difficult to implement in other instances. We expect this to have significant implications, e.g., for suppression of noise due to charge fluctuations in electrodes providing control voltages in quantum computation. Such a result has already been obtained experimentally in a Cooper-pair box experiment [16] and is predicted to apply to trapped-ion quantum computation as well [11].

Note added in proof. Subsequent to our work several papers have appeared that also deal with suppression of $1/f$ noise via BB decoupling [19–21], and that deal with modeling $1/f$ noise in solid-state qubit systems [22,23]. Most of these works [19–22] treat $1/f$ noise microscopically, as arising from a collection of bistable fluctuators (BSF), while Ref. [23] considers nonlinear coupling; the resulting model is hence not necessarily Gaussian. However, all these works reduce to our model in the appropriate limits (weak-coupling, and/or fast BSF, and/or many BSF). References [20,21] essentially follow the same decoherence suppression strategy as ours, but use the model for $1/f$ noise considered in Ref. [18].

The present study was sponsored by NSERC and the DARPA-QuIST program (managed by AFOSR under Agreement No. F49620-01-1-0468) (to D.A.L.).

[1] W.G. Unruh, Phys. Rev. A **51**, 992 (1995).
[2] L. Viola and S. Lloyd, Phys. Rev. A **58**, 2733 (1998).
[3] C. Uchiyama and M. Aihara, Phys. Rev. A **66**, 032313 (2002); e-print quant-ph/0303144.
[4] G.S. Agarwal, M.O. Scully, and H. Walther, Phys. Rev. Lett. **86**, 4271 (2001).
[5] C. Search and P.R. Berman, Phys. Rev. Lett. **85**, 2272 (2000).
[6] L. Viola, E. Knill, and S. Lloyd, Phys. Rev. Lett. **82**, 2417 (1999).
[7] L.-M. Duan and G. Guo, Phys. Lett. A **261**, 139 (1999).
[8] D. Vitali and P. Tombesi, Phys. Rev. A **59**, 4178 (1999); **65**, 012305 (2002).
[9] P. Zanardi, Phys. Lett. A **258**, 77 (1999); L. Viola, E. Knill, and S. Lloyd, Phys. Rev. Lett. **85**, 3520 (2000); L. Viola, Phys. Rev. A **66**, 012307 (2002); L. Viola and E. Knill, Phys. Rev. Lett. **90**, 037901 (2003); L.-A. Wu and D.A. Lidar, *ibid.* **88**, 207902 (2002); L.-A. Wu, M.S. Byrd, and D.A. Lidar, *ibid.* **89**, 127901 (2002); M.S. Byrd and D.A. Lidar, *ibid.* **89**, 047901 (2002); R. Alicki, M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. A **65**, 062101 (2002); S. Tasaki, A. Tokuse, P. Facchi, and S. Pascazio, e-print quant-ph/0210129.
[10] A.G. Kofman and G. Kurizki, Phys. Rev. Lett. **87**, 270405 (2001); P. Facchi, D.A. Lidar, and S. Pascazio, e-print quant-ph/0303132, Phys. Rev. A (to be published).
[11] D.A. Lidar and L.-A. Wu, Phys. Rev. A **67**, 032313 (2003).
[12] Q.A. Turchette *et al.*, Phys. Rev. A **61**, 063418 (2000).
[13] G. Burkard, D. Loss, and D.P. DiVincenzo, Phys. Rev. B **59**, 2070 (1999); J. Levy, Phys. Rev. A **64**, 052306 (2001).
[14] B.E. Kane, Nature (London) **393**, 133 (1998); R. Vrijen *et al.*, Phys. Rev. A **62**, 012306 (2000).
[15] P.M. Platzman and M.I. Dykman, Science **284**, 1967 (1999).
[16] Y. Nakamura *et al.*, Phys. Rev. Lett. **88**, 047901 (2002).
[17] J.M. Martinis *et al.*, Phys. Rev. B **67**, 094510 (2003).
[18] E. Paladino, L. Faoro, G. Falci, and R. Fazio, Phys. Rev. Lett. **88**, 228304 (2002).
[19] H. Gutmann, F.K. Wilhelm, W.M. Kaminsky, and S. Lloyd, e-print cond-mat/0308107.
[20] L. Faoro and L. Viola, e-print quant-ph/0312159.
[21] G. Falci, A. D'Arrigo, A. Mastellone, and E. Paladino, e-print cond-mat/0312442.
[22] Y.M. Galperin, B.L. Altshuler, and D.V. Shantsev, e-print cond-mat/0312490.
[23] Y. Maklin and A. Shnirman, e-print cond-mat/0308297.