Operator quantum error correction for continuous dynamics

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We study the conditions under which a subsystem code is correctable in the presence of noise that results from continuous dynamics. We consider the case of Markovian dynamics as well as the general case of Hamiltonian dynamics of the system and the environment, and derive necessary and sufficient conditions on the Lindbladian and system-environment Hamiltonian, respectively. For the case when the encoded information is correctable during an entire time interval, the conditions we obtain can be thought of as generalizations of the previously derived conditions for decoherence-free subsystems to the case where the subsystem is time dependent. As a special case, we consider conditions for unitary correctability. In the case of Hamiltonian evolution, the conditions for unitary correctability concern only the effect of the Hamiltonian on the system, whereas the conditions for general correctability concern the entire system-environment Hamiltonian. We also derive conditions on the Hamiltonian which depend on the initial state of the environment, as well as conditions for correctability at only a particular moment of time. We discuss possible implications of our results for approximate quantum error correction.

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I. INTRODUCTION

Operator quantum error correction (OQEC) [1] is a unified approach to error correction for noise represented by a completely positive trace-preserving (CPTP) linear map or noise channel. This approach uses the most general encoding for the protection of information—encoding in subsystems [2]. OQEC contains as special cases the standard quantum error-correction method [3] as well as the methods of decoherence-free subspaces [4] and subsystems [5]. Recently, the approach was generalized to include entanglement-assisted error correction [6], resulting in the most general quantum error-correction formalism presently known for CPTP maps [7].

In practice, however, noise is a continuous process and if it can be represented by a CPTP map, that map is generally a function of time. Correctability is therefore a time-dependent property. Furthermore, the evolution of an open system is completely positive if the system and the environment are initially uncorrelated [8], and necessary conditions for CPTP maps are not known. For more general cases one needs a notion of correctability that can capture non-CP transformations [9]. Whether completely positive or not, the noise map is a result of the action of the generator driving the evolution and possibly of the initial state of the system and the environment.

Perfect correctability is usually an idealization, since there is almost always a nonzero probability for uncorrectable errors. For example, if each qubit in a code undergoes independent errors, no matter how large the code is, there will always be a nonzero probability for multiqubit errors that are not correctable by the code (although, if this probability per unit time is sufficiently small, an arbitrarily long information processing task can be implemented reliably by the use of fault-tolerant techniques [10]). Nevertheless, perfect correctability is a fundamental concept in the theory of quantum error correction and its understanding is crucial for the understanding of error correction in realistic scenarios.

In this paper, we study the question of the conditions under which a subsystem code is perfectly correctable in the presence of noise that results from continuous dynamics. We first consider the case where the subsystem is correctable during an entire time interval following the encoding, i.e., when the information initially encoded in the subsystem does not leak out to the environment. Such conditions are needed in order to understand the mechanisms of information preservation during continuous processes. If the noise process is expressed as a CPTP map, the answer is simple—the Kraus operators have to satisfy the known error-correction conditions at every moment during the evolution. Our goal is, however, to understand these conditions in terms of the generator that drives the evolution—the system-environment Hamiltonian, or in the case of Markovian evolution the Lindbladian.

We also consider the case where a subsystem can be correctable at a given moment after the encoding without being correctable during the entire time interval between the encoding and that moment. This situation can arise in the case of non-Markovian dynamics, where the encoded information can flow out to the environment and later return to the system. We show that the conditions one obtains on the generator of evolution in this case do not provide nontrivial information about the properties of the instantaneous dynamics, except for the global requirement that the linear map resulting from the dynamics up to the moment in question satisfies the known error-correction conditions.

Conditions on the generator of evolution have been derived for the case of decoherence-free subsystems (DFSs) [11], which are a special type of operator codes. DFSs are *fixed* subsystems of the system's Hilbert space, inside which all states evolve unitarily. One generalization of this concept are the so-called *unitarily correctable subsystems* [1]. These are subsystems all states inside of which can be corrected via a unitary operation up to an arbitrary transformation inside the gauge subsystem. Unlike DFSs, the unitary evolution followed by states in a unitarily correctable code is not restricted to the initial subsystem. An even more general concept is that of *unitarily recoverable* subsystems [1], for which states can be recovered by a unitary transformation up to an expansion of the gauge subsystem. It was shown that any correctable subsystem is in fact a unitarily recoverable subsystem [12]. This result reflects the so-called subsystem principle [2], according to which protected information is always contained in a subsystem of the system's Hilbert space. The connection between DFSs and unitarily recoverable subsystems suggests that similar conditions on the generators of evolution to those for DFSs can be derived in the case of general correctable subsystems. This is the subject of the present paper.

The paper is organized as follows. In Sec. II, we review the definitions of correctable subsystems and unitarily recoverable subsystems. In Sec. III, we discuss the necessary and sufficient conditions for such subsystems to exist in the case of CPTP maps. In Sec. IV, we derive conditions for the case of Markovian dynamics. The conditions for general correctability in this case are essentially the same as those for unitary correctability except that the dimension of the gauge subsystem is allowed to suddenly increase. For the case when the evolution is noncorrectable, we conjecture a procedure for tracking the subsystem which contains the optimal amount of undissipated information and discuss its possible implications for the problem of optimal error correction. In Sec. V, we derive conditions on the system-environment Hamiltonian. In this case, the conditions for continuous unitary correctability concern only the effect of the Hamiltonian on the system, whereas the conditions for continuous general correctability concern the entire system-environment Hamiltonian. In the latter case, the state of the noisy subsystem plus environment belongs to a particular subspace which plays an important role in the conditions. We extend the conditions to the case where the environment is initialized inside a particular subspace. In Sec. VI, we discuss the conditions under which a subsystem is correctable only at a particular moment of time. We conclude in Sec. VII.

II. CORRECTABLE SUBSYSTEMS

For simplicity, we consider the case where information is stored in only one subsystem. Then there is a corresponding decomposition of the Hilbert space of the system,

$$\mathcal{H}^S = \mathcal{H}^A \otimes \mathcal{H}^B \oplus \mathcal{K}, \tag{1}$$

where the subsystem $\mathcal{H}^A$ is used for encoding of the protected information. The subsystem $\mathcal{H}^B$ is referred to as the gauge subsystem, and $\mathcal{K}$ denotes the rest of the Hilbert space. In the formulation of OQEC [1], the noise process is a CPTP linear map $\mathcal{E}:\mathcal{B}(\mathcal{H}^S)\to\mathcal{B}(\mathcal{H}^S)$, where $\mathcal{B}(\mathcal{H})$ denotes the set of linear operators on a finite-dimensional Hilbert space $\mathcal{H}$. Such a map can be written in the Kraus form [13]

$$\mathcal{E}(\sigma) = \sum_\alpha M_\alpha \sigma M_\alpha^\dagger \quad \text{for all} \quad \sigma \in \mathcal{B}(\mathcal{H}^S), \tag{2}$$

where the Kraus operators $\{M_\alpha\}\subseteq\mathcal{B}(\mathcal{H}^S)$ satisfy

$$\sum_\alpha M_\alpha^\dagger M_\alpha = I^S. \tag{3}$$

Let $\mathcal{P}^{AB}(\cdot)$ denote the superoperator projector on $\mathcal{B}(\mathcal{H}^A \otimes \mathcal{H}^B)$,

$$\mathcal{P}^{AB}(\cdot) = P^{AB}(\cdot)P^{AB}, \tag{4}$$

where P^{AB} is the projector of $\mathcal{H}^S$ onto $\mathcal{H}^A\otimes\mathcal{H}^B$,

$$P^{AB}\mathcal{H}^S = \mathcal{H}^A \otimes \mathcal{H}^B. \tag{5}$$

We now recall some of the key notions in correctability. The first and simplest version is one that does not require a recovery (or correction) step:

Definition 1 (noiseless subsystem). The subsystem $\mathcal{H}^A$ in Eq. (1) is called noiseless with respect to the noise process $\mathcal{E}$, if

$$\mathrm{Tr}_B\{(\mathcal{P}^{AB}\circ\mathcal{E})(\sigma)\} = \mathrm{Tr}_B\{\sigma\}$$

$$\text{for all} \quad \sigma \in \mathcal{B}(\mathcal{H}^S) \quad \text{such that} \quad \sigma = \mathcal{P}^{AB}(\sigma). \tag{6}$$

More general is the case when one invokes a correction map to correct the subsystem.

