Quantum computing in the presence of spontaneous emission by a combined dynamical decoupling and quantum-error-correction strategy

K. Khodjasteh
Department of Physics, University of Toronto, 60 St. George Street, Toronto, Ontario, Canada M5S 1A7

D. A. Lidar
Chemical Physics Theory Group, Chemistry Department, University of Toronto, 80 St. George Street, Toronto, Ontario, Canada M5S 3H6
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A method for quantum computation in the presence of spontaneous emission is proposed. The method combines strong and fast (dynamical decoupling) pulses and a quantum error correcting code that encodes n logical qubits into only $n+1$ physical qubits. Universal, fault-tolerant, quantum computation is shown to be possible in this scheme using Hamiltonians relevant to a range of promising proposals for the physical implementation of quantum computers.

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I. INTRODUCTION

Decoherence [1] remains the most daunting obstacle to the realization of quantum information processing, coherent control, and other applications requiring a high degree of quantum coherence. As quantum computation (QC) moves into the experimental realm, it becomes increasingly important to design methods for overcoming this main obstacle to realization, which are tailored to particular systems and the resulting errors that afflict them. Here we show how to perform universal, fault-tolerant QC in the presence of decoherence due to *spontaneous emission* (SE). Since SE is a consequence of the inevitable coupling to the vacuum field [2], it cannot be "engineered away" and must eventually be dealt with, in all QC proposals. Several methods have been designed to this end, which may roughly be classified as "hardware" and "software:" In the former category are proposals to construct quantum computers in materials where SE is strongly suppressed, e.g., placing atomic qubits in a photonic band-gap structure [3]. In the latter category are various error correction, avoidance, and suppression methods [4–10]. With the exception of the 2π pulsing method of Ref. [10], a unifying theme of these methods is to place the system under continuous observation. It is then well known that the Markovian quantum master equation can be unraveled into a set of quantum trajectories, consisting of a conditional evolution (governed by a non-Hermitian conditional Hamiltonian H_c, defined below), randomly interrupted by quantum jumps (wave-function collapse) into different observed decay channels [11–14]. The time evolution conditional to a *given* set of time-ordered observations is called "*a posteriori* dynamics" [15], and is *not* Markovian. The continuous observation can lead to a Zeno-effect type suppression of decoherence, a fact that was exploited in [9], in conjunction with an encoding into a decoherence-free subspace (DFS) [16,17], in order to resist SE. Quantum error correcting codes (QECCs) can correct both the conditional evolution and the jumps [5], but more efficient constructions are possible when one considers subspaces of the full system's Hilbert spaces that are *invariant under the conditional evolution*. It is then necessary to correct only the errors arising due to the quantum jumps [4–8]. The first proposal along these lines [4] did not consider QC. A simple, but non-fault-tolerant QC scheme, encoding a logical qubit into two physical qubits (four atomic levels), tailored to SE of phonons in trapped-ion QC, was subsequently presented in Ref. [5]. A QECC correcting one arbitrary single-qubit error and invariant under H_c was given in Ref. [6], using an encoding of one logical qubit into eight physical qubits. When one makes the assumption that the qubit undergoing the quantum jump can be identified ("detected jump"), a more efficient encoding is possible. A family of such detected jump codes (DJC) was first developed in Ref. [7], using a DFS to construct a subspace invariant under H_c. In Ref. [8] we showed how to perform fault-tolerant universal QC on a subclass of such codes encoding $n-1$ logical qubits into $2n$ physical qubits.

Here we present a method for reducing and correcting SE errors. Rather than constructing a code subspace invariant under H_c, we dynamically eliminate H_c by applying dynamical decoupling [or "bang-bang" (BB)] pulses [18,19]. We then construct a QECC that deals with the remaining jump errors, under the detected-jump assumption. The advantage of this method compared to the previous methods using encoding is that it is significantly more economical in qubit and pulse timing resources: It uses a QECC in which n logical qubits are encoded into only $n+1$ physical qubits; and, while in Ref. [10] the pulse interval has to satisfy the standard BB condition of being shorter than the inverse of the bath high-frequency cutoff [18,19], in our case the requirement is that the pulses are faster than the average time between photon emission events, which can be orders of magnitude longer. Furthermore, our method is fully compatible with universal QC using Hamiltonians that are naturally available in a large variety of quantum computer proposals [20], so unlike Ref. [3], it does not rely on one specific architecture.

