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Overview of quantum error prevention and leakage elimination

MARK S. BYRD†*, LIAN-AO WU‡ and DANIEL A. LIDAR‡

†Physics Department, Southern Illinois University, Carbondale, IL 62901-4401, USA

‡Chemical Physics Theory Group, Chemistry Department and Center for Quantum Information and Quantum Control, University of Toronto, 80 St. George Street, Toronto, Ontario M5S 3H6, Canada

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Abstract. Quantum error prevention strategies will be required to produce a scalable quantum computing device and are of central importance in this regard. Progress in this area has been quite rapid in the past few years. In order to provide an overview of the achievements in this area, we discuss the three major classes of error prevention strategies, the abilities of these methods and the shortcomings. We then discuss the combinations of these strategies which have recently been proposed in the literature. Finally, we present recent results in reducing errors on encoded subspaces using decoupling controls. We show how to generally remove mixing of an encoded subspace with external states (termed leakage errors) using decoupling controls. Such controls are known as 'leakage elimination operations' or 'LEOs'.

1. Introduction

Preventing errors in quantum information is an important part of quantum information theory and a central goal in quantum computing. Since efficient algorithms make use of many particle quantum states which are very fragile, this will be a key component of any working quantum computing device.

An idealistic goal would be the noiseless evolution of the quantum system. (We will take 'noise' to mean both undesirable unitary evolution and decoherence in a quantum system throughout this article and will specify if and when the need arises.) However, it is clear that no system is noiseless, since it will always interact with an environment and we cannot implement any operation perfectly. Thus after isolating a system to the best of our ability, we should aim for the realistic goals of the identification and correction of errors when they occur and/or avoiding noises when possible and/or suppressing noise in the system. To each of these tasks there corresponds an error prevention strategy developed for the specific purpose; quantum error correcting codes (QECCs), decoherence-free or noiseless subsystems (DFSs) and 'bang–bang' (BB) decoupling controls. All three of these classes of error prevention have limitations. Therefore, the choice of error prevention protocol depends on the system. Hybrid strategies appear to be required for

*Author for correspondence. e-mail: mbyrd@physics.siu.edu

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near-future experiments in which controllable qubits are available in a limited supply.

In the first part of this article we give an overview/review of the three error prevention strategies. This is designed to be a resource for novices as well as experts which conveys the ideas, objectives and several key references for quantum error prevention schemes. This includes an introduction to, and the range of applicability of, the three quantum error prevention classes. We then review some proposals for the combinations of these methods. Finally, we discuss recent results concerning the elimination of errors which serve to destroy the effectiveness of encoded qubits.

2. Quantum error correction strategies

2.1. *Quantum error correcting codes*

Very generally, we may describe a quantum error correcting code as a set of states which can be used to store information in a way that errors are able to be detected and corrected during a quantum information processing task. As with classical error correcting codes, the code is a repetition or redundancy code where information is stored in a state within a subspace. Errors are correctable as long as they map orthogonal states to orthogonal states. The first to show how to implement a quantum error correcting code (QECC) were Shor [1] and Steane [2], who proved the in-principle possibility of correcting errors in quantum computing devices. Shor's code uses nine physical qubits to encode one logical qubit and thus stores one qubit of information reliably. It protects the logical qubit against single independent errors on the physical qubits and is denoted [9, 1, 3]. The first entry is the number of physical qubits, the second the number of logical qubits and the third is the distance. ($d = 2t + 1$, where t is the number of errors which the code can protect against.) Subsequently, several methods were discussed for the construction of $[n, k, d]$ codes [2–5]. (A lucid account of the precise requirements is given in [6].)