Definition 2 (correctable subsystem). The subsystem $\mathcal{H}^A$ in Eq. (1) is called correctable if there exists a correcting CPTP map $\mathcal{R}:\mathcal{B}(\mathcal{H}^S)\to\mathcal{B}(\mathcal{H}^S)$, such that the subsystem is noiseless with respect to the map $\mathcal{R}\circ\mathcal{E}$:

$$\mathrm{Tr}_B\{(\mathcal{P}^{AB}\circ\mathcal{R}\circ\mathcal{E})(\sigma)\} = \mathrm{Tr}_B\{\sigma\}$$

$$\text{for all} \quad \sigma \in \mathcal{B}(\mathcal{H}^S) \quad \text{such that} \quad \sigma = \mathcal{P}^{AB}(\sigma). \tag{7}$$

A special case of this is unitary correction.

Definition 3 (unitarily correctable subsystem). The subsystem $\mathcal{H}^A$ in Eq. (1) is called unitarily correctable when there exists a unitary correcting map, i.e., when there exists a unitary map $\mathcal{U}:\mathcal{B}(\mathcal{H}^S)\to\mathcal{B}(\mathcal{H}^S)$ such that

$$\mathrm{Tr}_B\{(\mathcal{P}^{AB}\circ\mathcal{U}\circ\mathcal{E})(\sigma)\} = \mathrm{Tr}_B\{\sigma\}$$

$$\text{for all} \quad \sigma \in \mathcal{B}(\mathcal{H}^S) \quad \text{such that} \quad \sigma = \mathcal{P}^{AB}(\sigma). \tag{8}$$

A similar but more general notion is that of a unitarily recoverable subsystem, for which the unitary U need not bring the erroneous state back to the original subspace $\mathcal{H}^A \otimes \mathcal{H}^B$ but can bring it into a subspace $\mathcal{H}^A\otimes\mathcal{H}^{B'}$, with B not necessarily equal to B'.

Definition 4 (unitarily recoverable subsystem). The subsystem $\mathcal{H}^A$ in Eq. (1) is called unitarily recoverable when there exists a unitary map $\mathcal{U}:\mathcal{B}(\mathcal{H}^S)\to\mathcal{B}(\mathcal{H}^S)$ such that

$$\mathrm{Tr}_{B'}\{(\mathcal{P}^{AB'}\circ\mathcal{U}\circ\mathcal{E})(\sigma)\} = \mathrm{Tr}_B\{\sigma\}$$

$$\text{for all} \quad \sigma \in \mathcal{B}(\mathcal{H}^S) \quad \text{such that} \quad \sigma = \mathcal{P}^{AB}(\sigma). \tag{9}$$

Obviously, if $\mathcal{H}^A$ is unitarily recoverable, it is also correctable, since one can always apply a local CPTP map $\mathcal{E}^{B'\to B}:\mathcal{B}(\mathcal{H}^{B'})\to\mathcal{B}(\mathcal{H}^B)$ which brings all states from $\mathcal{H}^{B'}$ to $\mathcal{H}^B$. (In fact, if the dimension of $\mathcal{H}^{B'}$ is smaller than or equal to that of $\mathcal{H}^B$, this can always be done by a unitary map, i.e., $\mathcal{H}^A$ is unitarily correctable.) In Ref. [12] it was shown that the reverse is also true—if $\mathcal{H}^A$ is correctable, it is unitarily recoverable. This equivalence will provide the basis for our derivation of correctability conditions for continuous dynamics.

Before we proceed with our discussion, we point out that condition (9) can be equivalently written as [1]

$$\mathcal{U}\circ\mathcal{E}(\rho\otimes\tau) = \rho\otimes\tau', \quad \tau' \in \mathcal{B}(\mathcal{H}^{B'}),$$

$$\text{for all} \quad \rho \in \mathcal{B}(\mathcal{H}^A), \quad \tau \in \mathcal{B}(\mathcal{H}^B). \tag{10}$$

III. COMPLETELY POSITIVE LINEAR MAPS

An important class of transformations on quantum states consists of the so-called completely positive linear maps, also known simply as quantum operations [14]. Let $\mathcal{H}^S$ and $\mathcal{H}^E$ denote the Hilbert spaces of a system and its environment, and let $\mathcal{H}=\mathcal{H}^S\otimes\mathcal{H}^E$ be the total Hilbert space. A common example of a CP map is the transformation that the state of a system undergoes if the system is initially decoupled from its environment, $\rho(0)=\rho^S(0)\otimes\rho^E(0)$, and both the system and environment evolve according to the Schrödinger equation:

$$\frac{d\rho(t)}{dt} = -i[H(t),\rho(t)]. \tag{11}$$

(We work in units in which $\hbar=1$, and assume a generally time-dependent Hamiltonian.) Equation (11) gives rise to the unitary transformation

$$\rho(t) = V(t)\rho(0)V^\dagger(t), \tag{12}$$

with

$$V(t) = \mathcal{T}\exp\left(-i\int_0^t H(\tau)d\tau\right), \tag{13}$$

where $\mathcal{T}$ denotes time ordering. Under the assumption of an initially decoupled state of the system and the environment, the transformation of the state of the system is described by a CPTP map

$$\rho^S(0) \to \rho^S(t) \equiv \mathrm{Tr}_E\{\rho(t)\} = \sum_\alpha M_\alpha(t)\rho^S(0)M_\alpha^\dagger(t),$$

for which the time-dependent Kraus operators $M_\alpha(t) \in \mathcal{B}(\mathcal{H}^S)$ are given by

$$M_\alpha(t) = \sqrt{\lambda_\nu}\mathrm{Tr}_E\{I^S\otimes|\nu\rangle\langle\mu|V(t)\}, \quad \alpha=(\mu,\nu), \tag{14}$$

where $\{|\mu\rangle\}$ is a basis of the Hilbert space of the environment, in which the initial environment density operator is diagonal: $\rho^E(0)=\Sigma_\mu\lambda_\mu|\mu\rangle\langle\mu|$.

The Kraus representation (2) applies to any CP linear map, which need not necessarily arise from evolution of the type (11). This is why, in the following theorem, we derive conditions for discrete CP maps. For correctability under continuous dynamics, the same conditions must apply at all times, i.e., one can view the quantities M_α, U, and C_α, as well as the subsystem $\mathcal{H}^{B'}$ in the theorem as being implicitly time dependent.

Theorem 1. The subsystem $\mathcal{H}^A$ in the decomposition (1) is unitarily recoverable under a CP linear noise process in the form (2), if and only if there exists a unitary operator $U \in \mathcal{B}(\mathcal{H}^S)$ such that the Kraus operators satisfy

$$M_\alpha P^{AB} = U^\dagger I^A\otimes C_\alpha^{B\to B'}, \quad C_\alpha^{B\to B'}:\mathcal{H}^B\to\mathcal{H}^{B'},$$

$$\forall\ \alpha. \tag{15}$$

Proof. The sufficiency of condition (15) is obvious—using that $\rho\otimes\tau$ in Eq. (10) satisfies $\rho\otimes\tau=P^{AB}\rho\otimes\tau P^{AB}$, it can be immediately verified that Eq. (15) implies Eq. (10) with $\mathcal{U}=U(\cdot)U^\dagger$. Now assume that $\mathcal{H}^A$ is unitarily recoverable and the recovery map is $\mathcal{U}=U(\cdot)U^\dagger$. The map $\mathcal{U}\circ\mathcal{E}$ in Eq. (10) can then be thought of as having Kraus operators UM_α. In particular, condition (10) has to be satisfied for $\rho=|\psi\rangle\langle\psi|$, $\tau=|\phi\rangle\langle\phi|$ where $|\psi\rangle\in\mathcal{H}^A$ and $|\phi\rangle\in\mathcal{H}^B$ are pure states. Notice that the image of $|\psi\rangle\langle\psi|\otimes|\phi\rangle\langle\phi|$ under the map $\mathcal{U}\circ\mathcal{E}$ would be of the form $|\psi\rangle\langle\psi|\otimes\tau'$, only if all terms in Eq. (2) are of the form