The idea of using a hybrid BB-encoding approach to suppress decoherence was first proposed in Ref. [21], where it was pointed out that BB is fully compatible with encoding into a QECC or DFS. In particular, it was observed there that one could use BB to suppress phase-flip errors, thus leaving the QECC with the need only to correct bit-flip errors. However, no method specifically tailored for SE errors was given. An experimental nuclear-magnetic-resonance (NMR) implementation of a hybrid BB-QECC was presented in Ref. [22], where decoupling was used to remove coherent scalar coupling between protons (environment) and carbon qubits, together with QECC used to further correct for fast relaxation due to dipolar interactions modulated by random molecular motion.

Clearly, correcting for SE errors is only a part of a general procedure for offsetting decoherence, as additional decoherence sources will inevitably be present in any QC implementation. The methods we present here therefore will have to become part of this more general procedure, either as a first level of defense (in the case where SE is dominant), or at higher levels in a concatenated QECC scheme [23], after other more dominant errors have been accounted for. The importance of the results presented here lies in the fact that SE is *always* present, and therefore can never be ignored. A code that is optimized with respect to this type of error can potentially offer flexibility and significant savings in resources and overhead.

The structure of this paper is the following. In Sec. II, we show how the conditional evolution during SE can be eliminated using a sequence of simple, global BB decoupling pulses. In Sec. III, we construct a simple and economical QECC that corrects for the remaining quantum jump errors. We address fault tolerance and various imperfections in Sec. IV. We then show how to quantum compute in a universal and fault-tolerant manner over our QECC, using a variety of model Hamiltonians pertinent to a wide class of promising quantum computing proposals. We conclude in Sec. VI.

II. ELIMINATING THE CONDITIONAL EVOLUTION OF SPONTANEOUS EMISSION WITH BB PULSES

Consider N qubits that can each undergo SE, under the detected-jump assumption. This localizability of the SE events implies that the mean distance between qubits exceeds the wavelength of the emission. Note that this *optical* distinguishability between qubits does not limit our ability to couple the qubits via nonoptical interactions of the type we consider in Sec. V below.

The ground and excited states of each qubit are denoted by $|0\rangle$ and $|1\rangle$, respectively. Let $\sigma_i^- = |0\rangle_i\langle 1|$ denote the spontaneous emission error generator acting on the ith qubit and let κ_i denote the corresponding error rate. We use the quantum trajectories approach [11–14] to describe the dynamics of the decohering system. The evolution is decomposed into two parts: a conditional *non-Hermitian* Hamiltonian H_c, interrupted at random times by occurrence of random jumps, each corresponding to an observation of decay channels in a quantum optical setting. For errors such as SE, where the jump can be detected by observation of the emission, the quantum trajectories approach also provides us with a way to combine QECCs and BB, in analogy to the way this was done for QECC and DFS in Refs. [7,8]. The BB pulses take care of the conditional evolution, whereas the QECC deals with the random jumps. The conditional Hamiltonian is given in the SE case [11–14] by $H_c = -(i/2)\Sigma_{i=1}^N \kappa_i \sigma_i^+ \sigma_i^-$, where $\sigma_i^+ = (\sigma_i^-)^\dagger$. In Ref. [8] we assumed that the environment effectively does not distinguish among the qubits that undergo spontaneous emissions ($\kappa_i = \kappa$) and the conditional Hamiltonian would then become $-(i/2)\kappa \Sigma_i |1\rangle_i\langle 1|$. This assumption is *not* necessary in the current work. From here on, operators X_i, Y_i, Z_i refer to the corresponding Pauli matrices acting on the ith qubit. Now suppose that we apply a collective $X \equiv \otimes_{j=1}^N X_j$ pulse to the system, at intervals $T_c/2 \ll 1/\gamma$, where γ is the SE rate [24]. Under this condition and using $X_i \sigma_i^- X_i = \sigma_i^+$ we can write the evolution after a full T_c period as

