

## Holonomic Quantum Computation in Decoherence-Free Subspaces

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We show how to realize, by means of non-Abelian quantum holonomies, a set of universal quantum gates acting on decoherence-free subspaces and subsystems. In this manner we bring together the quantum coherence stabilization virtues of decoherence-free subspaces and the fault tolerance of all-geometric holonomic control. We discuss the implementation of this scheme in the context of quantum information processing using trapped ions and quantum dots.

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*Introduction.*—The implementation of quantum information processing (QIP) poses an unprecedented challenge to our capabilities of controlling the dynamics of quantum systems. The challenge is twofold and somewhat contradictory. On the one hand one must (i) maintain as much as possible the isolation of the computing degrees of freedom from the environment, in order to preserve their “quantumness”; on the other hand (ii) their dynamical evolution must be enacted with extreme precision in order to avoid errors whose propagation would quickly spoil the whole quantum computational process. To cope with the decoherence problem (i), active strategies such as quantum error correcting codes [1], as well as passive ones such as error avoiding codes [2], have been contrived. The latter are based on the symmetry structure of the system-environment interaction, which under certain circumstances allows for the existence of decoherence-free subspaces (DFSs), i.e., subspaces of the system Hilbert state-space over which the dynamics is still unitary. DFSs have been experimentally demonstrated in a host of physical systems (e.g., [3,4]). The DFS idea of symmetry-aided protection has been generalized to noiseless subsystems [5], experimentally tested in Ref. [6].

Holonomic quantum computation (HQC) [7] is an all-geometric strategy wherein QIP is realized by means of adiabatic non-Abelian quantum holonomies [8]. Quantum information is encoded in a degenerate eigenstate of a Hamiltonian depending on a set of controllable parameters, e.g., external laser fields. When the latter are adiabatically driven along a suitable closed path, the initial quantum state is transformed by a nontrivial unitary transformation (holonomy) that is geometrical in nature. Following the ion traps HQC-implementation proposal [9], several other schemes, based on a variety of physical setups, were proposed [10]. One expects the geometrical nature of quantum holonomies to endow HQC with inherent robustness against certain errors. This alleged fault tolerance has only recently been subjected to serious scrutiny [11]; the resulting, still developing, picture is that while stability

against decoherence must be further assessed (indeed the adiabatic theorem was only recently generalized to open quantum systems [12]), HQC seems to exhibit a strong robustness against stochastic errors in the control process generating the required adiabatic loops [13]. From this point of view HQC seems to be promising with respect to the general challenge (ii).

In this work we describe a QIP scheme which combines DFSs and HQC [14,15]. More specifically, we show how to perform universal quantum computation within a two-qubit DFS for collective dephasing by using non-Abelian holonomies only. The discussion is then extended to consider general collective decoherence as well. The appeal of such a strategy should, in view of the above, be evident: try to bring together the best of two worlds, namely, the resilience of the DFS approach against environment-induced decoherence and the operational robustness of HQC. Moreover, we formulate our results using rather generic Hamiltonians, so that the scheme proposed in this work appears to be a suitable candidate for experimental demonstration in a variety of systems, including trapped ions and quantum dots.

*Dark states in a decoherence-free subspace.*—Let us start by considering a four-qubit system with state-space  $\mathcal{H}_4 \cong (\mathbb{C}^2)^{\otimes 4}$ . We denote by  $X_l$ ,  $Y_l$ , and  $Z_l$  the three Pauli matrices acting on the  $l$ th qubit; for any pair  $l \neq m$  of qubits we define the operators  $R_{lm}^x := \frac{1}{2}(X_l X_m + Y_l Y_m)$ ,  $R_{lm}^y := \frac{1}{2}(X_l Y_m - Y_l X_m)$ , and  $R_{lm}^z := \frac{1}{2}(Z_m - Z_l)$ . These operators have a nontrivial action over the subspace of  $\mathcal{H}_{lm} \cong \mathbb{C}_l^2 \otimes \mathbb{C}_m^2$  spanned by  $|0_l 1_m\rangle$  and  $|1_l 0_m\rangle$  (where  $|0\rangle/|1\rangle$  are the  $\pm$  eigenstates of  $Z$ , respectively): they provide a faithful representation of the  $\mathfrak{su}(2)$  algebra over this subspace, where they act as the Pauli matrices, while they vanish on the orthogonal complement spanned by  $|0_l 0_m\rangle$  and  $|1_l 1_m\rangle$ . Let  $Z := \sum_{i=1}^4 Z_i$ , then  $[R_{lm}^\alpha, Z] = 0$  ( $\forall l, m$ ). It follows that every eigenspace of  $Z$  is invariant under the action of the  $R_{lm}^\alpha$ 's. In particular this holds true for the subspace

