Direct Characterization of Quantum Dynamics

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(Received 5 May 2006; published 24 October 2006)

The characterization of quantum dynamics is a fundamental and central task in quantum mechanics. This task is typically addressed by quantum process tomography (QPT). Here we present an alternative ''direct characterization of quantum dynamics'' (DCQD) algorithm. In contrast to all known QPT methods, this algorithm relies on error-detection techniques and does not require any quantum state tomography. We illustrate that, by construction, the DCQD algorithm can be applied to the task of obtaining *partial information* about quantum dynamics. Furthermore, we argue that the DCQD algorithm is experimentally implementable in a variety of prominent quantum-informatiom processing systems, and show how it can be realized in photonic systems with present day technology.

DOI: 10.1103/PhysRevLett.97.170501 PACS numbers: 03.67.Pp, 03.65.Wj

The characterization and identification of quantum systems are among the central modern challenges of quantum physics, and play an especially fundamental role in quantum-informatiom science [1], and coherent control [2]. A task of general and crucial importance is the characterization of the dynamics of a quantum system that has an unknown interaction with its embedding environment. Knowledge of this dynamics is indispensable, e.g., for verifying the performance of an information-processing device, and for the design of decoherence-mitigation methods. Such characterization of quantum dynamics is possible via the method of standard quantum process tomography (SQPT) [1,3]. SQPT consists of preparing an ensemble of identical quantum systems in a member of a set of quantum states, followed by a reconstruction of the dynamical process via quantum state tomography. We identify three main issues associated with the required physical resources in SQPT. (i) The number of ensemble measurements grows exponentially with the number of degrees of freedom of the system. (ii) Often it is not possible to prepare the complete set of required quantum input states. (iii) Information concerning the dynamical process is acquired *indirectly* via quantum state tomography, which results in an inherent redundancy of physical resources associated with the estimation of some superfluous parameters. To address (ii), the method of ancilla-assisted process tomography (AAPT) was proposed [4]. However, the number of measurements is the same in SQPT and (separable) AAPT [4].

Here we develop an algorithm for *direct characterization of quantum dynamics* (DCQD), which does not require quantum state tomography. The primary system is initially entangled with an ancillary system, before being subjected to the unknown dynamics. Complete information about the dynamics is then obtained by performing a certain set of error-detecting measurements. We demonstrate that for characterizing a non-trace-preserving quantum dynamical map on n qubits, the number of required experimental configurations is reduced from 2^{4n}, for SQPT and *separable* AAPT, to 2^{2n} in DCQD. For example, for a single qubit, we show that one can fully characterize the quantum dynamics by preparing one of four possible two-qubit entangled states, and a Bell-state measurement (BSM) at the output. This is illustrated in Fig. 1 and Table I. We also discuss the experimental feasibility of DCQD in a variety of quantum-informatiom processing (QIP) systems, and show how it can be realized with linear optics.

In principle, the number of required experimental configurations in the AAPT scheme can be reduced by utilizing *nonseparable* (global) measurements for the required state tomography, such as mutually unbiased basis (MUB) measurements [5], or a generalized measurement [6]. In general, these types of measurements require *many-body interactions* which are not available experimentally. Here, we demonstrate that the DCQD algorithm requires only $\mathcal{O}(n)$ single- and two-body operations per experimental configuration.

We demonstrate the inherent applicability of the DCQD algorithm to the task of partial characterization of quantum dynamics in terms of coarse-grained quantities. Specifically, we demonstrate that for a two-level system undergoing a sequence of amplitude and phase damping processes, the relaxation time T_1 and dephasing time T_2 can be simultaneously determined in one ensemble measurement.

Quantum dynamics.—The evolution of a quantum system (open or closed) can, under natural assumptions, be

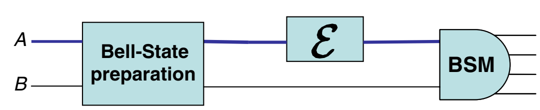

FIG. 1 (color online). Schematic of characterization algorithm for the single-qubit case, consisting of Bell-state preparations, applying the unknown quantum map, $\mathcal{E}$, and BSM.