$$\begin{aligned} U &= \exp\left(-i\frac{T_c}{2}H_c\right) X \exp\left(-i\frac{T_c}{2}H_c\right) X \\ &= \exp\left(-\frac{T_c}{4}\sum_i \kappa_i |1\rangle_i\langle 1|\right) \exp\left(-\frac{T_c}{4}\sum_i \kappa_i |0\rangle_i\langle 0|\right) \\ &= \exp\left(-\frac{T_c}{4}\sum_i \kappa_i\right) I, \end{aligned}$$

where I is the identity operator. Therefore the decohering effect of the conditional Hamiltonian (that distinguishes states with different numbers of 1's) is removed and replaced by an *overall* shrinking norm. When the jumps are included in the dynamics, the state must be renormalized [11–14], so this shrinking disappears. Note that we have *not* eliminated Markovian decoherence using BB pulses, since we have considered only a single trajectory. In fact, a comparison of the coherence $\mathcal{C} = \mathrm{Tr}(\rho^2)$ (where ρ is the qubit density matrix) shows that if the results are ensemble averaged over the *a posteriori* dynamics (recovering the Markovian master equation), and the jump errors are not corrected, then there is no advantage in using a BB sequence. More specifically, when comparing $\mathcal{C}$ for the (1) free evolution and (2) using X pulses at $T_c/2$ periods for a single qubit undergoing SE with rate γ, we find

$$\mathcal{C}_1 = 1 - \gamma T_c(\beta^2) + O(\gamma^2),$$

$$\mathcal{C}_2 = 1 - \gamma T_c(\alpha^4 + \beta^4) + O(\gamma^2)$$

where the initial qubit state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ is normalized: $\alpha^2 + \beta^2 = 1$. Averaging over a random sample of initial states chosen from a uniform distribution (of α and β subject to normalization) we have $\langle \mathcal{C}_1\rangle = \langle \mathcal{C}_2\rangle$; so as expected for purely Markovian dynamics, there is no improvement after using *just* BB pulses.

III. CORRECTING SPONTANEOUS EMISSION JUMPS WITH A QECC

We now introduce a very simple QECC that corrects the remaining part of the decoherence process, the random jumps. Since the error correction process by necessity takes place during the conditional evolution (the jump is instantaneous and the QECC takes time), we must ensure that the QECC keeps its error correcting properties under the conditional Hamiltonian and BB pulses. A minimal example of such a "decoupled-detected jump corrected" code is given by the subspaces of the $N=n+1$ qubit Hilbert space that have even Hamming weight, or parity, defined as $w(x)\equiv \oplus_{i=1}^{n} x_i = (\Sigma_{i=1}^{n} x_i)\mathrm{mod}2$, where $|x\rangle = |x_1,\ldots,x_n\rangle$ is a computational basis state ($x_i \in \{0,1\}$). We define a code subspace $\mathcal{C}_n$ with even parity as the linear span of the codewords,

$$|x\rangle_L \equiv |x_1,\ldots,x_n\rangle_L = |x_1,\ldots,x_n,w(x)\rangle. \tag{1}$$

For example, for $n=2$, the code $\mathcal{C}_2$ is

$$|\underline{00}\rangle_L = |\underline{00}0\rangle \quad |\underline{01}\rangle_L = |\underline{01}1\rangle$$
$$|\underline{10}\rangle_L = |\underline{10}1\rangle \quad |\underline{11}\rangle_L = |\underline{11}0\rangle. \tag{2}$$

As indicated by the underline, the first two qubits are the same in the physical and logical codewords, and the third qubit is set equal to the parity of the first two qubits. That $\mathcal{C}_n$ is a QECC against the jump errors follows from the fact that a spontaneous emission error always changes the parity of a given codeword, which is then taken to an orthogonal state, and that by construction no two codewords can be taken to the same state. More specifically, the sufficient condition that a QECC must satisfy is that orthogonal codewords must be mapped to orthogonal states after the occurrence of errors, so that the errors can be resolved and undone [25]. Recall that, here we are assuming that we know the location of the error, after recording the position of the spontaneous emission. Hence we need to only compare orthogonal codewords after the action of an error in a *known* location i:

