Bang-bang control of a qubit coupled to a quantum critical spin bath

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We analytically and numerically study the effects of pulsed control on the decoherence of a qubit coupled to a quantum spin bath. When the environment is critical, decoherence is faster and we show that the control is relatively more effective. Two coupling models are investigated, namely, a qubit coupled to a bath via a single link and a spin-star model, yielding results that are similar and consistent.

I. INTRODUCTION

Decoherence results from the unavoidable coupling between any quantum system and its environment, and is responsible for the dynamical destruction of quantum superpositions. It is detrimental for quantum information processing [1] since it leads to a loss of the quantum parallelism that is implicit in the superposition principle. The possibility of preventing or avoiding decoherence is hence of significant importance for any technological use of quantum systems, aimed at processing, communicating, or storing information. To this end, one must understand and model all of the relevant features characterizing the environment of the physical system to be protected. Understanding decoherence is also of fundamental interest in its own right, since it is at the basis of the description of the quantum-classical transition [2].

The study of open quantum systems has a long history, and many ingenious models have been proposed in order to describe the action of the environment in a quantum dynamical framework (see, e.g., Ref. [3]). Paradigmatic models represent the environment as a set of harmonic oscillators [4] or spins [5]. Recently there has been a renewed interest in the analysis of decoherence induced by such spin baths [6–24]; these are clearly relevant in a number of physically important situations, such as NMR [25] or spin qubits [26], where loss of coherence is induced by the coupling to nuclear spins [27]. Several questions have been addressed so far and the picture that emerges is rather rich. A possible, monogamylike relation between the entanglement in the bath and decoherence has been put forward in Ref. [11] and subsequently analyzed in different papers. The signatures of criticality of the environment in decoherence have been discussed through the study of solvable one-dimensional model systems [13,15,19]. A universal regime exists, in the strong coupling limit, in which the decay of the Loschmidt echo [28] does not depend on the system-bath coupling [14,22].

Several different protocols have been designed to protect quantum information. These include passive correction techniques, in which quantum information is encoded in such a way as to suppress the coupling with the environment [29,30], and active approaches such as quantum error correction [1] and dynamical decoupling techniques [31–34] (for an overview, see, e.g., [35]; for historical references see [36]). Dynamical decoupling strategies aim, by means of a dynamical control field, at averaging to zero the unwanted interaction with the environment. In its simplest version, which we consider here, the control field comprises a train of instantaneous pulses ("bang-bang" control). While previous work on dynamical decoupling has made clear distinctions between different environments, in particular, bosonic baths [31] versus spin baths [37,38], and fast versus slow $1/f$ noise [39], no attention has been paid so far to the impact a quantum critical environment might have on the efficacy of decoupling protocols. This is our goal in the present work: here we study bang-bang decoupling in the case where the quantum environment can become critical.

Many-body environments displaying critical behavior have been recently investigated in great detail, in order to study the sensitivity of decoherence to environmental dynamics (see, e.g., [13,15]). Close to a quantum critical point the environment becomes increasingly slower (a phenomenon known as critical slowing down). We analyze the decoherence process of a two level system (qubit) coupled to an environment modeled as a one-dimensional lattice of spins interacting through an Ising-type coupling. We focus on the suppression of qubit decoherence through a bang-bang control procedure, and study how the occurrence of a quantum phase transition (QPT) in the bath modifies the effectiveness of the control procedure. Our analysis is focused on the behavior of the Loschmidt echo (LE) [28], whose study has given new insights into the decoherence process of quantum spin chains. We discuss the application of a pulse train to the

qubit and show its effectiveness in quenching qubit dephasing, especially at the critical point.

The paper is organized as follows. In Sec. II we introduce the model and pertinent notation. The control procedure, based on a sequence of pulses that repeatedly flip the state of the system, is described in Sec. III, where we also derive an explicit expression for the LE in the presence of such control. We then provide a detailed analysis of its effects in the limiting cases of a single qubit-bath link (Sec. III A) and a spin-star model (Sec. III B). Finally, in Sec. IV we discuss our results. In the appendixes we provide an analytical formula for evaluating the LE in the presence of control (Appendix A), we perform a perturbative analysis in the pulse frequency of the LE (Appendix B), and discuss in detail a closed-form formula for the LE in the spin-star model (Appendix C).

II. MODEL AND NOTATION

We consider a two level quantum system S (qubit) coupled to an interacting spin bath E (environment), comprising a linear chain of N spin-1/2 particles, modeled by a transverse field Ising model. The Hamiltonian reads

$$\mathcal{H}_0 = \mathcal{H}_S + \mathcal{H}_E + \mathcal{H}_{\text{int}}, \tag{1}$$

where $\mathcal{H}_S$ and $\mathcal{H}_E$ are the free Hamiltonians of S and E as follows:

$$\mathcal{H}_S = -\frac{\omega_0}{2}(1 - \tau^z) = -\omega_0|\downarrow\rangle\langle\downarrow|, \tag{2}$$

$$\mathcal{H}_E = -J\sum_{j=1}^{N}(\sigma_j^x\sigma_{j+1}^x + \lambda\sigma_j^z). \tag{3}$$

Here σ_i^α and τ^α (with $\alpha = x, y, z$) indicate, respectively, the Pauli matrices of the ith spin of the chain E and of the qubit S, whose ground and excited states are denoted by $|\uparrow\rangle$ and $|\downarrow\rangle$. In this work we will use periodic boundary conditions, therefore we assume $\sigma_{N+1}^\alpha \equiv \sigma_1^\alpha$. The constants J and λ are the interaction strength between neighboring spins of the bath and an external transverse magnetic field, respectively (in the following, the energy and the time scale are taken in units of J, therefore, when not specified, we will implicitly assume $J = 1$). We suppose that the system is coupled to a given number of bath spins [15] as follows:

$$\mathcal{H}_{\text{int}} = -\epsilon|\downarrow\rangle\langle\downarrow| \otimes \sum_{j=j_1}^{j_m}\sigma_j^z, \tag{4}$$

where ϵ is the coupling constant and m is the number of environmental spins to which S is coupled. The LE can be calculated for a generic sequence $\{j_1, \ldots, j_m\}$ of system-bath links. In the following, however, we consider the cases $m = 1$ and $m = N$. We expect that the generic case will be a quantitative interpolation between these two extremes but no new qualitative features should emerge.