Several authors investigated quantum error correcting codes, showing that there are large classes of such codes. Two especially important classes are the CSS codes (Calderbank and Shor [3] and Steane [2]) and their generalization, the stabilizer codes [4, 5]. In addition to the descriptions of the classes of codes which protect against different types of errors, a bound was obtained, called the quantum Hamming bound, which describes the smallest set of states needed to protect against a given set of errors and defines efficient codes [6, 7]. This sets the limit of five for the number of physical qubits needed to protect one logical qubit against arbitrary single independent errors on the physical qubits. When errors are not independent or when gating errors are present, the number of physical qubits required to encode one logical qubit grows dramatically. In addition, storage and gating errors must be below a certain threshold for this scheme to work reliably. (See [8] and references therein for the threshold as well as fault-tolerant recovery requirements.) These constraints imply QECCs are the least qubit intensive when gating errors are low and the errors are truly independent.

In the near future (perhaps before 2010), we expect to have fewer than 50 physical qubits available in quantum computing experiments. Therefore, physical qubits will be a scarce resource. An encoding into a QECC would demand that 50 logical qubits are reduced to, at most, ten (neglecting ancilla qubits which are used for fault-tolerant recovery). Ten qubits can be created for use in NMR experiments at this time and proof-of-principle experiments have already been performed to exemplify the use of QECCs [9]. We therefore seek error prevention methods which provide a higher ratio of the number of logical qubits to physical qubits in order to investigate a wider range of scaling issues and algorithms. This is, in large part, the motivation for hybrid error prevention techniques, discussed below, which are sought for use in experiments which will be performed in the next ten years.

2.2. *Decoherence-free subspaces and (noiseless) subsystems*

A decoherence-free subspace and its generalization, a noiseless or decoherence-free subsystem (DFS), is a state or set of states which is not vulnerable to decoherence [10–15]. (For a recent review, see [16].) In this case, one takes advantage of a symmetry in a system–bath interaction in order to store information in a DFS which is invariant under the action of the interaction Hamiltonian. Under appropriate circumstances, one can expect such a symmetry to exist. However, identifying a useful symmetry and taking advantage of it can be very difficult. One must (1) identify the symmetry, (2) find the states which are invariant to the interaction and (3) construct, if possible, operations on the system which will serve as a universal set of gating operations *while* preserving the necessary symmetries. Although this may seem a daunting task, DFSs have been found which satisfy all of these requirements.

DFSs have shown promise in several experiments and have been observed to reduce noise in others [17–20], including computation in a DFS [21–22]. DFSs have also led to the concept of encoded universality (finding subspaces in which universal quantum computing can be performed even when it is not possible to perform universal computing on the whole space) [14, 23–32].

The simplest example of a DFS which shows promise for several experiments is a code which uses two physical qubits to encode one logical qubit and protects against collective phase errors [10–15]. The logical zero for this code is given by $|0_L\rangle = |01\rangle$ and the logical one state is $|1_L\rangle = |10\rangle$. It is clear that when the operation of a collective error, $S_z = a(\sigma_z \otimes \mathbb{1} + \mathbb{1} \otimes \sigma_z)$ (a is any constant), acts on this code, whether its origin is a unitary evolution or a collective error, the code is unaffected. (S_z gives zero when acting on the encoded qubit.) This code can also be shown to enable universal quantum computing using only the exchange interaction [26, 28, 33].

For near-future experiments, DFSs have advantages over quantum error correcting codes since the number of physical qubits required to encode one logical qubit is typically lower, and they do not require repeated identification and correction of errors. Once the qubits are encoded, they evolve noiselessly. The disadvantage is the difficulty in identifying the symmetry and exploiting it. (We may also note that DFSs are not inherently robust against all types of gating errors. If gates are chosen properly, the gates do not take the system out of the DFS. If ‘over- or under-rotation’ occurs, we may choose to concatenate a DFS with a QECC, as discussed in section 3.) Even when a symmetry cannot be found, in some cases it can be actively generated, a procedure known as encoded decoupling [28, 29, 33–38].

2.3. *Dynamical decoupling*

Bang–bang (BB) decoupling operations can be traced back to the decoupling operations used in NMR experiments [39, 40]. In the simplest case, one lets the Hamiltonian evolve for a time t, then changes the open-system evolution by acting only on the system, and lets it evolve for a second time t'. This produces an effective evolution after a total time $t + t'$. If the time evolution (the Hamiltonian) can be inverted for a time $t' = t$, then the two evolutions will cancel, producing zero net evolution after time $2t$. They may therefore be used to eliminate Hamiltonian evolutions.