$$_L\langle y|\sigma_i^+\sigma_i^-|x\rangle_L = \begin{cases} \delta_{xy} & \text{if} \quad y_i = x_i \\ 0 & \text{if} \quad y_i \neq x_i, \end{cases}$$

where the second line follows from the change in parity of $|x\rangle_L$ or $|y\rangle_L$. Thus the QECC condition is satisfied. To see that recovery from the errors is indeed possible, we describe a simple (non-fault-tolerant) scheme. To recover from an SE error on qubit j, we apply controlled-NOT gates from all other qubits (as controls) to qubit j (as target). That this unitary operation fixes the SE error, can be seen as follows. The codewords in which qubit j was in the state $|0\rangle$ before SE did not change after SE. In this case the number of remaining qubits in the state $|1\rangle$ was even, and the recovery operation will flip the erred qubit an even number of times, thus having no effect. If qubit j was in the state $|1\rangle$ before the SE error then it changed to $|0\rangle$. In this case the number of remaining qubits in the state $|1\rangle$ was odd, and the recovery operation flips the erred qubit. To illustrate this we discuss in detail the case of $\mathcal{C}_2$. The conditional evolution, under the collective BB pulse $X=X_1X_2X_3$, has the sole effect of shrinking the norm of all codewords in Eq. (2) *equally*. Thus the BB-modified conditional evolution does not change the orthogonality of the codewords. Now suppose SE from the first qubit has been observed. Then an arbitrary encoded state $|\psi\rangle_L = a|00\rangle_L + b|01\rangle_L + c|10\rangle_L + d|11\rangle_L$ changes into $|\psi_{\mathrm{err}}\rangle \equiv a|000\rangle + b|011\rangle + c|001\rangle + d|010\rangle$. To reverse the error we use the unitary operator $U = CN_{31}CN_{21}$, where CN_{ij} is a controlled-NOT gate with qubit i (j) as the control (target), i.e., $CN_{ij}|x_i,x_j\rangle = |x_i,x_j\oplus x_i\rangle$. The erred state is then transformed to $U|\psi_{\mathrm{err}}\rangle = a|000\rangle + b|011\rangle + c|101\rangle + d|110\rangle = |\psi\rangle_L$.

IV. FAULT-TOLERANT PREPARATION, MEASUREMENT, AND RECOVERY

So far we have assumed perfect error detection, recovery, and gates. Of course, in reality these assumptions must be relaxed. Here we discuss the implications of imperfections.

In general, a procedure is said to be fault tolerant if the occurrence of an error in one location does not lead (via the applied procedure) to the catastrophic multiplication of errors in other locations [23], an event that the code cannot correct.

Let us first discuss *preparation* of the encoded qubits. Since the state $|0\rangle_L = |0_1,\ldots,0_{n+1}\rangle$ is part of the code, preparation is as simple as preparing each physical qubit in its ground state, which can be done, e.g., via cooling, a strong polarizing field, or repeated strong measurements of all qubits. This step is manifestly fault tolerant. Once $|0\rangle_L$ has been prepared, computation proceeds using the fault tolerant logical operations given in Sec. V below, so any other state can be reached fault tolerantly. *Readout* is also very simple: a measurement of the first n physical qubits in $\mathcal{C}_n$ is easily seen to be equivalent to a direct measurement of the logical state. The measurement procedure must be tailored to the specific implementation, but our only assumption is that single-qubit measurements are possible, and that these measurements do not couple qubits. The measurement procedure is then fault tolerant.

Next consider *recovery*. The code $\mathcal{C}_n$ is an especially simple example of CSS stabilizer codes [26], with stabilizer generated by the single element $\otimes_{j=1}^{n+1} Z_j$. It is well known how in general to perform fault-tolerant recovery from this class of codes [23] (see also Ref. [27]), so we will not repeat the general construction here, which involves preparing and measuring encoded ancilla qubits (note that this typically doubles the number of physical qubits required, even before concatenation).