With the above choice of $\mathcal{H}_S$ and $\mathcal{H}_{\text{int}}$, the populations of the ground and excited state of the qubit do not evolve, since $[\tau^z, \mathcal{H}_0] = 0$, and we can study a model of pure dephasing.

As usual, we assume that the initial global state of the system is factorized as follows:

$$|\Psi(0)\rangle = (c_\uparrow|\uparrow\rangle + c_\downarrow|\downarrow\rangle) \otimes |G\rangle, \tag{5}$$

so that the qubit S is in a generic superposition of the ground and excited state, while the bath E is in its ground state (i.e., $|G\rangle$ is the ground state of the Hamiltonian $\mathcal{H}_E$). The evolution of such a state under the Hamiltonian (1) is dictated by the unitary operator $U_0 = e^{-i\mathcal{H}_0 t}$ and yields, at time t, the state

$$|\Psi(t)\rangle = c_\uparrow|\uparrow\rangle|\varphi_0(t)\rangle + c_\downarrow e^{i\omega_0 t}|\downarrow\rangle|\varphi_1(t)\rangle, \tag{6}$$

where $|\varphi_0(t)\rangle \equiv e^{-i\mathcal{H}_\uparrow t}|G\rangle$ and $|\varphi_1(t)\rangle \equiv e^{-i\mathcal{H}_\downarrow t}|G\rangle$ are the environment states evolved under an "unperturbed" and a "perturbed" Hamiltonian given, respectively, by

$$\mathcal{H}_\uparrow \equiv \mathcal{H}_E, \quad \mathcal{H}_\downarrow \equiv \mathcal{H}_E + \langle\downarrow|\mathcal{H}_{\text{int}}|\downarrow\rangle. \tag{7}$$

The density matrix of the qubit is $\rho = \text{Tr}_E|\Psi\rangle\langle\Psi|$. Its diagonal elements are constant, while off-diagonal elements decay in time as

$$\rho_{\downarrow\uparrow}(t) = \rho_{\downarrow\uparrow}(0)e^{i\omega_0 t}D(t), \tag{8}$$

with

$$D(t) = \langle\varphi_0(t)|\varphi_1(t)\rangle = \langle G|e^{i\mathcal{H}_\uparrow t}e^{-i\mathcal{H}_\downarrow t}|G\rangle. \tag{9}$$

The decoherence of the qubit is then fully characterized by the so-called *Loschmidt echo* $\mathcal{L}_0(t) \in [0, 1]$ of the environment as follows:

$$\mathcal{L}_0(t) \equiv |D(t)|^2 = |\langle G|e^{-i(\mathcal{H}_E + \langle\downarrow|\mathcal{H}_{\text{int}}|\downarrow\rangle)t}|G\rangle|^2. \tag{10}$$

The decay of the LE in the model (1)–(4) with $m = N$ (spin-star model) was first studied in detail in Ref. [13]; an extension to the more general case $m \neq N$, and for other spin baths—including the XY and Heisenberg models—can be found in Ref. [15]. It was pointed out that the echo decay is enhanced at criticality, due to the hypersensitivity to perturbations of the (time-evolved) unperturbed ground state $|\varphi_0(t)\rangle$. Indeed, at criticality the perturbation $\mathcal{H}_{\text{int}}$ is very effective at making the unperturbed state $|\varphi_0(t)\rangle$ orthogonal to $|\varphi_1(t)\rangle$, thus leading to a strong decay of the echo. Away from criticality, the perturbation is not so effective at orthogonalizing $|\varphi_0(t)\rangle$ and $|\varphi_1(t)\rangle$, whence the echo decays more slowly. In the following we investigate these effects when a control is also present. Details on how to evaluate the LE in both the absence and presence of such a control are given in Appendix A.

III. CONTROLLED DYNAMICS

Quantum dynamical decoupling procedures aimed at actively fighting decoherence hinge either on the action of frequent interruptions of the evolution or on the effect of a strong continuous coupling to an external field. These procedures are known to be physically and, to a large extent, mathematically equivalent [33]. Here we focus on one possible procedure, based on multipulse control [31]. Let us formally introduce the control scheme as

$$\mathcal{H}(\omega_0, t) = \mathcal{H}_0 + \mathcal{H}_P(\omega_0, t), \tag{11}$$

where $\mathcal{H}_P$ is an additional time-dependent Hamiltonian that causes spin flips of the qubit at regular time intervals through a monochromatic alternating magnetic field at resonance as follows:

$$\mathcal{H}_P(\omega_0, t) = \sum_n V^{(n)}(t)\{\cos[\omega_0(t - n\Delta t)]\tau_x$$