The first uses for the purpose of general noise reduction in quantum computing systems are found in [41, 42]. These showed that, within a spin-boson model, strong, fast operations can be used to eliminate the interaction with the environment which causes dephasing of a qubit. However, BB can be viewed more generally as a symmetrization, or averaging, technique which is more general than simply inverting the time evolution directly [43–46]. Several extensions of this method have been given, including conditions for computing in the presence of the decoupling controls [47] and for using empirical data to determine an appropriate set when computing or not [35].

The motivation for studying this technique more thoroughly is clear: *dynamical decoupling controls do not require extra qubits* for the reduction/elimination of noise and decoherence in the quantum system. This is a major advantage since, as stated above, physical qubits will be a scarce resource in near-future experiments. The limitation of dynamical decoupling is that the method assumes that the control operations are strong and fast [35, 41–45, 47–55]. In fact, there is a strong connection to the quantum Zeno effect (QZE) [56]. Systems certainly exist for which neither of these assumptions is difficult to satisfy and the strong assumption is less stringent [57]. However, the fast assumption can be quite difficult to satisfy, and, if it is not satisfied, decoherence can be accelerated rather than suppressed, as in the inverse QZE [56]. Roughly speaking, we require a complete set of control operations to be implemented within the correlation time of the bath [41, 44]. This is due to the fact that one aims to eliminate the system–bath interaction before the information is irretrievably lost, quite opposite of a Markovian assumption. There is a notable exception, however; in the case of $1/f$ noise, BB has been shown not to crucially depend on the high-frequency bath cut-off (which is the inverse of the correlation time of the bath) [58, 59]. This implies that the fast requirement is not difficult to satisfy in some important cases including trapped ions [37]. The strength requirement can also be relaxed if certain other conditions apply [60].

Soon after the initial research into BB, several authors sought to combine BB with DFSs [33–35, 43, 46, 50]. This is, in part, the subject of the next section: combining error prevention techniques.

3. Combining methods of error prevention

Given the three different error prevention methods described above, several possibilities for combinations exist. To be specific, we could combine

(1) DFSs and QECCs,
(2) QECCs and BB,
(3) DFSs and BB,
(4) QECCs, DFSs and BB.

At this time, all of these combinations have been explored to varying degrees in the literature. The first combination, DFSs and QECCs, is described in [61, 62]. In the case of [61], a perturbative independent error on a DFS structure may be detected and corrected. In the case of [62], computation takes the encoded information out of and then into the DFS, and errors that occur along this trajectory may be corrected by a compatible QECC. The subject was also studied for 'detected-jump error correcting' (detecting a spontaneously emitted excitation) in [63–66]. The idea is to let the DFS encoding protect against the non-unitary, conditional evolution that arises in the quantum trajectories picture, and let the QECC correct the errors that arise during quantum jumps in the same picture. The second combination, QECCs and BB, can be achieved in at least two ways. First, one can use BB on each physical qubit in the code to reduce noise on that particular independent qubit. One could also use BB on the logical qubits using the stabilizer formalism to determine the appropriate decoupling sequence [33, 67]. A third possibility is to use BB to suppress the conditional evolution and use QECC to correct quantum jumps, in the quantum trajectories picture [68]. This combination is interesting in the sense that it uses a minimal QECC ($n+1$ physical qubits per n logical qubits) and applies BB in the Markovian regime (though the trajectories themselves are non-Markovian). The third combination, DFSs and BB, has been the most thoroughly explored. There are several different motivations for this, but the primary objective is to produce an effective evolution which is compatible with a DFS [33–37, 43, 46, 50, 67, 69]. In all of these scenarios, the demands on the physical system have been drastically reduced by requiring only that the Hamiltonian be *modified* by BB in order to produce an interaction Hamiltonian which is compatible with an encoding method. This is in contrast to the original decoupling proposals which required that the interaction Hamiltonian be *eliminated*.