TABLE I. One possible set of input states and measurements for direct characterization of quantum dynamics (χ_{ij}) for a single qubit, where $|\alpha| \neq |\beta| \neq 0$, and $\{|0\rangle, |1\rangle\}$, $\{|\pm\rangle\}$, $\{|\pm i\rangle\}$ are eigenstates of the Pauli operators Z, X, and Y.

Input state	Measurement		Output
	Stabilizer	Normalizer	
$(\|0\rangle\|0\rangle + \|1\rangle\|1\rangle)/\sqrt{2}$	Z^AZ^B, X^AX^B	N/A	$\chi_{00}, \chi_{11}, \chi_{22}, \chi_{33}$
$\alpha\|0\rangle\|0\rangle + \beta\|1\rangle\|1\rangle$	Z^AZ^B	X^AX^B	χ_{03}, χ_{12}
$\alpha\|+\rangle\|0\rangle + \beta\|-\rangle\|1\rangle$	X^AZ^B	Z^AX^B	χ_{01}, χ_{23}
$\alpha\|+i\rangle\|0\rangle + \beta\|-i\rangle\|1\rangle$	Y^AZ^B	Z^AX^B	χ_{02}, χ_{13}

expressed in terms of a completely positive quantum dynamical map $\mathcal{E}$, which can be represented as [1]

$$\mathcal{E}(\rho) = \sum_{m,n=0}^{d^2-1} \chi_{mn} E_m \rho E_n^{\dagger}. \tag{1}$$

Here ρ is the system initial state, and the $\{E_m\}$ are a set of (error) operator basis elements in the Hilbert-Schmidt space of linear operators acting on the system satisfying $\mathrm{Tr}(E_m^{\dagger}E_n) = d\delta_{mn}$. The $\{\chi_{mn}\}$ are the matrix elements of the superoperator $\boldsymbol{\chi}$, which encodes all the information about the dynamics, relative to the basis set $\{E_m\}$ [1]. For an n qubit system, the number of independent matrix elements in $\boldsymbol{\chi}$ is 2^{4n} for a non-trace-preserving map, or $2^{4n} - 2^{2n}$ for a trace-preserving map. The matrix $\boldsymbol{\chi}$ is positive Hermitian, and $\mathrm{Tr}\boldsymbol{\chi} \leq 1$. Thus $\boldsymbol{\chi}$ can be thought of as a density matrix in Hilbert-Schmidt space, whence we often refer, below, to its diagonal and off-diagonal elements as "quantum dynamical population" and "coherence", respectively.