Finally, consider *detection of SE events*. Above we assumed that it is possible to perfectly identify the position of a qubit that underwent SE. Note that this measurement is in itself fault tolerant, in the sense that observing an SE event on a specific qubit cannot cause errors to multiply. Clearly, detecting which qubit emitted a photon is very demanding experimentally, and can in practice only be done to some finite precision (though there is no fundamental limit, provided the distance between the qubits is larger than the wavelengths of emitted photons), and at the cost of introducing a potentially cumbersome detection apparatus. The same difficulty is shared by previous detected-jump schemes [7–9].

More specifically, in reality there is a finite probability that the emitted photon will (i) Go undetected; (ii) be attributed to the wrong atom (misidentification). The latter possibility applies also to other qubit measurements; (iii) in case (ii), there is the additional possibility of an error by applying the correction step to the wrong qubit. In general, fault-tolerance results again come to the rescue: provided that the probability of an undetected photon and/or misidentification can be kept sufficiently low, concatenated QECC guarantees that the procedure will remain robust [23,28,29]. However, several additional comments are in order. First, we note that the performance of DJC codes in the presence of imperfections such as detection inefficiencies and time delay between error detection and recovery operations, has been analyzed in Ref. [30], with favorable conclusions regarding fidelity degradation. We expect similar conclusions for our current method. Second, unlike the case of DJC codes [7,8,30], we do not require equal error rates κ_i. Hence our qubits need not be identical: qubits can be tuned to different cavity modes and therefore emit distinguishable photons. This should enable a significant reduction in the misidentification error rate. Third, we can take advantage of the fixed-parity property of our code. An undetected SE event will change the overall parity of the codewords. Hence by adding one more ancilla qubit (initalized into the $|0\rangle$ state) that functions as a parity check bit, we can monitor for undetected SE. A non-fault tolerant procedure for enforcing the parity check is to make the ancilla qubit the target of successive controlled-NOT gates from the code qubits, and periodically check to see if it has changed to $|1\rangle$. In such a case the computation has to be restarted and the ancilla qubit has to be reset. Fault-tolerant procedures are known for parity check as well [23,28,29]. There is also the possibility of SE on the parity qubit, but this can only be caused by two successive spontaneous emissions (one on the code qubits and one on the parity qubit), which has a lower probability p^2, where p is the probability of a single SE error occurring during the same observation period, before the first one is detected. Note that the parity bit also helps preventing the error of applying a correction step without an SE event having taken place.

V. FAULT-TOLERANT COMPUTATION

So far we have described a fault-tolerant implementation of quantum *memory* in the presence of SE. Now we describe how to perform universal quantum *computation* fault tolerantly on our code. Formally, one can use the formalism of normalizer group operations, together with a non-normalizer element such as the $\pi/8$ or Toffoli gate [23]. However, here we are interested in how to carry this out from the perspective of *the naturally available interactions for a given physical system*. Similar questions have been raised recently under the heading of ‘‘encoded universality:’’ the ability to quantum compute universally directly in terms of a given and limited set of Hamiltonians, possibly by use of encoded qubits (see, e.g., Refs. [20,31] and references therein). The problem then translates into finding sets of Hamiltonians that generate a universal set of logic gates on the code. There are many options, depending on the set of naturally available interactions. Nevertheless, all encoded universality constructions rely on showing that the well-known universal set of all single-qubit operations and a single entangling gate can be generated on the encoded qubits. Underlying this are a few elementary identities. Let us define *conjugation* as:

$$A \xrightarrow{B,\varphi} \equiv e^{-i\varphi B} A e^{i\varphi B}.$$

Then for any three su(2) generators $\{J_x, J_y, J_z\}$ (e.g., $\{X/2, Y/2, Z/2\}$):

$$J_x \xrightarrow{J_z,\varphi} J_x \cos\varphi + J_y \sin\varphi. \tag{3}$$

This can be lifted to unitary evolutions using

$$U e^{A} U^{\dagger} = e^{UAU^{\dagger}}, \tag{4}$$

valid for any unitary U. Hence where convenient we present our arguments in terms of transformed Hamiltonians. Equations (3),(4) show that given two su(2) generators, one can generate a unitary evolution about any axis. This is also the basis for the well-known Euler angle construction, used to argue that all single qubit operations can be generated from σ^x and σ^z Hamiltonians: an arbitrary rotation by an angle ω around the unit vector $\mathbf{n}$ is given by three successive rotations around the z and x axes: $e^{-i\omega \mathbf{n}\cdot\boldsymbol{\sigma}} = e^{-i\beta\sigma^z} e^{-i\theta\sigma^x} e^{-i\alpha\sigma^z}$ [32]. Equations (3) and (4) show that this is true also for ‘‘encoded Hamiltonians’’, which we define as Hamiltonians that have the same effect on encoded states as do regular Hamiltonians on ‘‘bare’’ (unencoded) qubits. We denote encoded Hamiltonians by a bar. For the code states (1), these are given by