In principle, BB can be combined with any encoding [33, 35, 43, 69]. The codewords could then be DFS codewords, QECC codewords or any combination thereof. However, more specific results exist for the combination of all three methods in order to actively produce the conditions for a DFS and for correction of the departure from the symmetry required for a DFS using QECC techniques [34, 62]. In the next few sections we discuss methods of eliminating errors on a predefined qubit (encoded or physical) using BB.

4. Eliminating leakage

We now discuss a specific combination of encoding and BB methods called 'leakage elimination'. We begin by reviewing previous results, then presenting new results and finally we provide examples of physical systems where such techniques are useful.

An ideal qubit is a two-level system consisting of a pair of orthonormal quantum states. However, this idealization neglects other levels which are typically present and can mix with those defining the qubit. This mixing is what we will refer to as 'leakage'. Leakage may be the result of the application of logical operations, or induced by system–bath coupling. In the former case, a rather general solution was proposed in [70]. In the next few sections we will be interested in decoherence-induced leakage. The logical qubits of codes, as well as physical qubits, can undergo leakage errors, which are particularly serious: by mixing states from within the code and outside the code space, leakage completely invalidates the encoding. A simple procedure to detect and correct leakage, which can be incorporated into a fault-tolerant QECC circuit, was given in [71]. This scheme is, however, not necessarily compatible with all encodings [30]. Here we present a *universal* BB solution to leakage elimination in the limit of fast and strong 'bang–bang' (BB) pulses [41, 44, 49, 72].

4.1. *Leakage elimination operators*

Suppose that several multilevel systems are used to encode 2^K logical states representing K qubits (with the appropriate tensor product structure). Let us arrange the basis states $\{|k\rangle\}_{k=0}^{N}$, $N = 2^K$ of H_N so that $|0\rangle_i$ and $|1\rangle_i$ $(i \in K)$ represent the physical or encoded (logical) qubit states. The code subspace will be denoted C and its orthogonal complement $C^{\perp}$. For example, C could consist of the ground and first excited states of a quantum dot or atom representing $|0\rangle$ and $|1\rangle$ (the other states in the dot/atom would be $C^{\perp}$) or they could be $|0_L\rangle$ and $|1_L\rangle$ of a DFS or QECC.

We can classify all system operators as follows:

$$E = \begin{pmatrix} \sigma_L & 0 \\ 0 & 0 \end{pmatrix}, \qquad E^{\perp} = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_L^{\perp} \end{pmatrix}, \qquad L = \begin{pmatrix} 0 & D \\ F & 0 \end{pmatrix}, \tag{1}$$

where σ_L and $\sigma_L^{\perp}$ correspond to operations on the logical states and the orthogonal subspace, respectively. Operators of type E produce logical operations, i.e. they act entirely within the code subspace. These could be unitary evolutions in the space or non-unitary errors which arise in the logical subspace due to system–bath coupling. $E^{\perp}$ operators, on the other hand, have no effect on the code as they act entirely outside the qubit subspace. Finally, L represents the leakage operators, with D, F off-diagonal blocks which have the effect of creating superpositions between states within a code and outside of the code subspace. These algebraic elements correspond to the leakage from, or to, the logically encoded subspace.

Let us now recall the 'parity-kick' scheme [41, 49], which is a special case of BB. The total system–bath Hamiltonian can be written as $H_{\mathrm{SB}} = H_C + H_{\perp} + H_L$, where H_C $(H^{\perp}, H_L)$ is a linear combination of elements of the set E $(E^{\perp}, L)$ tensored with bath operators. Now consider a *leakage-elimination operator* (LEO)

$$R_L = \mathrm{e}^{i\phi}\begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}, \tag{2}$$

where the blocks have the same dimensions as in equation (1) and $\exp(i\phi)$ is an overall phase factor. This operator anticommutes with the leakage operators: $\{R_L, L\} = 0$, while $[R_L, E] = [R_L, E^{\perp}] = 0$. It is an LEO since it follows that the following (parity-kick) sequence eliminates the leakage errors:

$$\lim_{n\to\infty} (\mathrm{e}^{-iH_{\mathrm{SB}}t/n} R_L^{\dagger} \mathrm{e}^{-iH_{\mathrm{SB}}t/n} R_L)^n = \mathrm{e}^{-iH_C t} \mathrm{e}^{-iH^{\perp} t}. \tag{3}$$