$$+ \sin[\omega_0(t - n\Delta t)]\tau_y\}. \tag{12}$$

Here $V^{(n)}(t)$ is constant and equal to V for the entire duration τ_P of the nth pulse (i.e., for $n\Delta t \leq t \leq n\Delta t + \tau_P$), Δt being the time interval between two consecutive pulses. In this work we only deal with π pulses, satisfying the condition $2V\tau_P = \pm \pi$, and suppose that V is large enough to yield almost instantaneous spin flips, i.e., we take $\tau_P \ll \Delta t$. Therefore, in the ideal limit of instantaneous kicks of infinite strength ($\tau_P \rightarrow 0$, $V \rightarrow \infty$ such that $V\tau_P = \pm \pi/2$), the effect of each pulse on the qubit is simply a flip, which is described by the operator

$$U_P = \pm i\tau^x. \tag{13}$$

The evolution of the initial state (5) under the Hamiltonian (11) in one spin-flip cycle [i.e., two flips, from time $t=0$ to time $t_1 = 2(\Delta t + \tau_P) \simeq 2\Delta t$] is dictated by the unitary operator

$$U_C \equiv e^{-2i\mathcal{H}\Delta t} = U_P U_0(\Delta t) U_P U_0(\Delta t), \tag{14}$$

and it is such that

$$|\Psi(2\Delta t)\rangle = -c_\uparrow e^{-i\omega_0\Delta t}|\uparrow\rangle e^{-i\mathcal{H}_\downarrow \Delta t} e^{-i\mathcal{H}_\uparrow \Delta t}|G\rangle$$

$$- c_\downarrow e^{-i\omega_0\Delta t}|\downarrow\rangle e^{-i\mathcal{H}_\uparrow \Delta t} e^{-i\mathcal{H}_\downarrow \Delta t}|G\rangle. \tag{15}$$

This is again a pure dephasing phenomenon, so that all relevant information is contained in the off-diagonal element (8) of the system reduced density matrix. The behavior of decoherence is then fully captured by the LE as follows:

$$\mathcal{L}(2\Delta t) = |\langle G|(e^{i\mathcal{H}_\downarrow \Delta t} e^{i\mathcal{H}_\uparrow \Delta t})(e^{-i\mathcal{H}_\downarrow \Delta t} e^{-i\mathcal{H}_\uparrow \Delta t})|G\rangle|^2. \tag{16}$$

In general, at a certain time $t = 2M\Delta t + \tilde{t}$, the evolution operator of the global system is given by

$$U = \begin{cases} U_0(\tilde{t})[U_C]^M & \text{if } \tilde{t} < \Delta t \\ U_0(\tilde{t} - \Delta t)U_P U_0(\Delta t)[U_C]^M & \text{if } \tilde{t} \geq \Delta t \end{cases}, \tag{17}$$

where $M = [\frac{t}{2\Delta t}]$, [ ] denotes the integer part, and $\tilde{t} \equiv t - 2M\Delta t$ is the residual time after M cycles. It is now easy to write down the LE at a generic time t as follows:

$$\mathcal{L}(t) = \begin{cases} |\langle G|(e^{i\mathcal{H}_\downarrow \Delta t} e^{i\mathcal{H}_\uparrow \Delta t})^M e^{i\mathcal{H}_\downarrow \tilde{t}} e^{-i\mathcal{H}_\uparrow \tilde{t}}(e^{-i\mathcal{H}_\downarrow \Delta t} e^{-i\mathcal{H}_\uparrow \Delta t})^M|G\rangle|^2 & \text{if } \tilde{t} < \Delta t \\ |\langle G|(e^{i\mathcal{H}_\downarrow \Delta t} e^{i\mathcal{H}_\uparrow \Delta t})^M e^{i\mathcal{H}_\downarrow \Delta t} e^{i\mathcal{H}_\uparrow \tilde{t}} e^{-i\mathcal{H}_\downarrow \tilde{t}} e^{-i\mathcal{H}_\uparrow \Delta t}(e^{-i\mathcal{H}_\downarrow \Delta t} e^{-i\mathcal{H}_\uparrow \Delta t})^M|G\rangle|^2 & \text{if } \tilde{t} \geq \Delta t \end{cases}. \tag{18}$$

An explicit formula for evaluating the LE, also in the presence of pulses, is given in Appendix A.

In the limit of short pulse intervals, and when t is an integer multiple of the duration of a single spin-flip cycle $t = 2M\Delta t$, one can show (see Appendix B) that Eq. (18) can be rewritten as

$$\mathcal{L}(t = 2M\Delta t) = |\langle G|e^{it\mathcal{H}_{\text{eff}}}|G\rangle|^2 + MO(\Delta t^2), \tag{19}$$

where

$$\mathcal{H}_{\text{eff}} \equiv i\frac{\Delta t}{2}[\mathcal{H}_\downarrow, \mathcal{H}_\uparrow] = i\frac{\Delta t}{2}[\langle\downarrow|\mathcal{H}_{\text{int}}|\downarrow\rangle, \mathcal{H}_E] \tag{20}$$

is an effective Hamiltonian. By noting that $\langle\downarrow|\mathcal{H}_{\text{int}}|\downarrow\rangle = -\epsilon \sum_{j=j_1}^{j_m} \sigma_j^z$, we have, for arbitrary λ,

$$\mathcal{H}_{\text{eff}} = i\epsilon_{\text{eff}}\left[\sum_{j=j_1}^{j_m} \sigma_j^z, \sum_{j=1}^N \sigma_j^x \sigma_{j+1}^x\right], \tag{21}$$

where

$$\epsilon_{\text{eff}} \equiv \epsilon J\frac{\Delta t}{2}. \tag{22}$$

This is the renormalized system-bath coupling constant in the presence of multipulse control. We notice that $\mathcal{H}_{\text{eff}}$ does not depend on λ (which would appear at $O(\Delta t^3)$ through the double commutator $[[\mathcal{H}_\downarrow, \mathcal{H}_\uparrow], \mathcal{H}_\uparrow]$). Therefore, in the small Δt limit, the criticality of the model can manifest itself only through $|G\rangle$ in the LE expression (19).