$$UM_\alpha|\psi\rangle\langle\psi|\otimes|\phi\rangle\langle\phi|M_\alpha^\dagger U^\dagger$$
$$= |g_\alpha(\psi)|^2|\psi\rangle\langle\psi|\otimes|\phi'_\alpha(\psi)\rangle\langle\phi'_\alpha(\psi)|,$$

$$g_\alpha(\psi) \in C, \tag{16}$$

where for now we assume that g_α and $|\phi'_\alpha\rangle$ may depend on $|\psi\rangle$. This follows from the fact that each of the operators UM_α transforms pure states into pure states, and the (positive) reduced operator on $\mathcal{H}^A$ of each of the terms (16) must be proportional to the same pure state $|\psi\rangle\langle\psi|$ in order for the total reduced density operator on $\mathcal{H}^A$ to be pure. In other words,

$$UM_\alpha|\psi\rangle|\phi\rangle = g_\alpha(\psi)|\psi\rangle|\phi'_\alpha(\psi)\rangle, \quad g_\alpha(\psi)\in C, \quad \forall\ \alpha. \tag{17}$$

But if we impose (17) on a linear superposition $|\psi\rangle=a|\psi_1\rangle+b|\psi_2\rangle$ $(a,b\neq 0)$, we obtain $g_\alpha(\psi_1)=g_\alpha(\psi_2)$ and $|\phi'_\alpha(\psi_1)\rangle=|\phi'_\alpha(\psi_2)\rangle$, i.e.,

$$g_\alpha(\psi)\equiv g_\alpha, \quad |\phi'_\alpha(\psi)\rangle\equiv|\phi'_\alpha\rangle, \quad \forall\,|\psi\rangle\in\mathcal{H}^A, \quad \forall\ \alpha. \tag{18}$$

Since Eq. (17) has to be satisfied for all $|\psi\rangle\in\mathcal{H}^A$ and all $|\phi\rangle\in\mathcal{H}^B$, we obtain

$$UM_\alpha P^{AB} = I^A\otimes C_\alpha^{B\to B'}, \quad C_\alpha^{B\to B'}:\mathcal{H}^B\to\mathcal{H}^{B'},$$

$$\forall\ \alpha. \tag{19}$$

Applying $U^\dagger$ from the left yields condition (15). ∎

We remark that condition (15) is equivalent to the conditions obtained in Ref. [1].

IV. MARKOVIAN DYNAMICS

The most general continuous completely positive time-local evolution of the state of a quantum system is described by a semigroup master equation in the Lindblad form [15]:

$$\frac{d\rho(t)}{dt} = -i[H(t),\rho(t)] - \frac{1}{2}\sum_j (2L_j(t)\rho(t)L_j^\dagger(t) - L_j^\dagger(t)L_j(t)\rho(t) - \rho(t)L_j^\dagger(t)L_j(t)) \equiv \mathcal{L}(t)\rho(t). \tag{20}$$

Here $H(t)$ is a system Hamiltonian, $L_j(t)$ are Lindblad operators, and $\mathcal{L}(t)$ is the Liouvillian superoperator corresponding to this dynamics. The general evolution of a state is given by

$$\rho(t_2) = \mathcal{T}\exp\left(\int_{t_1}^{t_2}\mathcal{L}(\tau)d\tau\right)\rho(t_1), \quad t_2 > t_1. \tag{21}$$

Such evolution arises from a Hamiltonian interaction with the environment in the Markovian limit of short bath correlation times [16]. The evolution induced by (20) is completely positive and can be thought of as arising from an infinite sequence of infinitesimal completely positive maps of the form (2). These maps can depend on time, and therefore the operators in (20) are generally time dependent.

We first derive necessary and sufficient conditions for unitarily correctable subsystems under the dynamics (20), and then extend them to the case of unitarily recoverable subsystems.

A. Unitarily correctable subsystems

In the case of continuous dynamics, the error map $\mathcal{E}$ and the error-correcting map $\mathcal{U}$ in Eq. (8) are generally time dependent. If we set $t=0$ as the initial time at which the system is prepared, the error map resulting from the dynamics (20) is

$$\mathcal{E}(t)(\cdot) = \mathcal{T}\exp\left(\int_0^t \mathcal{L}(\tau)d\tau\right)(\cdot). \tag{22}$$

Our strategy is now to convert the problem into one of noiseless subsystems, for which necessary and sufficient conditions have already been found [11]. To this end let $\mathcal{U}(t) = U(t)(\cdot)U^\dagger(t)$ be the unitary error-correcting map in Eq. (8). We can define the rotating frame corresponding to $U^\dagger(t)$ as the transformation of each operator as

$$O(t) \to \widetilde{O}(t) = U(t)O(t)U^\dagger(t). \tag{23}$$

In this frame, the Lindblad equation (20) can be written as

$$\frac{d\widetilde{\rho}(t)}{dt} = -i[\widetilde{H}(t)+H'(t),\widetilde{\rho}(t)] - \frac{1}{2}\sum_j (2\widetilde{L}_j(t)\widetilde{\rho}(t)\widetilde{L}_j^\dagger(t) - \widetilde{L}_j^\dagger(t)\widetilde{L}_j(t)\widetilde{\rho}(t) - \widetilde{\rho}(t)\widetilde{L}_j^\dagger(t)\widetilde{L}_j(t)) \equiv \widetilde{\mathcal{L}}(t)\widetilde{\rho}(t), \tag{24}$$

where $H'(t)$ is defined through

$$i\frac{dU(t)}{dt} = H'(t)U(t), \tag{25}$$

i.e.,

$$U(t) = \mathcal{T}\exp\left(-i\int_0^t H'(\tau)d\tau\right). \tag{26}$$

The CPTP map resulting from the dynamics (24) is

$$\widetilde{\mathcal{E}}(t)(\cdot) = \mathcal{T}\exp\left(\int_0^t \widetilde{\mathcal{L}}(\tau)d\tau\right)(\cdot). \tag{27}$$

Theorem 2. Let $\widetilde{H}(t)$ and $\widetilde{L}_j(t)$ be the Hamiltonian and the Lindblad operators in the rotating frame (23) with $U(t)$ given by Eq. (25). Then the subsystem $\mathcal{H}^A$ in the decomposition (1) is correctable by $U(t)$ during the evolution (20), if and only if

$$\widetilde{L}_j(t)P^{AB} = I^A \otimes C_j^B(t), \quad C_j^B(t) \in \mathcal{B}(\mathcal{H}^B), \quad \forall\ j, \tag{28}$$

and

$$\mathcal{P}^{AB}(\widetilde{H}(t)+H'(t)) = I^A \otimes D^B(t), \quad D^B(t) \in \mathcal{B}(\mathcal{H}^B), \tag{29}$$

and

$$P^{AB}\left(\widetilde{H}(t)+H'(t)+\frac{i}{2}\sum_j \widetilde{L}_j^\dagger(t)\widetilde{L}_j(t)\right)P_{\mathcal{K}} = 0, \tag{30}$$

for all t, where $P_{\mathcal{K}}$ denotes the projector on $\mathcal{K}$.