To physically implement this, in practice one takes $n = 1$ and makes the time t very small compared to the bath correlation time [41, 49]. Equation (3) then holds to order t^2, and implies that one intersperses periods of free evolution for time t with R_L, $R_L^{\dagger}$ pulses which are so strong that H_{SB} is negligible during the BB pulses. The term $\mathrm{e}^{-iH^{\perp}t}$ in equation (3) has no effect on the qubit subspace. The term $\mathrm{e}^{-iH_C t}$ may result in logical errors, which will have to be treated by other methods, e.g., concatenation with a QECC [61, 71, 73], or additional pulses [33, 44, 45]. Therefore, in order to eliminate *leakage*, we seek an LEO for a given encoding, which is obtainable from a controllable system Hamiltonian H_S acting for a time τ, i.e. $R_L = \exp(-iH_S\tau)$.

4.2. *Generalized LEO*

The leakage operator given in [36] was termed canonical if the corresponding Hamiltonian was also a projection operator onto the code space C. The physically available logical operations may or may not be canonical in this sense. Here we show that we may relax this restriction and that one may obtain an LEO that need not also be a projective operation. In section 4.4 we will give an explicit example of such an operator.

In [36, 43] it was shown that an LEO R_L may be obtained through the exponentiation of a Hamiltonian in the following form:

$$R_L = \exp(\pm i\pi\sigma_L P), \tag{4}$$

where P is a projection and σ_L any operation such that $\sigma_L = \sigma_L^\dagger$ and $\sigma_L^2 = 1$, e.g. a logical operation. Note that the logical operations very often are already projective in the sense that they operate only on the code space. Examples will be given below.

However, not all LEOs have such a form and we now give a more general characterization of an LEO. Let the Hamiltonian for an LEO be given by

$$H = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix}, \tag{5}$$

where H_1 acts on the code subspace and H_2 on the orthogonal complement. If H_1 is diagonal with even (odd) integers as the diagonal elements and H_2 is diagonal with odd (even) integers as the diagonal elements, then one may write the LEO as

$$R_L = U\exp(-i\pi H)U^\dagger, \tag{6}$$

where $U = U_1 \oplus U_2$ is a direct sum (block diagonal). In this case, H is not projective since it may have non-zero eigenvalues when acting on the subspace orthogonal to the code. The effective LEO, however, is unchanged, i.e. the form equation (2) is obtained, which again produces and thus eliminates leakage errors as desired. Such is the case for the four-qubit DFS example in section 4.4.

We should note at this point that this can immediately be generalized to an arbitrary number of qubits (physical or logical qubits) [36]. We will review several physical examples of leakage elimination from [36] in the next section.

4.3. *Examples of leakage elimination in physical systems*

4.3.1. *Example 1.* As a simple first example, consider physical qubits (without encoding), such as electrons on liquid helium [74], or an electron-spin qubit in quantum dots [75–78], or a nuclear-spin qubit in donor atoms in silicon [79–80]. In those cases, a potential well at each site traps one fermion. Usually, the ground and first excited state are taken as a qubit for a given site: $|k\rangle = c_k^\dagger|\text{vac}\rangle$, where $c_k^\dagger$ is a fermionic creation operator for level $k = 0, 1$. Let $n_k = c_k^\dagger c_k$ be the fermion number operator. The logical operations for this qubit are $E = \{X = c_0^\dagger c_1 + c_0^\dagger c_1$, $Y = i(c_1^\dagger c_0 - c_0^\dagger c_1)$, $Z = n_0 - n_1\}$, whose elements satisfy $su(2)$ commutation relations. In this case, a general linear Hamiltonian which includes hopping terms, $H_{\mathrm{SB}} = \sum_{k,l=0}^{N-1} a_{kl} c_k^\dagger c_l$, where a_{kl} includes parameters and bath operators, and k, l denote all electron states, can leak the qubit states $k = 0, 1$ into any of the other states. Using parity-kicks, we can eliminate this leakage in terms of the LEO: $R_L = \exp[\pm i\pi(n_0 + n_1)]$. This LEO is implemented simply by controlling on-site energies.