$$C := \text{span}\{|1000\rangle, |0100\rangle, |0010\rangle, |0001\rangle\}. \quad (1)$$

$C$  is a DFS against collective dephasing [2]; i.e., states in  $C$  are immune from decoherence induced by system-bath interactions of the form  $Z \otimes B$ , where  $B$  is an arbitrary bath operator. Collective dephasing is known to be a major source of decoherence in ion-trap based QIP [3].

We assume that the system dynamics is generated by

$$H = \sum_{l>m} (J_{lm}^x R_{lm}^x + J_{lm}^y R_{lm}^y), \quad (2)$$

where  $l, m$  are qubit indices and the  $J_{lm}$ 's are *controllable* coupling constants. These are the parameters that will be driven along controlled adiabatic loops to enact quantum gates via non-Abelian holonomies. When  $J_{lm}^y = 0$ ,  $H$  is the  $XY$  Hamiltonian found in a variety of quantum computing proposals, e.g., the quantum Hall proposal [16], quantum dots [17], and atoms in cavities [18]. It also describes trapped ions subject to the Sørensen-Mølmer scheme [19]. The case  $J_{lm}^x = 0$  is related to the  $XY$  model via a unitary transformation.

By way of introduction to our HQC-DFS scheme, note that Hamiltonian (2) has a hidden multilevel structure with interesting properties (isomorphic to the one exploited in Ref. [9]). Indeed, the Hamiltonian  $H_{lmn} = \sum_{j=m,n} J_{jl} (R_{jl}^x \cos \varphi_{jl} + R_{jl}^y \sin \varphi_{jl})$ , in the basis  $\{|e\rangle := |100\rangle_{lmn}, |g_1\rangle := |010\rangle_{lmn}, |g_2\rangle := |001\rangle_{lmn}\}$ , with  $l \neq m \neq n$ , takes the form

$$H_{lmn} = J_{lm} e^{i\varphi_{lm}} |e\rangle\langle g_1| + J_{ln} e^{i\varphi_{ln}} |e\rangle\langle g_2| + \text{H.c.} \quad (3)$$

This is a so-called Lambda scheme, with  $|e\rangle$  at the top and  $|g_1\rangle, |g_2\rangle$  at the bottom. It is well known that for every value of the  $J_{jl}$ 's and  $\varphi_{jl}$ 's, Hamiltonian (3) has one *dark state*  $|D(J_{lm}, J_{ln})\rangle \propto J_{lm} e^{i\varphi_{lm}} |g_2\rangle - J_{ln} e^{i\varphi_{ln}} |g_1\rangle$ , i.e., a state satisfying  $H_{lmn} |D(J_{lm}, J_{ln})\rangle = 0$ .

The key idea is now as follows: moving in the control parameter space to the point  $J_{lm}/J_{ln} = 0$ , one has that the dark state is given by  $|g_1\rangle$ ; then, if one adiabatically changes the parameters along a closed loop, at the end the state  $|g_1\rangle$  will pick up a nontrivial geometric phase. This simple fact, suitably generalized, is the basic ingredient of our HQC scheme. We further supplement the dark state  $|g_1\rangle$  by another state, that is also annihilated by the system Hamiltonian (and thus does not acquire a dynamical phase either); these states together will form our qubit. In other words, the crucial observation is that one can embed within the DFS  $C$  [Eq. (1)] a manifold of dark states that, for specific values of the coupling constants in Eq. (2), coincide with logical encoded qubits. Holonomic manipulations are then used to generate a universal set of quantum logic gates.