Proof. Since by definition $U(t)$ is an error-correcting map for subsystem $\mathcal{H}^A$, if $\mathcal{P}^{AB}(\rho(0))=\rho(0)$, we have

$$\begin{aligned}\mathrm{Tr}_B\{\mathcal{P}^{AB}\circ\widetilde{\mathcal{E}}(\widetilde{\rho}(0))\} &= \mathrm{Tr}_B\{\mathcal{P}^{AB}(\widetilde{\rho}(t))\}\\ &= \mathrm{Tr}_B\{\mathcal{P}^{AB}\circ\mathcal{U}(t)\circ\mathcal{E}(t)(\rho(0))\}\\ &= \mathrm{Tr}_B\{\rho(0)\} = \mathrm{Tr}_B\{\widetilde{\rho}(0)\},\end{aligned}$$

i.e., $\mathcal{H}^A$ is a noiseless subsystem under the evolution in the rotating frame (24). Then the theorem follows from Eq. (24) and the conditions for noiseless subsystems under Markovian dynamics obtained in [11]. ∎

Remark. Conditions (29) and (30) can be used to obtain the operator $H'(t)$ [and hence $U(t)$] if the initial decomposition (1) is known. Note that there is a freedom in the definition of $H'(t)$. For example, $D^B(t)$ in Eq. (29) can be any Hermitian operator. In particular, we can choose $D^B(t)=0$. Also, the term $P_{\mathcal{K}}H'(t)P_{\mathcal{K}}$ does not play a role and can be chosen arbitrarily. Using that $P_{\mathcal{K}}=I-P^{AB}$, we can choose

$$H'(t) = -\widetilde{H}(t) - \frac{i}{2}P^{AB}\left(\sum_j \widetilde{L}_j^\dagger(t)\widetilde{L}_j(t)\right) + \frac{i}{2}\left(\sum_j \widetilde{L}_j^\dagger(t)\widetilde{L}_j(t)\right)P^{AB}, \tag{31}$$

which satisfies Eqs. (29) and (30). Using Eqs. (23), (25), and (31), we obtain the following first-order differential equation for $U(t)$:

$$i\frac{dU(t)}{dt} = - U(t)H(t) - \frac{i}{2}P^{AB}U(t)\Big(\sum_j L_j^\dagger(t)L_j(t)\Big) + \frac{i}{2}U(t)\Big(\sum_j L_j^\dagger(t)L_j(t)\Big)U^\dagger(t)P^{AB}U(t). \quad (32)$$

This equation can be used to solve for $U(t)$ starting from $U(0)=I$.

Notice that, since $\mathcal{H}^A$ is unitarily correctable by $U(t)$, at time t the initially encoded information can be thought of as contained in the subsystem $\mathcal{H}^A(t)$ defined through

$$\mathcal{H}^A(t) \otimes \mathcal{H}^B(t) \equiv U^\dagger(t)\mathcal{H}^A \otimes \mathcal{H}^B, \quad (33)$$

i.e., this subsystem is obtained from $\mathcal{H}^A$ in Eq. (1) via the unitary transformation $U^\dagger(t)$. One can easily verify that the fact that the right-hand side of Eq. (28) acts trivially on $\mathcal{H}^A$ together with Eq. (29) are necessary and sufficient conditions for an arbitrary state encoded in subsystem $\mathcal{H}^A(t)$ to undergo trivial dynamics at time t. Therefore, these conditions can be thought of as the conditions for lack of noise in the instantaneous subsystem that contains the protected information. On the other hand, the fact that the right-hand side of Eq. (28) maps states from $\mathcal{H}^A\otimes\mathcal{H}^B$ to $\mathcal{H}^A\otimes\mathcal{H}^B$ together with Eq. (30) are necessary and sufficient conditions for states inside the time-dependent subspace $U^\dagger(t)\mathcal{H}^A\otimes\mathcal{H}^B$ not to leave this subspace during the evolution. Thus the conditions of the theorem can be thought of as describing a time-varying noiseless subsystem $\mathcal{H}^A(t)$.

B. Unitarily recoverable subsystems

We now extend the above conditions to the case of unitarily recoverable subsystems. As we pointed out earlier, the difference between a unitarily *correctable* and a unitarily *recoverable* subsystem is that in the latter the dimension of the gauge subsystem may increase. Since the dimension of the gauge subsystem is an integer, this increase can happen only in a jumplike fashion at particular moments. Between these moments, the evolution is unitarily correctable. Therefore, we can state the following theorem.

Theorem 3. The subsystem $\mathcal{H}^A$ in Eq. (1) is correctable during the evolution (20), if and only if there exist times t_i, $i=0,1,2,\ldots$, $t_0=0$, $t_i<t_{i+1}$, such that for each interval between t_{i-1} and t_i there exists a decomposition

$$\mathcal{H}^S = \mathcal{H}^A \otimes \mathcal{H}^B_i \oplus \mathcal{K}_i, \quad \mathcal{H}^B_i \ni \mathcal{H}^B_{i-1}, \quad (34)$$

with respect to which the evolution during this interval is unitarily correctable.

Remark. An increase of the gauge subsystem at time t_i happens if the operator $C_j(t)$ in Eq. (28) obtains nonzero components that map states from $\mathcal{H}^B_i$ to $\mathcal{H}^B_{i+1}$. From that moment on, $t_i\leqslant t\leqslant t_{i+1}$, Eq. (28) must hold for the new decomposition $\mathcal{H}^S=\mathcal{H}^A\otimes\mathcal{H}^B_{i+1}\oplus\mathcal{K}_{i+1}$. The unitary $U(t)$ is determined from Eqs. (29) and (30), as described earlier.

The conditions derived in this section provide insights into the mechanism of information preservation under Markovian dynamics, and thus could also have implications for the problem of error correction when perfect recovery is not possible [17,18]. For example, it is conceivable that the unitary operation constructed according to Eq. (25) with the appropriate modification for the case of increasing gauge subsystem may be useful for error correction also when the conditions of the theorems are only approximately satisfied. Notice that the generator driving the effective evolution of the subspace $U^\dagger(t)\mathcal{H}^A\otimes\mathcal{H}^B$, whose projector we denote by $P^{AB}(t)\equiv U^\dagger(t)P^{AB}U(t)$, can be written as

$$\mathcal{L}(t)(\cdot) = - i[H_{\mathrm{eff}}(t), \cdot\,] + \mathcal{D}(t)(\cdot) + \mathcal{S}(t)(\cdot), \quad (35)$$

where

$$H_{\mathrm{eff}}(t) = H(t) + \frac{i}{2}P^{AB}(t)\Big(\sum_j L_j^\dagger(t)L_j(t)\Big) - \frac{i}{2}\Big(\sum_j L_j^\dagger(t)L_j(t)\Big)P^{AB}(t) \quad (36)$$

is an effective Hamiltonian,

$$\mathcal{D}(t)(\cdot) = \sum_j L_j(t)(\cdot)L_j^\dagger(t)$$

is a dissipator, and

$$\mathcal{S}(t)(\cdot) = - \frac{1}{2}P^{AB}(t)\Big(\sum_j L_j^\dagger(t)L_j(t)\Big)P^{AB}(t)(\cdot) - \frac{1}{2}(\cdot)P^{AB}(t)\Big(\sum_j L_j^\dagger(t)L_j(t)\Big)P^{AB}(t) \quad (37)$$

is a superoperator acting on $\mathcal{B}(U^\dagger(t)\mathcal{H}^{AB})$. The dissipator most generally causes an irreversible loss of the information contained in the current subspace, which may involve loss of the information stored in subsystem $\mathcal{H}^A(t)$ as well as an increase of the gauge subsystem. The superoperator $\mathcal{S}(t)(\cdot)$ gives rise to a transformation solely inside the current subspace. In the case when the evolution is correctable, this operator acts locally on the gauge subsystem, but in the general case it may act nontrivially on $\mathcal{H}^A(t)$. The role of the effective Hamiltonian is to rotate the current subspace by an infinitesimal amount. If one could argue that the information lost under the action of $\mathcal{D}(t)$ and $\mathcal{S}(t)$ is in principle irretrievable, then heuristically one could expect that, after a single time step dt, the corresponding factor of the infinitesimally rotated (possibly expanded) subspace will contain the maximal amount of the remaining encoded information. Note that to keep track of the increase of the gauge subsystem one would need to determine the operator C_j on the right-hand side of Eq. (28) that optimally approximates the left-hand side. Of course, since the dissipator generally causes leakage of states outside the current subspace, the error-correcting map at the end would have to involve more than just a unitary recovery followed by a CPTP map on the gauge subsystem. In order to maximize the fidelity [19] of the encoded information with a perfectly encoded state, one would have to bring the state of the system fully inside the subspace $\mathcal{H}^A\otimes\mathcal{H}^B$. These heuristic arguments, however, require a rigorous analysis. It is possible that the action of the superoperators $\mathcal{D}(t)$ and $\mathcal{S}(t)$ may be partially correctable and thus one may have to modify the unitary (25) in order to optimally track the retrievable information. We leave this as a problem for future investigation.