In the next two subsections we turn to a numerical study of the LE for the cases of a qubit coupled to one spin of the chain [$m=1$ in Eq. (4)], and the spin-star model [$m=N$ in Eq. (4)].

A. Qubit coupled to a single bath spin

When $m=1$, the system-bath Hamiltonian of Eqs. (1)–(4) can be rewritten as

$$\mathcal{H}_0 = -|\downarrow\rangle\langle\downarrow|(\omega_0 + \epsilon\sigma_1^z) - J\sum_{j=1}^N (\sigma_j^x \sigma_{j+1}^x + \lambda\sigma_j^z), \tag{23}$$

and corresponds to a situation in which the qubit is directly coupled to only one spin of an Ising chain with periodic boundary conditions (the coupled bath-spin qubit is assumed

for simplicity and with no loss of generality to be the first one). In Fig. 1 we show the behavior of the LE in Eq. (18) as a function of time, for different values of the pulse frequency Δt. The three panels refer to different values of the transverse magnetic field λ; the thick dashed lines represent the case $\mathcal{L}_0(t)$ with no external control [$\Delta t \rightarrow \infty$ in Eq. (18), or simply Eq. (10)]. Here the environment consists of $N=100$ Ising spins, and the system-bath coupling has been set at $\epsilon=0.25$.

We notice a very different behavior as λ is varied. Away from criticality [i.e., for $\lambda=0.5$, Fig. 1(a), and $\lambda=1.5$, Fig. 1(c)] the LE in absence of control quickly reaches its asymptotic (saturation) value $\mathcal{L}_\infty$, as indicated by the dashed black lines. Very fast control pulses do improve the situation, but only in the sense that this asymptotic value becomes slightly closer to unity. In contrast, slow pulses make the situation much worse: when $J\Delta t$ is larger than a certain value, the pulses act as an additional source of noise and, as a consequence, the coherence decays (exponentially). On the other hand, when the chain is critical [$\lambda=1$, Fig. 1(b)] and there is no control, the LE decays (albeit only logarithmically [15]), as can be seen from the dashed curve. In this case the pulses can be very effective, as a control procedure: when $J\Delta t \lesssim 0.375$ decay is suppressed. Again, when Δt exceeds this threshold, decay is enhanced. This situation is reminiscent of the transition between a quantum Zeno and an inverse Zeno effect [40].

In Fig. 2 we show the values of the LE at a fixed time t^* (we performed an average of $\mathcal{L}(t)$ for $Jt \in [Jt^*-5, Jt^*+5]$ in order to eliminate fast oscillations) as a function of Δt. The different curves are obtained for different values of the transverse field λ. We set $Jt^*=25$ so that (i) in the absence of pulse control and for noncritical λ, $\mathcal{L}_0$ has already reached its saturation value $\mathcal{L}_\infty$; (ii) at criticality, the minimum of $\mathcal{L}_0(t)$ is found exactly at $Jt^* \simeq N/4$ (in this case $N=100$) [15]. In the panel (b), bars denote the corresponding value of $\mathcal{L}_0(t^*)$ without external control.