$$\bar{X}_i = X_i X_{n+1}, \quad \bar{Z}_i = Z_i, \tag{5}$$

and generate su(2). Therefore controllable $X_i X_{n+1}$ and Z_i Hamiltonians suffice to generate arbitrary single encoded-qubit transformations. To complete the set of universal logic gates we require *some* nontrivial (entangling) gate [33], such as controlled phase (CP), CP $=\mathrm{diag}(1,1,1,-1)$, in the computational basis. CP can be generated from the Ising interaction $Z_i Z_j$ as follows: $\mathrm{CP}_{ij} = e^{-i(\pi/4)(Z_i+Z_j)} e^{-i(3\pi/4)Z_i Z_j}$. An entangling gate can also be generated from the Hamiltonian $X_i X_j$ [one way to see this is to note that it can be rotated to $Z_i Z_j$ using Y_i and Y_j in Eqs. (3) and (4)]. Encoded CP can thus be generated from the encoded Hamiltonians $\bar{Z}_i \bar{Z}_j = Z_i Z_j$ or $\bar{X}_i \bar{X}_j = X_i X_j$. Note that in both cases the physical interaction is also the corresponding encoded Hamiltonian. Thus the sets of controllable Hamiltonians $\{Z_i, X_i X_j\}$ or $\{Z_i, X_i X_{n+1}, Z_i Z_j\}$ suffice for encoded universal QC on our code. Importantly, these sets moreover exhibit ‘‘natural fault tolerance’’ [17]: they *preserve the code subspace* and hence will not expose the code to uncorrectable errors. An *accuracy* error in the time over which the Hamiltonians are turned on can be dealt with using the technique of concatenated QECCs [23]. The question now is how to generate these sets, or an equivalent fault-tolerant universal set, from the given, naturally available interactions. We will consider here the most important cases, extending methods developed in Refs. [20,34,35]. Note that the decoupling procedure requires us to assume in any case the ability to apply a *global* (nonselective) X pulse, and the recovery procedure requires the ability to apply a controlled-NOT gate. We comment on these requirements in each of the cases we next analyze.

A. Case 1: Natural $\{Z_i, X_iX_j\}$

The Hamiltonians Z_i, X_iX_j are naturally available, e.g., in the Sørensen-Mølmer scheme for trapped-ion QC [36], and in proposals using Josephson charge qubits [37]. This is a universal set for our code, so that encoded computation is automatically compatible with these proposals. Regarding decoupling, there are at least two ways to obtain an X pulse: (i) to generate it by simultaneously turning on all interactions $\{X_{2i-1}X_{2i}\}_{i=1}^{N/2}$ for a time $\pi/(2J_i)$, where J_i is the coupling between spins $2i-1,2i$. However, $\{Z_i, X_iX_j\}$ alone is insufficient for generating a controlled-NOT gate and hence we must (ii) assume the ability to turn on spin-selective X_i Hamiltonians. In case (ii) it is clear that controlled-NOT can be generated, since $\{X_i, Z_i, X_iX_j\}$ is a universal set of Hamiltonians.