Let us stress that even if during state manipulation the system described by Eq. (2) leaks out of the logical encoding subspace (the states  $|0\rangle_L$  and  $|1\rangle_L$  defined below), e.g., due to the breakdown of adiabaticity [12], this results in

just the kind of errors that HQC is robust against [11]. Moreover, during such leakage the system *never* abandons the DFS  $C$ . Thus protection against collective dephasing is maintained throughout the whole gating period, which in turns allows one to stretch the gating time in such a way as to fulfill the adiabatic constraint for a longer period of time than would be possible without the DFS. These remarks at least partially counter a standard objection to HQC, that the use of slow gates gives decoherence more time to exert its detrimental effects.

*One-qubit holonomic gates.*—We now show how to realize the single-qubit phase gate  $\exp(i\varphi Z_L)$ , where  $Z_L |\alpha\rangle = (-1)^\alpha |\alpha\rangle_L$  (note that  $Z_L = R_{12}^z$ ). We encode a logical qubit in  $C$  by  $|1\rangle_L := |0010\rangle$  and  $|0\rangle_L := |0001\rangle$ , while the remaining states  $|a_1\rangle := |1000\rangle$  and  $|a_2\rangle := |0100\rangle$  play the role of ancillae. Let us set all the  $J_{lm}$  to zero, except  $J_{24}$  and  $J_{34}$ , such that the Hamiltonian (2) reduces to

$$H_Z = J_{34} R_{34}^x + J_{24} (R_{24}^x \cos \varphi_{24} + R_{24}^y \sin \varphi_{24}). \quad (4)$$

We can also write  $H_Z = J_{34} |a_1\rangle\langle a_2| + J_{24} e^{i\varphi} |a_1\rangle \times \langle 1|_L + \text{H.c.}$  This is a Lambda configuration with  $|a_1\rangle$  at the top and  $|a_2\rangle, |1\rangle_L$  at the bottom. Therefore, as in our discussion above,  $H_Z$  has a zero-eigenvalue eigenstate (dark state) given by  $|\Psi_1\rangle = \cos \theta |1\rangle_L - \sin \theta e^{i\varphi} |a_2\rangle$  where  $\theta = \tan^{-1}(J_{24}/J_{34})$  and  $\varphi = \varphi_{24}$ . The state  $|0\rangle_L$  is also a zero-eigenvalue eigenstate that does not depend on the parameters  $\theta$  and  $\varphi$ . By adiabatically changing  $\varphi$  in such a way as to have a loop starting from  $\theta = \varphi = 0$ , the state  $|1\rangle_L$  acquires a Berry phase which is proportional to the solid angle  $\Omega_Z$  swept out by the vector  $(\varphi, \theta)$  [20]. Therefore after this adiabatic loop one has that:  $|0\rangle_L \rightarrow |0\rangle_L$  and  $|1\rangle_L \rightarrow e^{-i\Omega_Z/2} |1\rangle_L$ , which is clearly equivalent to the operation  $\exp(i\Omega_Z Z_L/2)$ .

In order to obtain a universal set of gates we need to generate at least two noncommuting single-qubit gates. Therefore we next show how to implement  $\exp(i\varphi X_L) = \exp(i\varphi R_{12}^x)$ . One way is to once again establish an isomorphism between the system governed by Eq. (2), restricted to  $C$ , and the HQC model of Ref. [9]. Here we provide an independent derivation. We turn on the couplings in such a way as to obtain the Hamiltonian

$$H_X = J_{34} R_{34}^x + J_{24} \left( \cos \varphi \frac{R_{24}^x - R_{14}^x}{\sqrt{2}} + \sin \varphi \frac{R_{24}^y - R_{14}^y}{\sqrt{2}} \right). \quad (5)$$