The behavior at large pulse intervals Δt is nontrivial and rather interesting: we note that the echo has a minimum and has an almost complete recovery, and that the LE for $\lambda=0.5$ rises higher than for $\lambda=0.9,1,1.1$. The large Δt regime is nonperturbative (in the sense of the perturbation theory of Sec. III and Appendix B). Nevertheless, the rise of the LE for large Δt can be understood as being due to the fact that, after a short transient time $\bar{t}$, the LE *without control* saturates around a constant value (see the black dashed curves in the insets of Fig. 1, or Ref. [15]). Therefore, if the pulse frequency is such that $\Delta t > \bar{t}$, the effect of the bang-bang control procedure will be progressively reduced as Δt grows, until, in the limit $\Delta t \rightarrow +\infty$, it will completely disappear. In other words, the detrimental effect of the control for large Δ is offset by the gradual diminishing of its effect as Δt grows, which allows the LE to recover to its saturation value. Moreover, as the insets of Fig. 1 show, for $\lambda=1.5$ the saturation is truly at a constant value; for $\lambda=0.5$ the saturation is an oscillation around a constant value; at criticality ($\lambda=1$) there is a logarithmic decay of the LE, but for a finite system size this decay will eventually stop and revivals of quantum coherence will appear. The oscillation at $\lambda=0.5$ explains why this curve rises higher than the other curves in Fig. 2(a); at a time $t^* \approx 1.5$, the uncontrolled LE in Fig. 1 at $\lambda=0.5$ is larger than for other values of λ.
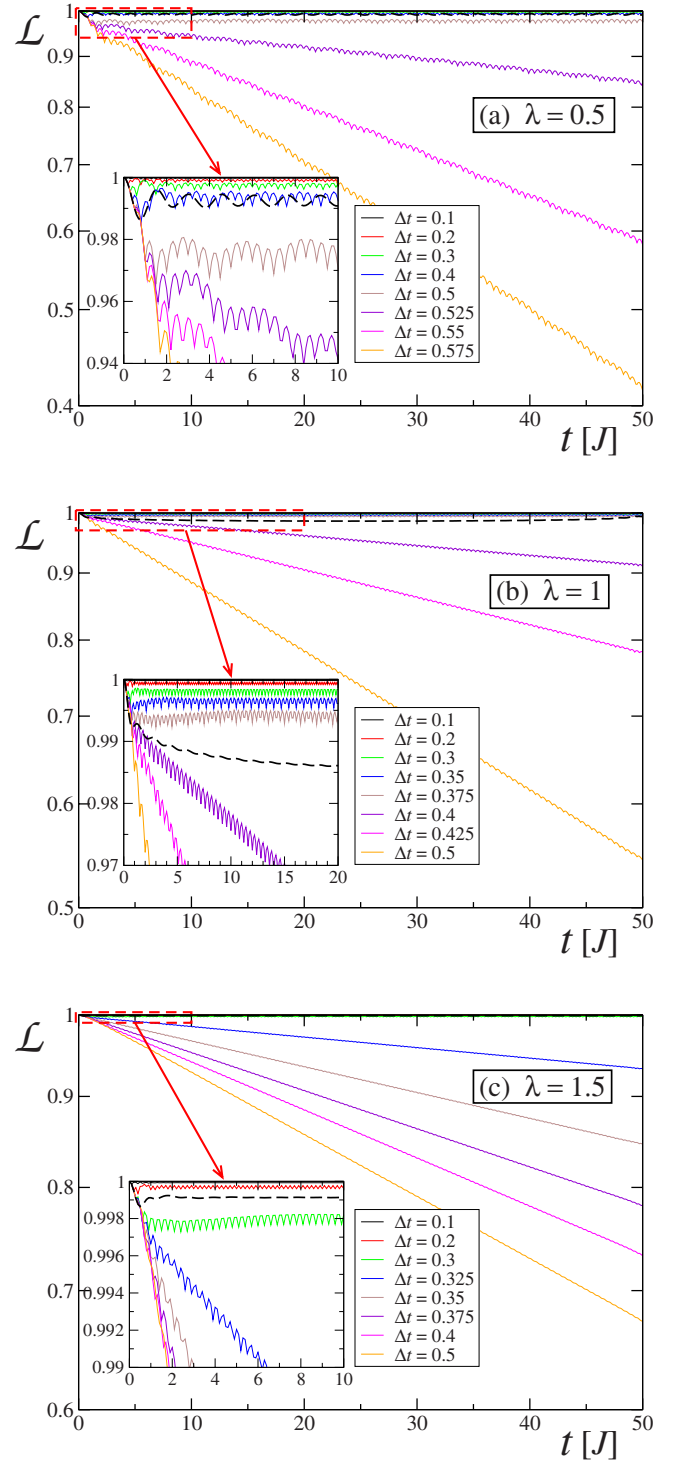

FIG. 1. (Color online) Loschmidt echo as a function of time for a qubit coupled to a $N=100$ spin Ising chain, with $\epsilon=0.25$. Panels stand for different values of the transverse field: (a) $\lambda=0.5$, (b) $\lambda=1$, (c) $\lambda=1.5$; the various curves in each panel are for decreasing pulse intervals Δt, from top to bottom. Insets: magnification at small times t (axes units are the same as in the main panels); notice that, when $\lambda=1$, frequent pulses suppress decay for $J\Delta t \lesssim 0.375$ (here and in the following figures Δt values are expressed in units of J).
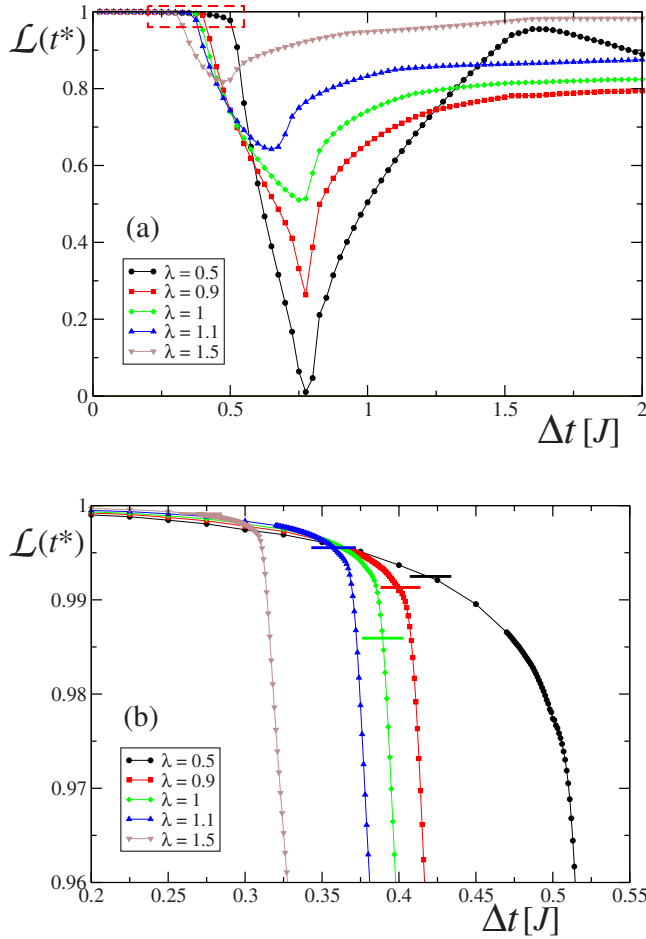
FIG. 2. (Color online) (a) LE as a function of the pulse frequency Δt at a given time t^*, for different λ. (b) Magnification of panel (a) in the highlighted zone; the bars denote the corresponding values of $\mathcal{L}_0(t^*)$ without pulsing. Here we set $Jt^*=25$, $N=100$, and $\epsilon=0.25$.