4.3.2. *Example 2.* We can also treat bosonic systems, such as the linear optics quantum computing (QC) proposal [81]. In this case, a qubit is encoded into two modes. The first qubit has states $|0\rangle_1 = b_1^\dagger|\mathrm{vac}\rangle$ and $|1\rangle_1 = b_2^\dagger|\mathrm{vac}\rangle$, and the second qubit is $|0\rangle_2 = b_3^\dagger|\ \mathrm{vac}\rangle$ and $|1\rangle_2 = b_4^\dagger|\mathrm{vac}\rangle$, where $b_i^\dagger$ are bosonic creation operators. Encoded two-qubit states are $|00\rangle = b_1^\dagger b_3^\dagger|\mathrm{vac}\rangle$, $|01\rangle = b_1^\dagger b_4^\dagger|\mathrm{vac}\rangle$, $|10\rangle = b_2^\dagger b_3^\dagger|\mathrm{vac}\rangle$ and $|11\rangle = b_2^\dagger b_4^\dagger|\mathrm{vac}\rangle$. But the linear optics Hamiltonian $H = \sum_{k,l=1}^{4} a_{kl} b_k^\dagger b_l$ contains beam-splitter terms like $b_1^\dagger b_3$ and $b_2^\dagger b_3$, which can cause leakage into states such as $b_1^\dagger b_2^\dagger|\mathrm{vac}\rangle$ or $b_1^{\dagger 2}|\mathrm{vac}\rangle$. By using the LEO $R_L = \exp[\pm i\pi(b_1^\dagger b_1 + b_2^\dagger b_2)]$, we can eliminate the leakage terms. This LEO can be implemented simply using a phase shifter. However, generalizing this LEO to multiple encoded qubits requires a photon–photon interaction, which is not readily available.

4.3.3. *Example 3.* A substantial number of promising solid-state QC proposals, e.g. [74–80, 82], are governed by effective isotropic and anisotropic exchange interactions, which, quite generally, can be written as

$$H_{\mathrm{ex}} = \sum_{i<j} J_{ij}^x X_i X_j + J_{ij}^y Y_i Y_j + J_{ij}^z Z_i Z_j, \tag{7}$$

where X_i is the Pauli σ_x matrix on the ith qubit, etc. The encoding $|0\rangle_L = |01\rangle$, $|1\rangle_L = |10\rangle$ (using two physical qubits per logical qubit) is highly compatible with H_{ex} in the sense that universal QC can be performed by controlling the *single* parameter J_{ij}^x in the Heisenberg ($J_{ij}^x = J_{ij}^y = J_{ij}^z$), XXZ ($J_{ij}^x = \pm J_{ij}^y \neq J_{ij}^z$), and XY ($J_{ij}^x = J_{ij}^y$, $J_{ij}^z = 0$) instances of H_{ex}, provided there is a Zeeman splitting that distinguishes single-qubit Z_i terms. This is done using the 'encoded selective recoupling' method [29]. Furthermore, the $\{|01\rangle, |10\rangle\}$ encoding is a DFS for collective dephasing (where the bath couples only to system $Z_{2i-1} + Z_{2i}$ operators) [11, 14, 61]. A set of logical operations on the code is $E = \{\overline{X}_1 = (X_1X_2 + Y_1Y_2)/2,\ \overline{Y}_1 = (X_2Y_1 - Y_2X_1)/2,\ \overline{Z}_1 = (Z_1 - Z_2)/2\}$. Only the $\overline{X}_1$ term is assumed to be directly controllable (by manipulation of J_{12}^x), whereas the $\overline{Z}_1$ term can be turned on/off using recoupling [29]. The $\overline{Y}_1$ term can then be reached in a few steps:

$$\mathrm{e}^{-i\theta\overline{Y}_1} = \mathrm{e}^{i(\pi/4)\overline{X}_1}\mathrm{e}^{-i\theta\overline{Z}_1}\mathrm{e}^{-i(\pi/4)\overline{X}_1}.$$