V. CONDITIONS ON THE SYSTEM-ENVIRONMENT HAMILTONIAN

We now derive conditions for correctability of a subsystem when the dynamics of the system and the environment is described by the Schrödinger equation (11). While the CP-map conditions can account for such dynamics when the states of the system and the environment are initially uncorrelated, they depend on the initial state of the environment. Below, we first derive conditions on the system-environment Hamiltonian that hold for any state of the environment, and then extend them to the case when the environment is initialized inside a particular subspace.

We point out that the equivalence between unitary recoverable subsystems and correctable subsystems has been proven for CPTP maps. Here, we could have a non-CP evolution since the initial state of the system and the environment may be entangled. Nevertheless, since correctability must hold for the case when the initial states of the system and the environment are uncorrelated, the conditions we obtain are necessary. They are obviously also sufficient since unitary recoverability implies correctability.

Let us write the system-environment Hamiltonian as

$$H_{\mathrm{SE}}(t) = H_S(t) \otimes I^E + I^S \otimes H_E(t) + H_I(t), \tag{38}$$

where $H_S(t)$ and $H_E(t)$ are the system and the environment Hamiltonians, respectively, and

$$H_I(t) = \sum_j S_j(t) \otimes E_j(t) \tag{39}$$

is the interaction Hamiltonian. From the point of view of the Hilbert space of the system plus environment, the decomposition (1) reads

$$\mathcal{H} = (\mathcal{H}^A \otimes \mathcal{H}^B \oplus \mathcal{K}) \otimes \mathcal{H}^E = \mathcal{H}^A \otimes \mathcal{H}^B \otimes \mathcal{H}^E \oplus \mathcal{K} \otimes \mathcal{H}^E. \tag{40}$$

A. Conditions independent of the state of the environment

We will again consider conditions for unitary correctability first, and then conditions for general correctability.

1. Unitary correctability

In the rotating frame (23), the Schrödinger equation (11) becomes

$$\frac{d\tilde{\rho}(t)}{dt} = - i[\tilde{H}_{\mathrm{SE}}(t) + H'(t), \tilde{\rho}(t)]. \tag{41}$$

Since in this picture a unitarily correctable subsystem is noiseless, we can state the following theorem.

Theorem 4. Consider the evolution (11) driven by the Hamiltonian (38). Let $\tilde{H}_S(t)$ and $\tilde{S}_j(t)$ be the system Hamiltonian and the interaction operators (39) in the rotating frame (23) with $U(t)$ given by Eq. (25). Then the subsystem $\mathcal{H}^A$ in the decomposition (1) is unitarily correctable by $U(t)$ during this evolution, if and only if, for all t,

$$\tilde{S}_j(t) P^{AB} = I^A \otimes C_j^B(t), \quad C_j^B(t) \in \mathcal{B}(\mathcal{H}^B), \quad \forall\ j, \tag{42}$$

and

$$(\tilde{H}_S(t) + H'(t)) P^{AB} = I^A \otimes D^B(t), \quad D^B(t) \in \mathcal{B}(\mathcal{H}^B). \tag{43}$$

Proof. With respect to the evolution in the rotating frame (23), the subsystem $\mathcal{H}^A$ is noiseless. The theorem follows from the conditions for noiseless subsystems under Hamiltonian dynamics [11] applied to the Hamiltonian in the rotating frame. Note that the fact that the operator on the right-hand side of Eq. (43) sends states from $\mathcal{H}^A \otimes \mathcal{H}^B$ to $\mathcal{H}^A \otimes \mathcal{H}^B$ implies that the off-diagonal terms of $\tilde{H}_S(t)+H'(t)$ in the block basis corresponding to the decomposition (1) vanish, i.e., $P^{AB}(\tilde{H}_S(t)+H'(t))P_{\mathcal{K}}=0$. ■

Remark. The Hamiltonian $H'(t)$ can be obtained from conditions (42) and (43). We can choose $D^B(t)=0$ and define $H'(t)=-\tilde{H}_S(t)$, which together with Eq. (25) yields

$$i\frac{dU(t)}{dt} = - U(t) H_S(t), \tag{44}$$

i.e.,

$$U^\dagger(t) = \mathcal{T}\exp\left(- i \int_0^t H_S(\tau) d\tau \right). \tag{45}$$

This simply means that the evolution of the subspace that contains the encoded information is driven by the system Hamiltonian.

The conditions again can be separated into two parts. The fact that the right-hand sides of Eqs. (42) and (43) act trivially on $\mathcal{H}^A$ is necessary and sufficient for the information stored in the instantaneous subsystem $\mathcal{H}^A(t)$ to undergo trivial dynamics at time t. The fact that the right-hand sides of these equations do not take states outside $\mathcal{H}^A \otimes \mathcal{H}^B$ is necessary and sufficient for states not to leave the subspace $U^\dagger(t)\mathcal{H}^A \otimes \mathcal{H}^B$ as it evolves.

2. Unitary recoverability

The conditions for unitary recoverability are not obtained directly from Theorem 4 in analogy to the case of Markovian dynamics. Such conditions would certainly be sufficient, but it turns out that they are not necessary. If after the unitary recovery operation the dimension of the gauge subsystem $\mathcal{H}^{B'}$ is larger than that of the initial gauge subsystem $\mathcal{H}^B$, the state of the gauge subsystem plus environment must belong to a proper subspace of $\mathcal{H}^{B'} \otimes \mathcal{H}^E$ (because the overall evolution is unitary and the dimension of the subspace occupied by the possible states of the system and the environment must be preserved). Thus it is not necessary that the Hamiltonian acts trivially on the factor $\mathcal{H}^A$ in $\mathcal{H}^A \otimes \mathcal{H}^{B'} \otimes \mathcal{H}^E$, but only on the factor $\mathcal{H}^A$ in $\mathcal{H}^A \otimes \tilde{\mathcal{H}}^{\mathrm{BE}}$, where $\tilde{\mathcal{H}}^{\mathrm{BE}}$ is the proper subspace in question.