Let  $|\pm\rangle_L := (|1\rangle_L \pm |0\rangle_L)/\sqrt{2}$ . Then one can readily check that under the action of  $(R_{24}^x - R_{14}^x)/\sqrt{2}$  the state  $|+\rangle_L \mapsto 0$ ,  $|-\rangle_L \mapsto |a_1\rangle$ , and  $|a_1\rangle \mapsto |-\rangle_L$ , etc., Therefore  $H_X = J_{34} |a_1\rangle\langle a_2| + J_{24} e^{i\varphi} |a_1\rangle\langle -|_L + \text{H.c.}$  This is a Lambda configuration with  $|a_1\rangle$  at the top and  $|a_2\rangle, |-\rangle_L$  at the bottom, so that  $H_X$  supports a dark state  $|\Psi_2\rangle = \cos \theta |-\rangle_L - \sin \theta e^{i\varphi} |a_2\rangle$  where  $\theta = \tan^{-1}(J_{24}/J_{34})$ . The similarity between  $H_X$  and  $H_Z$  is evident. Then, by executing an adiabatic loop in the parameter space in analogy to

the  $H_Z$  case, one obtains the geometric evolution  $|+\rangle_L \rightarrow |+\rangle_L$ ,  $|-\rangle_L \rightarrow e^{-i\Omega_X/2}|-\rangle_L$ , where now  $\Omega_X$  is the solid angle swept out by the vector  $(\theta, \varphi)$ . Switching back to the computational basis this transformation amounts to the map  $|0\rangle_L \rightarrow \cos\frac{\Omega_X}{4}|0\rangle_L + i\sin\frac{\Omega_X}{4}|1\rangle_L$ , and  $|1\rangle_L \rightarrow \cos\frac{\Omega_X}{4}|1\rangle_L + i\sin\frac{\Omega_X}{4}|0\rangle_L$ , which is equivalent to  $\exp(i\Omega_X R_{12}^x/4)$ .

*Two-qubit holonomic gates.*—A crucial, and typically rather demanding part, of any QIP implementation proposal is the realization of an entangling two-qubit gate. We next consider this problem and demonstrate how to solve it following a strategy relying on the same abstract holonomic structure as that discussed above for one-qubit gates. The total state-space is now  $\mathcal{H}_4^{\otimes 2}$ , while the two-qubit code is spanned by  $|\alpha\rangle_L \otimes |\beta\rangle_L$ ,  $(\alpha, \beta = 0, 1)$ . The states  $|\alpha_i\rangle^{\otimes 2}$  ( $i = 1, 2$ ) will function as ancillae. Importantly, once again all the relevant states belong to a DFS against collective dephasing.

Let us suppose that one can engineer the following controllable four-qubit interaction

$$H_4 = J_{24,68}(R_{24}^x \cos\varphi + R_{24}^y \sin\varphi)(R_{68}^x \cos\varphi + R_{68}^y \sin\varphi) + J_{34,78}R_{34}^x R_{78}^x, \quad (6)$$

which should be recognizable as a straightforward extension of the one-qubit Hamiltonian (4). Below we discuss the implementation of such a Hamiltonian. To explicitly exhibit the dark state structure, we write this Hamiltonian in the form  $H_4 = J_{34,78}R_{34}^x(|a_1\rangle\langle a_2|)^{\otimes 2} + J_{24,68}e^{i\varphi}(|a_1\rangle \times \langle 1_L|)^{\otimes 2} + \text{H.c.}$ , from which it is easily seen that there is one dark state, given by  $\cos\theta|1\rangle_L^{\otimes 2} - \sin\theta e^{i\varphi}|a_2\rangle^{\otimes 2}$ , where  $\theta = \tan^{-1}(J_{24,68}/J_{34,78})$ . After the same kind of adiabatic cyclic evolution as described in the single-qubit gate case, one obtains in  $C^{\otimes 2}$  the controlled phase-shift gate  $CP = \text{diag}(1, 1, 1, e^{i\Omega_P/2})$ , where, as usual,  $\Omega_P$  is the solid angle swept out by the vector  $(\theta, \varphi)$ .