Characterization of quantum dynamical population.—Here we demonstrate how to determine $\{\chi_{mm}\}$ in a single (ensemble) measurement for a single qubit. Let us maximally entangle two qubits A (primary) and B (ancillary) as $|\psi\rangle = (|0_A 0_B\rangle + |1_A 1_B\rangle)/\sqrt{2}$, and then subject only qubit A to a map $\mathcal{E}$. From here on, for simplicity, we denote $\mathcal{E} \otimes I$ by $\mathcal{E}$. In this case the error basis $\{E_m\}_{m=0}^3$ becomes the identity operator and the Pauli operators: $\{I, X, Y, Z\}$. The state $|\psi\rangle$ is a $+1$ eigenstate of the commuting operators Z^AZ^B and X^AX^B, i.e., it is stabilized under the action of these "stabilizer operators", and it is referred to as a "stabilizer state" [1]. Any nontrivial operator acting on the state of the first qubit anticommutes with at least one of Z^AZ^B and X^AX^B, and therefore by measuring these operators we can detect an arbitrary error on the first qubit, i.e., by finding the eigenvalues of either one or both operators to be -1. Measuring the observables Z^AZ^B and X^AX^B is equivalent to a BSM, and can be represented by the four projection operators $P_0 = |\phi^+\rangle\langle\phi^+|$, $P_1 = |\psi^+\rangle\langle\psi^+|$, $P_2 = |\psi^-\rangle\langle\psi^-|$, $P_3 = |\phi^-\rangle\langle\phi^-|$, where $|\phi^{\pm}\rangle = (|00\rangle \pm |11\rangle)/\sqrt{2}$, $|\psi^{\pm}\rangle = (|10\rangle \pm |01\rangle)/\sqrt{2}$, form the Bell basis of the two-qubit system. The probabilities of obtaining the no error outcome I, bit flip error X^A, phase flip error Z^A, and both phase flip and bit flip errors Y^A, on the first qubit, become $p_m = \mathrm{Tr}[P_m\mathcal{E}(\rho)] = \chi_{mm}$, for $m = 0, 1, 2, 3$, respectively. Therefore, we can determine the quantum dynamical population, $\{\chi_{mm}\}_{m=0}^3$, *in a single ensemble measurement* (e.g., by simultaneously measuring the operators Z^AZ^B and X^AX^B) on multiple copies of the state $|\psi\rangle$.

Characterization of quantum dynamical coherence.—In order to preserve the coherence ($\chi_{m\neq n}$) in the quantum dynamical process, we perform a set of measurements such that we always obtain partial information about the nature of the errors. The simplest case is a measurement of the projection operators $P_{+1} = |\phi^+\rangle\langle\phi^+| + |\phi^-\rangle\langle\phi^-|$ and $P_{-1} = |\psi^+\rangle\langle\psi^+| + |\psi^-\rangle\langle\psi^-|$ corresponding to the eigenvalues $+1$ and -1 of measuring the stabilizer Z^AZ^B. The outcomes of this measurement represent the probabilities of no bit flip error and bit flip error on qubit A, without telling us anything about a phase flip error. Therefore, we preserve only the coherence between operators I and Z^A, and also between the X^A and Y^A (which are represented by the off-diagonal elements χ_{03} and χ_{12}, respectively). The required input state is a nonmaximally entangled state $|\phi_C\rangle = \alpha|00\rangle + \beta|11\rangle$, with $|\alpha| \neq |\beta| \neq 0$, whose sole stabilizer is Z^AZ^B. Now, by separating real and imaginary parts, the probabilities of no bit flip error and bit flip error events become: $\mathrm{Tr}[P_{+1}\mathcal{E}(\rho)] = \chi_{00} + \chi_{33} + 2\,\mathrm{Re}(\chi_{03}) \times \langle Z^A\rangle$ and $\mathrm{Tr}[P_{-1}\mathcal{E}(\rho)] = \chi_{11} + \chi_{22} + 2\,\mathrm{Im}(\chi_{12})\langle Z^A\rangle$, where $\langle Z^A\rangle \equiv \mathrm{Tr}(\rho Z^A) \neq 0$ (because $|\alpha| \neq |\beta| \neq 0$), with $\rho = |\phi_C\rangle\langle\phi_C|$. We already know the $\{\chi_{mm}\}$ from the population measurement described above, so we can determine $\mathrm{Re}(\chi_{03})$ and $\mathrm{Im}(\chi_{12})$. After measuring Z^AZ^B the system is in either of the states $\rho_{\pm 1} = P_{\pm 1}\mathcal{E}(\rho)P_{\pm 1}/\mathrm{Tr}[P_{\pm 1}\mathcal{E}(\rho)]$. Now we measure the expectation value of a (commuting) normalizer operator U (such as X^AX^B) [7]. Thus we obtain $\mathrm{Tr}[U\rho_{+1}] = [(\chi_{00} - \chi_{33})\langle U\rangle + 2i\,\mathrm{Im}(\chi_{03})\langle Z^AU\rangle]/\mathrm{Tr}[P_{+1}\mathcal{E}(\rho)]$ and $\mathrm{Tr}[U\rho_{-1}] = [(\chi_{11} - \chi_{22})\langle U\rangle - 2i\,\mathrm{Re}(\chi_{12})\langle Z^AU\rangle]/\mathrm{Tr}[P_{-1}\mathcal{E}(\rho)]$, where $\langle Z^A\rangle$, $\langle U\rangle$, and $\langle Z^AU\rangle$ are all nonzero and already known. Therefore, we can obtain the four independent real parameters needed to calculate the coherence components χ_{03} and χ_{12}, by simultaneously measuring, e.g., Z^AZ^B and X^AX^B.