The leakage errors are due to system–bath interactions where the system terms include any of X_i, Y_j, X_iZ_j and Y_iZ_j, since, as is easily seen, such terms do not preserve the $\{|01\rangle, |10\rangle\}$ code subspace. As pointed out first in [33], the LEO can be expressed as $R_L = \exp(i\pi\overline{X}_1) = Z_1Z_2$, which means that it is implementable using just the controllable J_{12}^x parameter in the instances of H_{ex} mentioned above. This form for R_L is an instance of equation (4). Note that, in agreement with our general comments above, R_L commutes with every element of E, meaning that logical operations can be performed on the encoded subspace *while eliminating leakage*.

4.3.4. *Example 4*. The most economical encoding of a DFS against collective decoherence requires three physical qubits per logical qubit. In the tensor product of three qubits, there exists two doublets and a quadruplet. In this case the logical qubit states are stored in the two doublet states which represent the logical zero and one. Under the action of collective errors, the two doublets mix within their respective subspaces, but not with each other. The logical operations are formed from the Heisenberg exchange interaction, which is known to be universal for this code [14]. This example is given in detail in [36, 83], where it is shown that it is possible to construct efficient, canonical LEOs by using only the Heisenberg exchange interaction. This follows, in principle, from the theorem in [33] and can be done in practice using the constructions in [36, 83].

4.4. *Leakage elimination on the 4-qubit DFS*

The 4-qubit DFS [10] contains two singlet states for the representative qubit, three triplets and a spin 2, or quintuplet. The singlet states represent the logical zero and one of the DFS encoded qubit. It was shown in [14, 23] that the exchange operation between the first and second qubits in the computational basis will provide a logical Z operation which we denote $\bar{Z}$. However, on further analysis, we find that we cannot create a 'canonical' LEO in the sense described in [36]. In this case, we require the more general characterization of the LEO given above.

First we give a LEO that is appropriate in the Kempe *et al*. [14] basis. Let the (square of the) total angular momentum operator be denoted $\mathbf{S}^2$ with eigenvalue $S(S+1)$. Then

$$4\mathbf{S}^2 = \left(\sum_i \boldsymbol{\sigma}_i\right)^2,$$

where $\boldsymbol{\sigma}_i = (\sigma_i^x, \sigma_i^y, \sigma_i^z)$ are the Pauli matrices acting on the ith qubit. Therefore,

$$\frac{1}{2}\mathbf{S}^2 = \frac{1}{8}\left(12i + 2\sum_{i<j} \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j\right)$$

gives an appropriate LEO of the form given in equation (2). This can be seen as follows. On the $S = 0$ (singlet) subspaces the operator gives zero. On the $S = 1$ (triplet) subspaces the operator gives 1 and on the $S = 2$ (quintuplet) subspace it gives 3. Therefore, the appropriate LEO can be obtained using

$$R_L = \exp(-i\pi S^2/2).$$

Since the operator S^2 is composed of exchange interactions, it is also experimentally available.

5. Concluding remarks

We have provided an overview of the quantum error preventing strategies for quantum computing devices. While the methods of QECC were motivated by error correction methods for classical computing, the other methods are more physically motivated. We believe the theory and practice of error prevention in quantum computing systems is converging based upon strategies which combine more than one of the error prevention techniques discussed here. The progress is motivated by the desire to construct practical error prevention schemes for near-future experiments. We hope that the overview given here will provide a resource/review for novices/experts working in quantum computing and that the progress concerning the elimination of leakage from encoded spaces will aid in the development of practical error prevention strategies.

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