*Extensions and generalizations.*—The scheme described so far for the case of collective dephasing is straightforwardly scalable to an arbitrary number  $N$  of encoded DF qubits. The total space is now given by  $C^{\otimes N} \cong (C^4)^{\otimes N}$ , with  $C$  given in Eq. (1). This, of course, is still an eigenspace of the collective spin  $z$  component, i.e.,  $Z = \sum_{l=1}^{4N} Z_l$ . Following the procedure established above, the controllable Hamiltonian used to generate a controlled phase shift between the  $i$ th and the  $j$ th encoded qubits, has the same structure as  $H_4$  [Eq. (6)], where, with obvious notation,  $J_{24,68} \rightarrow J_{4(i-1)+2,4(i-1)+4;4(j-1)+2,4(j-1)+4}$  and  $J_{34,78} \rightarrow J_{4(i-1)+3,4(i-1)+4;4(j-1)+3,4(j-1)+4}$ , and similarly for the  $R_{kl}^x, R_{kl}^y$  operators.

Next, by making use of *noiseless subsystems* (NSs) [5], we show that our combined HQC-DFS strategy can also be applied against *general* collective decoherence [2]. Our arguments will be existential in nature, with constructive details to be discussed elsewhere. Let  $\mathcal{H}_J \cong C^{2^{J+1}}$  denote the total spin- $J$  irreducible representation of  $SU(2)$ . The state-space of five qubits, i.e.,  $\mathcal{H}_{1/2}^{\otimes 5}$ , decomposes, with

respect to the collective  $SU(2)$ -representation (Clebsch-Gordan decomposition), as follows:

$$\mathcal{H}_{1/2}^{\otimes 5} \cong C^4 \otimes \mathcal{H}_{3/2} \oplus C^5 \otimes \mathcal{H}_{1/2} \oplus C \otimes \mathcal{H}_{5/2}. \quad (7)$$

Each of the  $C^{n_j}$  factors represents the multiplicity of the total spin- $J$  irreducible representation and corresponds to a NS against collective general decoherence [5]. Consider the first term in (7): the multiplicity factor  $C^4$  for the  $J = 3/2$  representation provides a four-dimensional NS. It might then encode two noiseless qubits, but, since we wish to perform QIP with holonomies, we will instead use this  $C^4$  space as a code for just one noiseless qubit  $|\tilde{\alpha}\rangle_L$ ,  $(\alpha = 0, 1)$  and two ancillary states  $|\tilde{a}_i\rangle$ ,  $(i = 1, 2)$ . Suppose now that one is able to enact the controllable Hamiltonian  $H_{NS} = J'|\tilde{a}_1\rangle\langle\tilde{a}_2| + J''_0 e^{i\varphi}|\tilde{a}_1\rangle \times \langle\tilde{0}|_L + J''_1 e^{i\varphi}|\tilde{a}_1\rangle\langle\tilde{1}|_L + \text{H.c.}$ , which, when  $J''_0 = 0$ , admits a dark state given by  $|\Psi_3\rangle = \cos\theta|\tilde{1}\rangle_L - \sin\theta e^{i\varphi}|\tilde{a}_2\rangle$ , where  $\theta = \tan^{-1}(J''_1/J')$ . By resorting to the same considerations as previously developed for the collective dephasing case, it should be clear that this allows us to enact a phase gate  $Z_L$  between  $|\tilde{0}\rangle_L$  and  $|\tilde{1}\rangle_L$ . By choosing  $J''_0, J''_1$  such that  $H_{NS}$  becomes  $J'|\tilde{a}_1\rangle\langle\tilde{a}_2| + J'' e^{i\varphi}|\tilde{a}_1\rangle\langle\tilde{-}|_L + \text{H.c.}$  we can enact the  $X_L$  gate, so that universal single-qubit control by holonomies can be achieved in this case as well.