In order to characterize the remaining coherence elements of $\boldsymbol{\chi}$ we make an appropriate change of basis in the preparation of the two-qubit system. For characterizing χ_{01} and χ_{23}, we can perform a Hadamard transformation on the first qubit, as $H^A|\phi_C\rangle = \alpha|+\rangle|0\rangle + \beta|-\rangle|1\rangle$, where $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$. We then measure the stabilizer operator X^AZ^B, and a normalizer such as Z^AX^B. For characterizing χ_{02} and χ_{31}, we prepare the system in the stabilizer state $S^AH^A|\phi_C\rangle = \alpha| + i\rangle|0\rangle + \beta| - i\rangle|1\rangle$, and measure the stabilizer operator $Y^A Z^B$ and a normalizer such as $Z^A X^B$, where S is the single-qubit phase gate, and $|\pm i\rangle = (|0\rangle \pm i|1\rangle)/\sqrt{2}$. Therefore, we can completely characterize the quantum dynamical coherence with three BSM's overall. We note that the bases $\{|0\rangle, |1\rangle\}$, $\{|\pm\rangle\}$, and $\{|\pm i\rangle\}$ are *mutually unbiased*, i.e., the inner products of each pair of elements in these bases have the same magnitude [5].

A summary of the scheme for the case of a single qubit is presented in Fig. 1 and Table. I. This Table implies that the required resources in DCQD are as follows: (a) preparation of a maximally entangled state (for population characterization), (b) preparation of three other (nonmaximally) entangled states (for coherence characterization), and (c) a fixed Bell-state analyzer. We remark that a generalized DCQD algorithm for qudits, with d being a power of a prime, is possible, and will be the subject of a future publication [8].

An additional feature of DCQD is that all the required ensemble measurements, for measuring the expectation values of the stabilizer and normalizer operators, can also be performed in a temporal sequence on the *same* pair of qubits with only *one* Bell-state generation. This is because at the end of each measurement, the output state is in fact in one of the four possible Bell states, which can be utilized as an input stabilizer state.

For characterizing a quantum dynamical map on n qubits we need to perform a measurement corresponding to a tensor product of the required measurements for single qubits. An important example is a QIP unit with n qubits which has a 2^n-dimensional Hilbert space ($\mathcal{H}$). DCQD requires a total of 4^n experimental configurations for a complete characterization of the dynamics. This is a *quadratic advantage* over SQPT and separable AAPT, which require a total of 16^n experimental configurations. In general, the required state tomography in AAPT could also be realized by nonseparable (global) quantum measurements. These measurements can be performed either in the same Hilbert space, with $4^n + 1$ measurements, e.g., using a MUB measurement [5], or in a *larger* Hilbert space, with a single generalized measurement [6]. A comparison of the required physical resources is presented in Table II. A detailed resource cost analysis comparing DCQD to other QPT methods will be reported in a future publication [9].