Example. To illustrate this point, consider the following example. Let $\mathcal{H}^S=\mathcal{H}^1\otimes\mathcal{H}^2$ be the Hilbert space of two qubits with Hilbert spaces $\mathcal{H}^1$ and $\mathcal{H}^2$, respectively. Let the environment consist of a single qubit, i.e., $\dim(\mathcal{H}^E)=2$. We will work in the rotating frame (23) defined through the recovering unitary (25) but will drop the tilde for simplicity of notation and will include the Hamiltonian $H'(t)$ in the definition of the overall Hamiltonian $H_{\mathrm{SE}}(t)$. Let us denote the basis states of each of the qubit systems by $|0\rangle^\sigma$ and $|1\rangle^\sigma$ where the superscript σ labels the qubit ($\sigma=1,2,E$). Consider the encoding (1) with $\mathcal{H}^A=\mathcal{H}^1$ and $\mathcal{H}^B=\mathrm{Span}\{|0\rangle^2\}$. In our basis, the system-environment Hamiltonian is such that it leaves the state of qubit 1 invariant, i.e., its effect on the initial state of the system plus environment is equivalent to a unitary transformation on $\mathcal{H}^2\otimes\mathcal{H}^E$. Since the initial state of the joint system of qubits 2 and E belongs to the two-dimensional subspace $\mathrm{Span}\{|0\rangle^2\}\otimes\mathcal{H}^E$ of $\mathcal{H}^2\otimes\mathcal{H}^E$, the state of these two qubits at any later time must belong to a two-dimensional subspace of $\mathcal{H}^2\otimes\mathcal{H}^E$. Let us imagine that the action of the Hamiltonian up to a given time t_* results in the effective unitary transformation $I^2\otimes|0\rangle\langle 0|^E+X^2\otimes|1\rangle\langle 1|^E$ (here X denotes the σ^X Pauli matrix). Then the state of qubits 2 and E at this moment will belong to the subspace $\widetilde{\mathcal{H}}^{\mathrm{BE}}=\mathrm{Span}\{|0\rangle^2|0\rangle^E,|1\rangle^2|1\rangle^E\}$. If we assume that there is no restriction on the initial state of the environment, the reduced density operator on subsystem 2 at time t_* can have support on the entire Hilbert space $\mathcal{H}^2$. In particular, if the environment is initially in the maximally mixed state $(|0\rangle\langle 0|^E+|1\rangle\langle 1|^E)/2$, the reduced density operator on subsystem 2 at time t_* will also be in the maximally mixed state $(|0\rangle\langle 0|^2+|1\rangle\langle 1|^2)/2$. In other words, the state of qubit 2 is not restricted to any particular proper subspace of $\mathcal{H}^2$ and similarly the state of the environment is not restricted to any proper subspace of $\mathcal{H}^E$. However, we cannot argue that the operators $S_j(t_*)$ of the interaction Hamiltonian (39) must have the form $S_j(t_*)=I^1\otimes C_j^2(t_*)$ [the analog of Eq. (42)] or that the system Hamiltonian in Eq. (38) must have the form $H_S(t_*)=I^1\otimes H^2(t_*)$ [the analog of Eq. (43)]. The reason is that, since the state of the entire system plus environment at time t_* belongs to the subspace $\mathcal{H}^1\otimes\widetilde{\mathcal{H}}^{\mathrm{BE}}$, it is necessary only that $H_{\mathrm{SE}}(t_*)$ acts trivially on qubit 1 when acting on states in this subspace. For example, if the system Hamiltonian is $H_S(t_*)=Z^1\otimes Z^2$ (where Z denotes the σ^Z Pauli matrix) and the interaction Hamiltonian is $H_I(t_*)=Z^1\otimes(|11\rangle\langle 11|^{2E}-|00\rangle\langle 00|^{2E})$, qubit 1 will be effectively acted upon trivially by the Hamiltonian because the effect of $H_S(t_*)\otimes I^E$ on states in the subspace $\mathcal{H}^1\otimes\mathrm{Span}\{|0\rangle^2|0\rangle^E,|1\rangle^2|1\rangle^E\}$ is equivalent to that of $-Z^1\otimes Z^2\otimes I^E$. Note that in the case of unitary correctability, we can derive necessary conditions only in terms of the part of the Hamiltonian that acts on the system because $\mathcal{H}^{B'}=\mathcal{H}^B$ and $\mathcal{H}^{B'}\otimes\mathcal{H}^E$ is fully occupied, so the condition that $\mathcal{H}^A$ in $\mathcal{H}^A\otimes\mathcal{H}^{B'}$ is acted upon trivially must hold for any state of the environment.

We now proceed to formulate general conditions for unitary recoverability under Hamiltonian evolution. Let

$$\mathcal{H}^S=\mathcal{H}^A\otimes\mathcal{H}^{B'}\oplus\mathcal{K}' \tag{46}$$

be a decomposition of the Hilbert space of the system such that the factor $\mathcal{H}^{B'}\supset\mathcal{H}^B$ is such that it has the largest possible dimension, i.e., $\dim(\mathcal{H}^{B'})\equiv d^{B'}$ is the largest integer such that

$$\dim(\mathcal{H}^S)=\dim(\mathcal{H}^A)d^{B'}+d_{\mathcal{K}'}, \tag{47}$$

where $d_{\mathcal{K}'}$ is a non-negative integer.

Since the evolution of the state of the system plus the environment is unitary, at time t the initial subspace $\mathcal{H}^A\otimes\mathcal{H}^B\otimes\mathcal{H}^E$ will be transformed to some other subspace of $\mathcal{H}^S\otimes\mathcal{H}^E$ which is unitarily related to the initial one. Applying the unitary recovery operation $U(t)$ returns this subspace to the form $\mathcal{H}^A\otimes\widetilde{\mathcal{H}}^{\mathrm{BE}}(t)$, where $\widetilde{\mathcal{H}}^{\mathrm{BE}}(t)$ is a subspace of $\mathcal{H}^{B'}\otimes\mathcal{H}^E$. Clearly, there exists a unitary operator $W_0(t):\mathcal{H}^{B'}\otimes\mathcal{H}^E\to\mathcal{H}^{B'}\otimes\mathcal{H}^E$ that maps this subspace to the initial subspace $\mathcal{H}^B\otimes\mathcal{H}^E$:

$$W_0(t)\widetilde{P}^{BE}(t)W_0^\dagger(t)=P^{BE}. \tag{48}$$

[Here $\widetilde{P}^{\mathrm{BE}}(t)$ denotes the projector on $\widetilde{\mathcal{H}}^{\mathrm{BE}}(t)$.] Moreover, since the overall unitary that describes the evolution is a differentiable function of time, if $U(t)$ is chosen as differentiable, $W_0(t)$ can also be chosen differentiable. Note that, as an operator on the entire Hilbert space, this unitary has the form $W_0(t)\equiv I^A\otimes W_0^{B'E}(t)\oplus I_{\mathcal{K}'}\otimes I^E$.

Let us define the frame

$$\hat{O}(t)=W(t)O(t)W^\dagger(t), \tag{49}$$

where

$$i\frac{dW(t)}{dt}=H''(t)W(t). \tag{50}$$

Then the evolution driven by a Hamiltonian $G(t)$, in this frame will be driven by $\hat{G}(t)+H''(t)$.

Theorem 5. Let $\widetilde{O}(t)$ denote the image of an operator $O(t)\in\mathcal{B}(\mathcal{H})$ under the transformation (23) with $U(t)\in\mathcal{B}(\mathcal{H}^S)$ given by Eq. (25) $[H'(t)\in\mathcal{B}(\mathcal{H}^S)]$, and let $\hat{O}(t)$ denote the image of $O(t)$ under the transformation (49) with $W(t)$ given by Eq. (50). Let P^{ABE} be the projector on $\mathcal{H}^A\otimes\mathcal{H}^B\otimes\mathcal{H}^E$. The subsystem H^A in the decomposition (40) is unitarily recoverable by $U(t)$ during the evolution driven by the system-environment Hamiltonian $H_{\mathrm{SE}}(t)$, if and only if there exists $H''(t)\in\mathcal{B}(\mathcal{H}^{B'}\otimes\mathcal{H}^E)$, where $\mathcal{H}^{B'}$ was defined in (46), such that

$$(\hat{\widetilde{H}}_{\mathrm{SE}}(t)+\hat{H}'(t)+H''(t))P^{ABE}=I^A\otimes D^{BE}(t),$$

$$D^{BE}(t)\in\mathcal{B}(\mathcal{H}^B\otimes\mathcal{H}^E),\quad\forall\ t. \tag{51}$$