To realize the required multilevel controllable Hamiltonian, we observe that the  $C^4$  space under consideration is a four-dimensional irreducible representation of the permutation group  $S_5$  (acting over the whole space as  $\hat{\sigma} \otimes_{j=1}^5 |\tilde{\alpha}_j\rangle = \otimes_{j=1}^5 |\tilde{\alpha}_{\sigma(j)}\rangle$ , where  $\sigma \in S_5$ ). Following Ref. [21], universal control over this irreducible representation space amounts to the ability to switch on and off a pair of *generic* Hamiltonians in the group algebra of  $S_5$ . An important example is provided by Heisenberg *exchange* Hamiltonians, i.e.,  $\sum_{l<m} J_{lm} \mathbf{S}_l \cdot \mathbf{S}_m$  [where  $\mathbf{S}_l = (X_l, Y_l, Z_l)/2$ ], which are naturally available interactions in several spin-based proposals for QIP [22,23]. The construction of two-encoded-qubit holonomic gates, and the generalization to arbitrary numbers of such qubits, again follow the same pattern as in the collective dephasing case.

*Implementation.*—We note that all required Hamiltonians,  $H_Z, H_X$ , and  $H_4$ , have a similar form. These Hamiltonians all involve control over both  $\theta$  and  $\varphi$ . However, we are free to choose any loop  $C(\theta, \varphi)$  in the  $(\theta, \varphi)$ -parameter space, and we can choose a loop which toggles between  $\theta$  and  $\varphi$ . For example, the loop  $C(\theta, \varphi): (0, 0) \rightarrow (\frac{\pi}{2}, 0) \rightarrow (\frac{\pi}{2}, \varphi_0) \rightarrow (0, \varphi_0) \equiv (0, 0)$  has this property, where  $\varphi_0$  is the solid angle  $\Omega$  swept out by the vector  $(\theta, \varphi)$  for this specific loop.

Let us briefly address the feasibility of the control of  $H_Z, H_X$ , and  $H_4$  in the context of actual quantum computing proposals. In proposals based on electron spins in quantum dots [22], the  $H_Z$  and  $H_X$  Hamiltonians are available when one takes into account the spin-orbit interaction, for then the effective spin-spin interaction becomes  $H_{ij}(t) = J_{ij}(t)[\beta(t)(X_i Y_j - Y_i X_j) + (1 + \gamma(t))(X_i X_j + Y_i Y_j) + Z_i Z_j]$

[24,25]. This Hamiltonian has enough degrees of freedom to implement  $H_Z$  and  $H_X$ , via separate control of  $J$  and the dimensionless anisotropy parameters  $\beta$  and  $\gamma$ , assuming the latter can be made  $\gg 1$ , e.g., when the interdot distance is large [24]. The parameters  $\beta$  and  $\gamma$  can be controlled via the confining potential or via pulse shaping, as has been discussed in detail in Ref. [25], and  $\varphi = \tan^{-1}\beta/(1 + \gamma)$ . The parameter  $\theta$  is controllable via, e.g.,  $J_{24}$  and  $J_{34}$ . Implementation of  $H_4$  is undoubtedly more challenging, since four-body interactions are involved. Recent results indicate that such terms arise by simultaneously coupling four quantum dots [26]; their relative strength can be controlled, e.g., by adjusting the confining potentials [27].

In proposals based on trapped ions, the implementation of  $H_Z$  and  $H_X$  is directly possible using the Sørensen-Mølmer (SM) scheme [19]. The SM gate between two ions implements  $H_Z$  and  $H_X$  with control over the various terms achieved via the phases of two lasers [28]. It is also possible to implement  $H_4$  using the SM scheme, via control over two pairs of ions [28]. Given that a geometric two ion-qubit phase gate has already been demonstrated [29], trapped ions seem to be particularly favorable for the implementation of our proposed HQC-DFS scheme.

*Conclusions.*—We have combined the DFS and HQC techniques, and have shown how to realize universal quantum computation over a scalable DFS against collective dephasing by using adiabatic holonomies only. We discussed an extension to the general collective decoherence case, arguing that controllability of exchange Hamiltonians would suffice. Remarkably, the whole computational process is carried out *within* the DFS. The DFS embedding, along with the all-geometrical nature of HQC, promises to give to this scheme a twofold resilience, against decoherence and stochastic control errors. The proposed universal quantum gates are carried out via adiabatic manipulation of Hamiltonians that are controllable in several proposals for QIP implementations. We are therefore hopeful that the theoretical ideas presented here may stimulate corresponding experimental activity.

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