FIG. 3. (Color online) [(a) and (b)] LE at a fixed time t^* as a function of the transverse field, for different values of Δt. [(c) and (d)] Rescaled LE, $\mathcal{L}(t^*)/\mathcal{L}_0(t^*)$. Notice the widely different scales in the y axes of (a) and (c) (small Δt), with respect to (b) and (d) (large Δt). Here we set $Jt^*=25$, $N=100$, and $\epsilon=0.25$.

The panels (a)–(b) of Fig. 3 display $\mathcal{L}(t^*)$ as a function of λ for different values of Δt. In panels (c)–(d) we plot the rescaled quantities, obtained by dividing $\mathcal{L}(t^*)$ by the corresponding value in the absence of pulse control $\mathcal{L}_0(t^*)$. The LE has a maximum not at $\lambda=1$, but at $\lambda>1$, while at $\lambda=1$ there is an inflexion point. At criticality, the rescaled LE displays a cusp. The cusp disappears at $J\Delta t\gtrsim0.375$ in agreement with Fig. 1(b), where we observed, at the same value of Δt, an increase of the LE when the control is present. A qualitative explanation of this phenomenon is straightforward: for short time pulses, the renormalized coupling constant ϵ_{eff} in Eq. (22) and therefore the LE, are only weakly dependent on λ at leading order in the perturbative expansion. In contrast, the free echo $\mathcal{L}_0$ has a downward cusp [15] [present also in Fig. 4(a) for the spin-star case]. The ratio must therefore display an upward cusp, as seen in Fig. 3. Another way to state this explanation is the following. For sufficiently small values of Δt the bang-bang protocol succeeds at effectively eliminating the environment action. The only remnant of criticality is then the weak signature of an inflexion point seen in Fig. 3(a). The echo of the uncon-
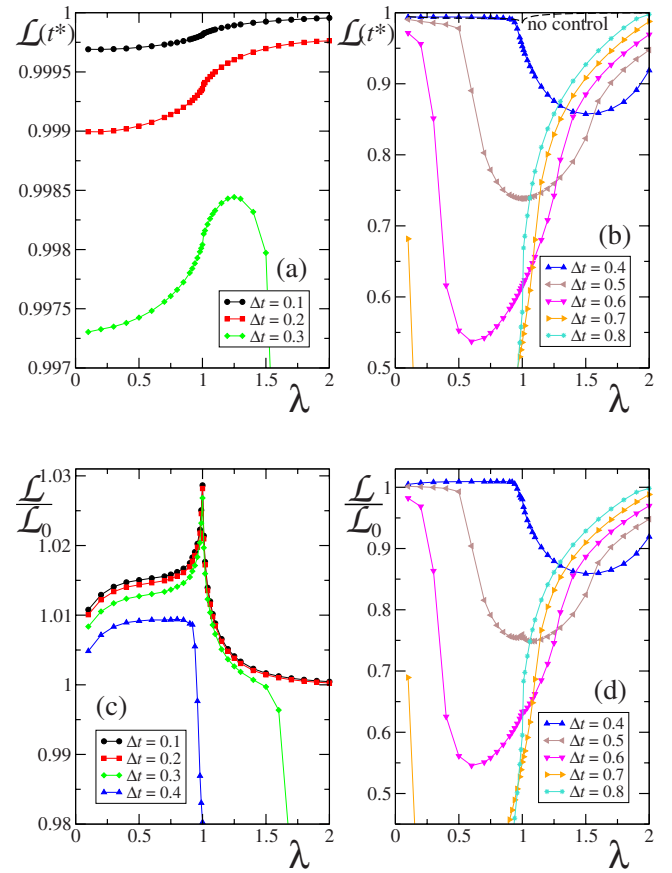
trolled system, however, is hypersensitive to criticality, as indicated by the cusp. On the other hand, when Δt is too large [Figs. 3(b)–3(d)], the bang-bang protocol fails at removing the coupling of the qubit to the environment, and the controlled and uncontrolled echoes behave similarly.

There are other interesting features in Fig. 3. Panels (a)–(b) show that the LE rises for sufficiently large λ, and (c)–(d) show that the ratio between the decoupled and free echoes approaches unity for large λ. This can be understood as being due to the dominance of the uniform magnetic field term $\lambda\Sigma_{j=1}^N\sigma_j^z$ over the transverse Ising term $\Sigma_{j=1}^N\sigma_j^x\sigma_{j+1}^x$ in Eq. (23). Indeed, in the limit of large λ, this means that $\mathcal{H}_\downarrow\simeq\mathcal{H}_E$ [recall Eq. (7)], so that $[\mathcal{H}_\uparrow,\mathcal{H}_\downarrow]\simeq0$ and the LE $\simeq1$ by Eqs. (9) and (10). Thus, at large λ, decoupling is not needed to obtain a large LE.