Proof. Assume that the information encoded in $\mathcal{H}^A$ is unitarily recoverable by $U(t)$. Consider the evolution in the frame defined through the unitary operation $W(t)U(t)$, where $W(t)=W_0(t)$ for some differentiable $W_0(t)$ that satisfies the property (48). In this frame, which can be obtained by consecutively applying the transformations (23) and (49), the Hamiltonian is $\hat{\widetilde{H}}_{\mathrm{SE}}(t)+\hat{H}'(t)+H''(t)$. Under this Hamiltonian, the subsystem $\mathcal{H}^A$ must be noiseless and no states should leave the subspace $\mathcal{H}^A\otimes\mathcal{H}^B\otimes\mathcal{H}^E$. It is straightforward to see that the first requirement means that $\mathcal{H}^A$ must be acted upon trivially by all terms of the Hamiltonian, hence the factor I^A on the right-hand side of Eq. (51). At the same time, the subspace $\mathcal{H}^B \otimes \mathcal{H}^E$ must be preserved by the action of the Hamiltonian, which implies that the factor $D^{\mathrm{BE}}(t)$ on the right-hand side of Eq. (51) must send states from $\mathcal{H}^B \otimes \mathcal{H}^E$ to $\mathcal{H}^B \otimes \mathcal{H}^E$. Note that this implies that the off-diagonal terms of the Hamiltonian in the block form corresponding to the decomposition (40) must vanish, i.e., $P^{ABE}(\hat{\tilde{H}}_{\mathrm{SE}}(t) + \hat{H}'(t) + H''(t))P_{\perp}^{ABE} = 0$, where $P_{\perp}^{ABE}$ denotes the projector on $\mathcal{K} \otimes \mathcal{H}^E$. Obviously, these conditions are also sufficient, since they ensure that in the frame defined by the unitary transformation $W(t)U(t)$, the evolution of $\mathcal{H}^A$ is trivial, and states inside the subspace $\mathcal{H}^B \otimes \mathcal{H}^E$ evolve unitarily under the action of the Hamiltonian $D^{\mathrm{BE}}(t)$. Since $W(t)$ acts on $\mathcal{H}^{B'} \otimes \mathcal{H}^E$, subsystem $\mathcal{H}^A$ is invariant also in the rotating frame (23). This means that $\mathcal{H}^A$ is recoverable by the unitary $U(t)$. ■

Remark. Similarly to the previous cases, the unitary operators $U(t)$ and $W(t)$ can be obtained iteratively from Eq. (51) if the decomposition (1) is given. Since $H''(t)$ acts on $\mathcal{H}^{B'} \otimes \mathcal{H}^E$, from Eq. (51) it follows that the operator $\hat{\tilde{H}}_{\mathrm{SE}}(t) + \hat{H}'(t)$ must satisfy

$$(\hat{\tilde{H}}_{\mathrm{SE}}(t) + \hat{H}'(t))P^{ABE} = I^A \otimes F^{B'E}(t),$$

$$F^{B'E}(t) \in \mathcal{B}(\mathcal{H}^{B'} \otimes \mathcal{H}^E). \tag{52}$$

At the same time, we can choose $H''(t)$ so that $D^{\mathrm{BE}}(t) = 0$. This corresponds to

$$W(t)\tilde{\mathcal{H}}^{BE}(t) = \mathcal{H}^B \otimes \mathcal{H}^E, \tag{53}$$

where $\tilde{\mathcal{H}}^{\mathrm{BE}}(t)$ was defined in the discussion before Theorem 5. To ensure $D^{\mathrm{BE}}(t) = 0$, we can choose

$$H''(t) = -\hat{\tilde{H}}_{\mathrm{SE}}(t) - \hat{H}'(t) + \mathcal{P}_{\perp}^{ABE}(\hat{\tilde{H}}_{\mathrm{SE}}(t) + \hat{H}'(t)), \tag{54}$$

where $\mathcal{P}_{\perp}^{ABE}(\cdot) = P_{\perp}^{ABE}(\cdot)P_{\perp}^{ABE}$. For $t=0$ [$U(0)=I$, $W(0)=I$], we can find a solution for $\hat{H}'(0) = H'(0)$ from Eq. (52), given the Hamiltonian $\hat{\tilde{H}}_{\mathrm{SE}}(0) = H_{\mathrm{SE}}(0)$. Plugging the solution in Eq. (54), we can obtain $H''(0)$. For the unitaries after a single time step dt we then have

$$U(dt) = I - iH'(0)dt + O(dt^2), \tag{55}$$

$$W(dt) = I - iH''(0)dt + O(dt^2). \tag{56}$$

Using $U(dt)$ and $W(dt)$ we can calculate $\hat{\tilde{H}}_{\mathrm{SE}}(dt)$ according to Eqs. (23) and (49). Then we can solve Eq. (52) for $\hat{H}'(dt) = W(dt)H'^{\dagger}(dt)$, which we can use in Eq. (54) to find $H''(dt)$, and so on. Note that here we cannot specify a simple expression for $\hat{H}'(t)$ in terms of $\hat{\tilde{H}}_{\mathrm{SE}}(t)$, since we do not have the freedom to choose fully $F^{B'E}(t)$ in Eq. (52) due to the restriction that $H'(t)$ acts locally on $\mathcal{H}^S$.

We point out that condition (51) again can be understood as consisting of two parts—the fact that the right-hand side acts trivially on $\mathcal{H}^A$ is necessary and sufficient for the instantaneous dynamics undergone by the subsystem $U^{\dagger}(t)W^{\dagger}(t)\mathcal{H}^A$ at time t to be trivial, while the fact that it preserves $\mathcal{H}^A \otimes \mathcal{H}^B \otimes \mathcal{H}^E$ is necessary and sufficient for states not to leave $U^{\dagger}(t)W^{\dagger}(t)\mathcal{H}^A \otimes \mathcal{H}^B \otimes \mathcal{H}^E$ as it evolves.

It is tempting to perform an argument similar to the one we presented for the Markovian case about the possible relation of the specified recovery unitary operation $U(t)$ and the optimal error-correcting map in the case of approximate error correction. If the encoded information is not perfectly preserved, we can construct the unitary operation $U(t)$ as explained in the comment after Theorem 5 by optimally approximating Eqs. (52) and (54). However, in this case the evolution is not irreversible and the information that leaks out of the system may return to it. Thus we cannot argue that the unitary map specified in this manner would optimally track the remaining encoded information.

B. Conditions depending on the initial state of the environment

We can easily extend Theorem 5 to the case when the initial state of the environment belongs to a particular subspace $\mathcal{H}^{E_0} \in \mathcal{H}^E$. The only modification is that, instead of P^{ABE} in Eq. (51), we must have P^{ABE_0}, where P^{ABE_0} is the projector on $\mathcal{H}^A \otimes \mathcal{H}^B \otimes \mathcal{H}^{E_0}$, and on the right-hand side we must have $D^{BE_0}(t) \in \mathcal{B}(\mathcal{H}^B \otimes \mathcal{H}^{E_0})$.

The following two theorems follow by arguments analogous to those for Theorem 5. We assume the same definitions as in Theorem 5 [Eqs. (23), (25), (49), and (50)], except that in the second theorem we restrict the definition of $H''(t)$.

Theorem 6. Let P^{ABE_0} be the projector on $\mathcal{H}^A \otimes \mathcal{H}^B \otimes \mathcal{H}^{E_0}$, where $\mathcal{H}^{E_0} \in \mathcal{H}^E$. The subsystem $\mathcal{H}^A$ in the decomposition (40) is unitarily recoverable by $U(t) \in \mathcal{B}(\mathcal{H}^S)$ during the evolution driven by the system-environment Hamiltonian $H_{\mathrm{SE}}(t)$ when the state of the environment is initialized inside $\mathcal{H}^{E_0}$, if and only if there exists $H''(t) \in \mathcal{B}(\mathcal{H}^{B'} \otimes \mathcal{H}^E)$ such that

$$(\hat{\tilde{H}}_{\mathrm{SE}}(t) + \hat{H}'(t) + H''(t))P^{ABE_0} = I^A \otimes D^{BE_0}(t),$$

$$D^{BE_0}(t) \in \mathcal{B}(\mathcal{H}^B \otimes \mathcal{H}^{E_0}), \quad \forall \ t. \tag{57}$$

The conditions for unitary correctability in this case require the additional restriction that $W(t)$ acts on $\mathcal{H}^B \otimes \mathcal{H}^E$ and not on $\mathcal{H}^{B'} \otimes \mathcal{H}^E$, since in this case $U(t)$ brings the state inside $\mathcal{H}^A \otimes \mathcal{H}^B \otimes \mathcal{H}^E$. Notice that when the state of the environment is initialized in a particular subspace, we cannot use conditions for unitary correctability similar to those in Theorem 4. This is because, after the correction $U(t)$, the state of the gauge subsystem plus environment may belong to a proper subspace of $\mathcal{H}^B \otimes \mathcal{H}^E$, and tracing out the environment would not yield necessary conditions.