Physical realization.—For qubit systems, the resources required in order to implement the DCQD algorithm are Bell-state preparation and measurement, and single-qubit rotations. These tasks play a central role in quantum-informatiom science, since they are prerequisites for quantum teleportation, quantum dense coding, and quantum key distribution [1]. Because of their importance, these tasks have been studied extensively and successfully implemented in a variety of different quantum systems, e.g., nuclear magnetic resonance (NMR) [10] and trapped ions [11]. Thus, the DCQD algorithm is already within experimental feasibility of essentially all systems which have been used to demonstrate QIP principles to date.

TABLE II. Comparison of the required physical resources for characterizing a non-trace-preserving CP quantum dynamical map on n qubits. N_{in} and N_{m}, respectively, denote the number of required input states and the number of *noncommutative* measurements for each input state, and $N_{\mathrm{exp}} \equiv N_{\mathrm{in}} N_{\mathrm{m}}$ is the number of required experimental configurations. "NAAPT" denotes nonseparable AAPT.

Scheme	$\dim(\mathcal{H})$	N_{in}	N_{m}	N_{exp}
SQPT	2^n	4^n	4^n	16^n
NAAPT	2^{2n}	1	$4^n + 1$	$4^n + 1$
DCQD	2^{2n}	4^n	1	4^n

Here, we propose a specific linear-optical implementation, realizable with present day technology—see Fig. 2. Using only linear-optical elements, at most 50% efficiency in discriminating among Bell states is possible [12]. The number of ensemble preparations in this specific setup is thus effectively increased by a factor of 2 over an implementation of the DCQD algorithm using an ideal Bell-state analyzer. However, even in this optical realization, the number of experimental configurations is still reduced by a factor of 2^n (for characterizing a quantum dynamical process on n qubits) over SQPT and separable AAPT schemes. Alternatively, a small-scale and deterministic linear-optical implementation of DCQD can in principle be realized by a multirail representation of the qubits [13].

To increase the efficiency of an all-optical Bell-state one can, e.g., employ hyperentanglement between pairs of photons for a complete and deterministic linear-optical Bell-state discrimination [14,15]. This method is based on the fact that the pairs of polarization-entangled photons generated by parametric down-conversion have intrinsic time-energy and momentum correlations. These additional degrees of freedom can be exploited in order to distinguish between the subsets of Bell states which otherwise cannot be discriminated by a standard Hong-Ou-Mandel interfer-

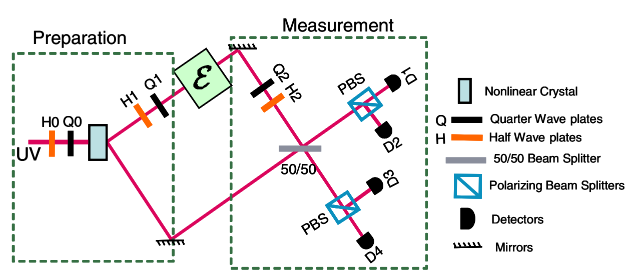

FIG. 2 (color online). A schematic layout of a linear-optical realization of the DCQD algorithm. A pair of entangled photons, $(|H_A H_B\rangle + e^{i\varphi}|V_A V_B\rangle)/\sqrt{2}$, is generated via parametric down-conversion in a nonlinear crystal from a UV pump laser, where $|H\rangle$ and $|V\rangle$ represent horizontal and vertical polarization [12]. The quarter and half wave plates are used to create nonmaximally entangled states [18], and change the preparation and measurement bases. Two-photon interferometry occurs at the 50/50 beam splitters. This allows for discriminating two of the four Bell states deterministically, by utilizing polarizing beam splitters and photodetectors as analyzers.

ometer. Note that employing the time-energy correlations in the context of implementing the DCQD algorithm is feasible if the quantum dynamical map acts only on the polarization degrees of freedom.