More interesting is the monotonic rise of the LE visible in panel (a) as a function of λ for $J\Delta t=0.1,0.2$, in contrast to the maximum around $\lambda\sim1.25$ for $J\Delta t=0.3$. Indeed, panel (c) shows that decoupling makes the situation worse for $J\Delta t=0.3$ and $\lambda\gtrsim1.25$, and a similar trend continues in panels (b)–(d). Thus, in our model decoupling is fully effective (i.e., for all values of λ) for $J\Delta t\lesssim0.2$.
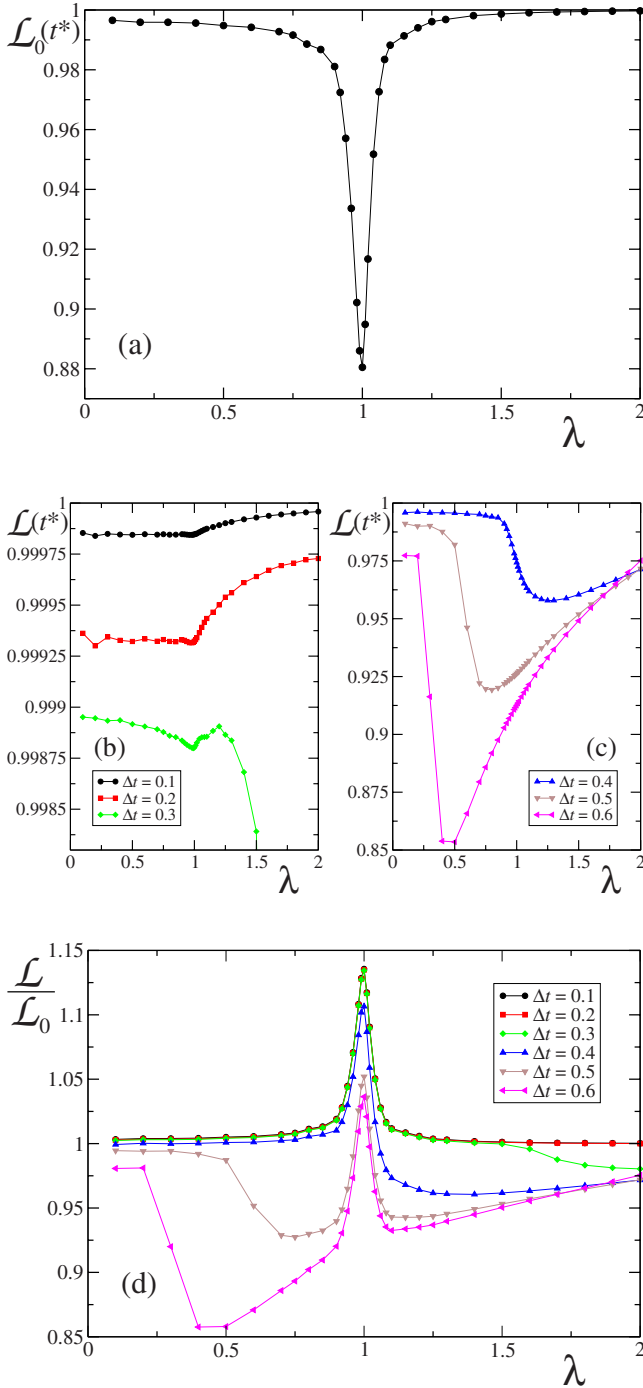
FIG. 4. (Color online) LE for the spin-star model (the parameters of the simulation are $Jt^* = 10$, $\epsilon = 0.01$, and $N = 300$). (a) Dependence on λ of the LE without external control at fixed time. [(b) and (c)] LE in the presence of pulsed control with frequency Δt. (d): Renormalized controlled LE.

B. Spin-star model

The "spin-star" model corresponds to the case when the qubit is equally coupled to all the spins of the chain [$m = N$ in Eq. (4)]. This situation is opposite to the one considered in the previous section. Interestingly, in this limit the model is almost solvable. The system-bath Hamiltonian of Eq. (1) reads

$$\mathcal{H}_0 = -\omega_0 |\downarrow\rangle\langle\downarrow| - J \sum_{j=1}^{N} \left[\sigma_j^x \sigma_{j+1}^x + \left(\lambda + \frac{\epsilon}{J}\right) \sigma_j^z \right]. \quad (24)$$

We first notice that $\mathcal{H}_\downarrow(\lambda) = \mathcal{H}_\uparrow(\tilde{\lambda}) \equiv \mathcal{H}_E(\tilde{\lambda})$, where $\tilde{\lambda} = \lambda + \epsilon/J$. Therefore, both the perturbed and the unperturbed Hamiltonians describe an Ising model with a uniform transverse field, and can be diagonalized analytically by means of a standard Jordan-Wigner-Fourier transformation, followed by a Bogoliubov rotation. Details on how to evaluate the LE of Eq. (19) for a spin-star model can be found in Appendix C, where we show that

$$\langle G | e^{it\mathcal{H}_{\text{eff}}} | G \rangle = \prod_{k>0} \cos(8t\epsilon_{\text{eff}}\Delta_k), \quad (25)$$

with $\Delta_k = \sin(2\pi k/N)$ and ϵ_{eff} defined in Eq. (22). In the limit of small ϵ_{eff}, while keeping t finite, we can approximate this as

$$\langle G | e^{it\mathcal{H}_{\text{eff}}} | G \rangle \simeq \prod_{k>0} e^{-1/2(8t\epsilon_{\text{eff}}\Delta_k)^2} = e^{-\Gamma/2(t\epsilon_{\text{eff}})^2}, \quad (26)$$

where we have defined

$$\Gamma \equiv 64 \sum_{k>0} \Delta_k^2. \quad (27)$$

We notice that the dependence on λ in Eq. (26) has entirely disappeared. This means that, to leading order in the pulse interval Δt, dynamical decoupling is not sensitive to criticality [Eq. (19) for the LE and Eq. (25) lead to $\mathcal{L} \approx e^{-\Gamma(t\epsilon_{\text{eff}})^2}$; we explicitly checked that, for small Δt and at short times, this formula exactly reproduces the data obtained from numerical simulations, which are completely insensitive to λ in that regime]. This is consistent with the data shown in Fig. 4. In panel (a) we see the behavior of the LE in absence of control; we notice a slight dip at $\lambda = 1$. In panels (b)–(c) we study the LE for various Δt; we observe strong similarities with Fig. 3, in particular, the weak dependence of the LE on λ for very small Δt. In panel (d) the rescaled LE again displays a cusp.