B. Case 2: $\{Z_i$, XY Model$\}$

Members of a relatively large class of promising QC proposals (quantum dots [38,39], atoms in a cavity [40], quantum Hall qubits [41], subradiant dimers in a solid host [42], and capacitively coupled superconducting qubits [43]) have a controllable Hamiltonian of the XY form: $H_{ij}^{XY} = J_{ij}(X_iX_j + Y_iY_j)$. Let $T_{ij} \equiv \frac{1}{2}(X_iX_j + Y_iY_j)$. Then $|01\rangle \overset{T_{12}}{\leftrightarrow} |10\rangle$, and annihilates $|00\rangle, |11\rangle$; i.e., the XY Hamiltonian cannot change the total number of 1's in a computational basis state [34,35]. Therefore by itself, or even if supplemented with Z_i Hamiltonians, it cannot generate su(2) on our code. This conclusion is unchanged even if one considers conjugating H_{ij}^{XY} with H_{ik}^{XY}: then $\{T_{12}, T_{13}, -Z_1Z_2T_{23}\}$ close as su(2), and still preserve the total number of 1's. Therefore in this case we must assume the ability to tune X_i Hamiltonians as well, to obtain universality. However, in order to preserve the code space we must ensure that only the *pulses* (unitary transformations) $X_iX_j = e^{-i(\pi/2)(X_i+X_j)}$ are applied using these Hamiltonians, since such pulses preserve parity. Now, $X_iX_j(T_{jk})X_iX_j = \frac{1}{2}(X_jX_k - Y_jY_k)$, which commutes with T_{jk}. Therefore, using Eq. (4), we have $X_iX_je^{-i\theta T_{jk}}X_iX_je^{-i\theta T_{jk}} = e^{-i\theta X_jX_k}$, showing that the *Hamiltonian* X_jX_k can be generated in four steps. At this point we have the same set of Hamiltonians as in Case 1, so that universal encoded computation is possible, as are the global X pulse and recovery.

C. Case 3: $\{Z_i$, Heisenberg interaction$\}$

Next we consider the case of single-qubit Z_i control together with the Heisenberg interaction $H_{ij}^{\mathrm{Heis}} = J_{ij}(X_iX_j + Y_iY_j + Z_iZ_j)$. Heisenberg interactions prevail in QC proposals using spin-coupled quantum dots [44–46] and donor atoms in Si [47,48]. This case is similar to that of the XY model, since H_{ij}^{Heis} also preserves the total number of 1's in the computational basis states. Therefore, as in the XY case, we must assume the ability to generate an X_iX_j *pulse*. Then, $X_iX_j(H_{jk}^{\mathrm{Heis}})X_iX_j = J_{jk}(X_jX_k - Y_jY_k - Z_jZ_k)$, which commutes with H_{jk}^{Heis}, so that $X_iX_je^{-itH_{jk}^{\mathrm{Heis}}}X_iX_je^{-itH_{jk}^{\mathrm{Heis}}} = e^{-2itJ_{jk}X_jX_k}$, and we are back to Case 1. There is now another option for generating an entangling gate: we can generate a pure ZZ interaction using $ZI(H_{\mathrm{Heis}})ZI = -XX - YY + ZZ$, which commutes with H_{Heis}, so that $e^{-itH_{\mathrm{Heis}}}e^{-i(\pi/2)ZI}e^{-itH_{\mathrm{Heis}}}e^{-i(\pi/2)ZI} = e^{-2itJZZ}$. This is a four-step, naturally fault-tolerant procedure. The decoupling pulse and recovery are now the same as in Case 1.

Finally, there remains the issue of compatibility between the encoded logic operations and the decoupling pulses that are being constantly applied to the system. All three interaction Hamiltonians we have considered commute with the global X BB pulse, so are fully compatible with the BB operations. On the other hand, the single-qubit $Z_i^L = Z_i$ terms anti-commute with the X pulse. Hence when using Z_i, one must be extra careful only to apply it after an even number of X BB pulses, so that the effect of the BB pulse is neutralized, before and after the Z_i Hamiltonian is used.

VI. CONCLUSIONS

We have proposed a method for performing universal, fault-tolerant quantum computation in the presence of spontaneous emission. The method combines dynamic decoupling pulses with a particularly simple and efficient quantum error correcting code, encoding n logical qubits into $n+1$ physical qubits. Computation is performed by controlling single-qubit σ^x and σ^z terms together with any of three major examples of qubit-qubit interaction Hamiltonians, applicable to a wide range of quantum computing proposals. The proposed method offers an improvement over previous schemes for protecting quantum information against spontaneous emission in that the code is at least twice as efficient in terms of qubit resources, and the method is fully compatible with computation using physically reasonable resources and interactions.

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