Partial characterization of dynamics.—DCQD can be applied, *by construction*, to the task of partial characterization of quantum dynamics, where we cannot afford or do not need a full characterization of the system, or when we have some *a priori* knowledge about the dynamics. In particular, we can substantially reduce the number of measurements, when we are interested in estimating the coherence elements of the superoperator for only specific *subsets* of the operator basis and/or subsystems of interest. For example, we need to perform a single ensemble measurement if we are required to identify only the coherence elements χ_{03} and χ_{12} of a particular qubit. Here, we present an example of such a task. Specifically, we demonstrate that the DCQD algorithm enables the simultaneous determination of coarse-grained physical quantities, such as the longitudinal relaxation time T_1 and the transverse relaxation (or dephasing) time T_2. Assume that we prepare a two-qubit system in the nonmaximally entangled state $|\phi_C\rangle = \alpha|0_A 0_B\rangle + \beta|1_A 1_B\rangle$. Then we subject qubit A to an amplitude damping process for duration t_1, followed by a phase damping process, for duration t_2. The elements of the final density matrix for qubit A then read $\langle 0|\rho_f|0\rangle = 1 - \exp(-\frac{t_1}{T_1})(1 - |\alpha|^2)$ and $\langle 0|\rho_f|1\rangle = \exp(-\frac{t'}{2T_2'})\alpha^*\beta$, where $\frac{t'}{T_2'} = \frac{t_2}{T_2} + \frac{t_1}{T_1}$. In order to determine T_1, we measure the eigenvalues of the stabilizer operator $Z^A Z^B$. We obtain either $+1$ or -1, corresponding to the projective measurement P_{+1}, or P_{-1}. The probabilities of either of these outcomes, e.g., $\mathrm{Tr}(P_{-1}\rho_f)$ are related to T_1 through the relation $\frac{1}{T_1} = -\frac{1}{t_1}\ln\{1 - 2\mathrm{Tr}(P_{-1}\rho_f)/[1 - \mathrm{Tr}(Z^A\rho)]\}$. In order to obtain information about T_2, we measure the expectation value of any normalizer of the input state, such as $X^A X^B$, yielding $\frac{t'}{T_2'} = -2\ln[\mathrm{Tr}(X^A X^B \rho_f)/\mathrm{Tr}(X^A X^B \rho)]$. Since the operators $Z^A Z^B$ and $X^A X^B$ commute, we can measure them simultaneously. Therefore we can find both T_1 and T_2 in a Bell-state measurement.

Outlook.—One can combine the DCQD algorithm with the method of maximum-likelihood estimation [16], in order to minimize the statistical errors in each experimental configuration. Moreover, a new scheme for *continuous* characterization of quantum dynamics can be introduced, by utilizing weak measurements for the required quantum error detections in DCQD [17].

For quantum systems with controllable two-body interactions (e.g., trapped-ion and NMR systems), DCQD could have near-term experimental applications for complete verification of *small QIP units*. For example, DCQD, reduces the number of required experimental configurations for systems of 3 or 4 physical qubits from 5×10^3 and 6.5×10^4 (in SQPT) to 64 and 256, respectively. A similar scale up can only be achieved by utilizing nonseparable AAPT methods. Complete characterization of such dynamics would be essential for verification of quantum key distribution procedures, teleportation units, quantum repeaters, and more generally, in any situation in quantum physics where a few qubits have a common local bath and interact with each other. Furthermore, as demonstrated here, DCQD is inherently suited to extract partial information about quantum dynamics. Several other examples of such applications have been demonstrated. Specifically, it has been shown that DCQD can be used for realization of generalized quantum dense coding [8]. Moreover, DCQD can be efficiently applied to (single- and two-qubit) Hamiltonian identification tasks [9]. Finally, the general techniques developed here could be further utilized for closed-loop control of open quantum systems.

This work was supported by NSERC (to M. M.), NSF No. CCF-0523675, ARO No. W911NF-05-1-0440, and the Sloan Foundation (to D. A. L.). We thank R. Adamson, J. Emerson, D. F. V. James, K. Khodjasteh, D. W. Leung, A. T. Rezakhani, A. M. Steinberg, and M. Ziman for useful discussions.

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