Theorem 7. Let P^{ABE_0} be the projector on $\mathcal{H}^A \otimes \mathcal{H}^B \otimes \mathcal{H}^{E_0}$, where $\mathcal{H}^{E_0} \in \mathcal{H}^E$. The subsystem $\mathcal{H}^A$ in the decomposition (40) is unitarily correctable by $U(t) \in B(\mathcal{H}^S)$ during the evolution driven by the system-environment Hamiltonian $H_{\mathrm{SE}}(t)$ when the state of the environment is initialized inside $\mathcal{H}^{E_0}$, if and only if there exists $H''(t) \in \mathcal{B}(\mathcal{H}^B \otimes \mathcal{H}^E)$ such that

$$(\hat{\hat{H}}_{\rm SE}(t) + \hat{H}'(t) + H''(t))P^{ABE_0} = I^A \otimes D^{BE_0}(t),$$

$$D^{BE_0}(t) \in \mathcal{B}(\mathcal{H}^B \otimes \mathcal{H}^{E_0}), \quad \forall \ t. \tag{58}$$

Notice that the conditions of Theorems 6 and 7 do not depend on the particular initial state of the environment but only on the subspace to which it belongs. This can be understood by noticing that different environment states inside the same subspace give rise to Kraus operators (14) which are linear combinations of each other. The discretization of errors in operator quantum error correction [1] implies that all such maps will be correctable.

The conditions for correctable dynamics dependent on the state of the environment could be useful if we are able to prepare the state of the environment in the necessary subspace. The environment, however, is generally outside the experimenter's control. Nevertheless, it is conceivable that the experimenter may have some control over the environment (for example, by varying its temperature), which for certain Hamiltonians could bring the environment state close to a subspace for which the evolution of the system is correctable. It is important to point out that, according to a result derived in Ref. [19], the error due to imperfect initialization of the environment will not increase under the evolution.

VI. CORRECTABILITY AT A PARTICULAR MOMENT OF TIME

So far, we have looked at the conditions under which the encoded information is preserved during an entire time interval following the encoding. As pointed out earlier, this is not the most general form of correctability that can occur. It is possible that the encoded information is lost for a while but is later retrieved. This clearly cannot happen in the case of Markovian dynamics because Markovian dynamics is irreversible. However, it can occur in the case of Hamiltonian dynamics where the information can flow out to the environment and later return to the system. In this section, we discuss the conditions on the system-environment Hamiltonian for this most general type of correctability.

Let the unitary transformation generated by the action of the system-environment Hamiltonian from time $t=0$ to time $t=T$ be

$$V_{\rm SE}(T) = \mathcal{T}\exp\left(-i\int_0^T H_{\rm SE}(t)dt\right). \tag{59}$$

The following theorem follows directly from the definition of unitary recoverability.

Theorem 8. Let P^{ABE_0} denote the projector on $\mathcal{H}^A \otimes \mathcal{H}^B \otimes \mathcal{H}^{E_0}$, where $\mathcal{H}^{E_0} \in \mathcal{H}^E$. Let $\mathcal{H}^{B'}$ be defined as in Eq. (46). The subsystem H^A in the decomposition (40) is unitarily recoverable by $U=U^S \otimes I^E$ at time $t=T$ under the evolution driven by the system-environment Hamiltonian $H_{\rm SE}(t)$ when the state of the environment is initialized inside $\mathcal{H}^{E_0}$, if and only if

$$UV_{\rm SE}(T)P^{ABE_0} = I^A \otimes C^{BE_0 \to B'E},$$

$$C^{BE_0 \to B'E}: \mathcal{H}^B \otimes \mathcal{H}^{E_0} \to \mathcal{H}^{B'} \otimes \mathcal{H}^E. \tag{60}$$

The theorems for the case of unitary correctability or correctability independent of the state of the environment can be obtained from Theorem 8 by substituting $\mathcal{H}^{B'}=\mathcal{H}^B$ and $\mathcal{H}^{E_0}=\mathcal{H}^E$, respectively.

Theorem 8 can be regarded as a generalization of Theorem 1, which concerns the process that leads to a particular transformation at a given moment, rather than the transformation itself. More specifically, condition (60) is equivalent to the condition that all possible CPTP maps obtained through Eq. (14), for the different possible initial density matrices of the environment with support on $\mathcal{H}^{E_0}$, satisfy Eq. (15). This equivalence can be obtained by sandwiching both sides of Eq. (60) between all pairs of vectors $|\mu\rangle$ and $|\nu\rangle$ from an orthonormal basis which spans $\mathcal{H}^E$ and a subset of which spans $\mathcal{H}^{E_0}$.

Note that Theorem 8 imposes conditions on the Hamiltonian $H_{\rm SE}(t)$ only indirectly—through a condition on the resulting unitary (59). At first sight this may not seem too different from the situation we had before for the case of continuous correctability, because the conditions in that case (e.g., Theorem 6) could be regarded as equivalent to the requirement that Theorem 8 holds at every moment of time. But precisely because in that case condition (60) was imposed for all times, we obtained nontrivial conditions on the Hamiltonian for all times. Those nontrivial conditions ensured that, at every moment of time, the Hamiltonian does not take the information of interest outside the system.

In this case, the only restriction on the resulting unitary is the global requirement that at time T the unitary $V_{\rm SE}(T)$ satisfies Eq. (60). But up to any moment t_0, $0<t_0<T$, the unitary $V_{\rm SE}(t)$ can be completely arbitrary because it can always become of the form that satisfies Theorem 8 during the interval between t_0 and T. Therefore, if we write conditions on the Hamiltonian similar to those for continuous correctability, up to any moment $t_0<T$ these conditions will be trivial. The only nontrivial condition has a global character and it is expressed through the condition on $V_{\rm SE}(T)$ as given by Theorem 8.

VII. CONCLUSION

We have derived conditions for correctability of subsystems under continuous dynamics. We first presented conditions for the case when the evolution can be described by a CPTP linear map. These conditions are equivalent to those known for operator codes [1] except that we consider them for time-dependent noise processes. We then derived conditions for continuous correctability for the case of Markovian dynamics and general Hamiltonian dynamics of the system and the environment. We derived conditions for both unitary correctability and general correctability, using the fact that correctable subsystems are unitarily recoverable [12]. For the case of Hamiltonian dynamics, we also considered conditions for correctability at only a particular moment of time.

The conditions for continuous correctability under both Markovian and Hamiltonian evolution can be understood as consisting of two parts—the first is necessary and sufficient for the absence of noise inside the instantaneous subsystem that contains the information, and the second is necessary and sufficient for states not to leave the subsystem as it evolves with time. In this sense, the new conditions can be thought of as generalizations of the conditions for noiseless subsystems to the case where the subsystem is time dependent.

In the Hamiltonian case, the conditions for continuous unitary correctability concern only the action of the Hamiltonian on the system, whereas the conditions for continuous unitary recoverability concern the entire system-environment Hamiltonian. The reason for this is that the state of the gauge subsystem plus the environment generally belongs to a particular subspace, which does not factor into a sector belonging to the system and a sector equal to the Hilbert space of the environment.

We also derived conditions in the Hamiltonian case that depend on the initial state of the environment. These conditions could be useful, in principle, since errors due to imperfect initialization of the environment do not increase under the evolution. Furthermore, these conditions could provide a better understanding of correctability under CPTP maps, since a CPTP map that results from Hamiltonian evolution depends on both the Hamiltonian and the initial state of the environment.

Finally, we discussed the conditions for correctability at only a particular moment of time. This most general form of correctability can occur in the case of non-Markovian dynamics where the information can flow out to the environment but later return to the system. We showed that the conditions on the generator of evolution in this case amount to a condition on the overall transformation and do not provide nontrivial information about the time-local properties of the dynamics.

We also discussed possible implications of the conditions for continuous correctability for the problem of optimal recovery in the case of imperfectly preserved information. We hope that the results obtained in this paper will provide insight into the mechanisms of information flow under decoherence that could be useful in the area of approximate error correction as well.

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