It is remarkable how similar the results are for $m = 1$ (qubit coupled to a single spin of the chain) and $m = N$ (spin-star model). The consistency of these results and the analogies between these two opposite situations lead us to conclude that general features of the decoherence of the qubit under bang-bang control are largely independent of the number of chain spins coupled to it, at least when the chain is close to criticality.

IV. DISCUSSION AND CONCLUSIONS

We have studied the efficacy of pulsed control of a qubit when it is coupled to a spin bath. It is well known that, without control pulses, the qubit decoheres particularly fast in the vicinity of the critical point. The reason for this is that the evolution takes the initial state $|\Psi(0)\rangle$, in the form of Eq. (5), into a superposition of the type $|\uparrow\rangle|\varphi_0(t)\rangle + |\downarrow\rangle|\varphi_1(t)\rangle$ and the two bath states become rapidly orthogonal near the critical point. The application of decoupling pulses to the qubit removes the dependence of decoherence on the criticality of the environment. On the other hand, we also found a regime

(larger interval Δt between pulses) such that the control can increase the effects of decoherence. Away from criticality the perturbation is not as effective at orthogonalizing $|\varphi_0(t)\rangle$ and $|\varphi_1(t)\rangle$, leading to a slow decay of the echo and to relatively less effective control. Therefore, we can conclude that, in general, decoupling is *relatively* more effective near the critical point, since there it results in the largest enhancement of coherence.

From the quantum information processing perspective, there is another positive message in these results: suppose we are trying to preserve the coherence of a qubit in the presence of a spin bath. Without decoupling we know that the spin decoheres particularly fast in the vicinity of the critical point. Therefore, not knowing whether we are close to criticality when trying to operate a quantum computer coupled to a spin bath, is a problem. But in light of the results presented here, it follows that application of dynamical decoupling pulses removes this concern: for sufficiently frequent pulses, decoupling works independently of the value of the system-bath coupling λ, so closeness to criticality does not matter.

Our analytical and numerical calculations suggest that these results seem to be largely independent of the details of the model of qubit-environment coupling. Indeed, we have considered two extreme situations (qubit coupled to a single spin of the chain and qubit coupled to all spins in the chain), and obtained the same qualitative behavior.

Finally, a comparison of different control strategies (Zeno effect, decoupling pulses, and strong continuous coupling) [41] has shown that, although these procedures are physically equivalent, there are important practical differences among them. Future attention will be directed toward the exploration of these similarities and differences in the context of coupling of a qubit to a critical system.

APPENDIX A

We explain here how to evaluate the LE for the Hamiltonian in Eq. (1), and then extend some of these results to the case of pulsed control, Eq. (11). This technique can be easily generalized to the case of an *XY* spin bath, as has been done in Ref. [15].

By means of the Jordan-Wigner transformation

$$\sigma_j^+ = c_j^\dagger \exp\left(i\pi \sum_{k=1}^{j-1} c_k^\dagger c_k\right), \quad \sigma_j^z = 2c_j^\dagger c_j - 1, \quad \text{(A1)}$$

we first map the Hamiltonians $\mathcal{H}_\downarrow$ and $\mathcal{H}_\uparrow$ of the spin bath onto a free-fermion model that can be expressed in the form

$$\mathcal{H}_{\uparrow/\downarrow} = \frac{1}{2}\boldsymbol{\Psi}^\dagger \mathbf{C}\boldsymbol{\Psi}, \quad \text{(A2)}$$

where $\boldsymbol{\Psi}^\dagger = (c_1^\dagger \cdots c_N^\dagger c_1 \cdots c_N)$ (c_i being the corresponding spinless fermion operators) and

$$\mathbf{C} = \sigma^z \otimes \mathbf{A} + i\sigma^y \otimes \mathbf{B} \quad \text{(A3)}$$

is a tridiagonal block matrix with

$$A_{j,k} = -J(\delta_{k,j+1} + \delta_{j,k+1}) - 2(\lambda + \epsilon_j)\delta_{j,k}, \quad \text{(A4)}$$

$$B_{j,k} = -J(\delta_{k,j+1} - \delta_{j,k+1}), \quad \text{(A5)}$$

such that $\epsilon_j = 0$ for $\mathcal{H}_\uparrow$, while $\epsilon_j = \epsilon \delta_{j,j_m}$ for $\mathcal{H}_\downarrow$. The LE can then be evaluated exactly, by rewriting it in terms of the determinant of a $2N \times 2N$ matrix [15] as follows:

$$\mathcal{L}_0(t) = |\det(\mathbb{1} - \mathbf{r} + \mathbf{r}e^{i\mathbf{C}_\downarrow t})|, \quad \text{(A6)}$$

where $\mathbf{r}$ is a matrix whose elements $r_{i,j} = \langle\Psi_i^\dagger\Psi_j\rangle$ are the two-point correlation functions of the spin chain, evaluated in the ground state of the Hamiltonian $\mathcal{H}_\uparrow$. Equation (A6) can be obtained from the following trace formula [42]:

$$\text{Tr}(e^{\Gamma(A)}e^{\Gamma(B)}) = \det(\mathbb{1} + e^{\mathbf{A}}e^{\mathbf{B}}), \quad \text{(A7)}$$

where $\Gamma(A) = \sum_{i,j}\mathbf{A}_{ij}a_i^\dagger a_j$ and $a_i^\dagger, a_i$ are the creation and annihilation operators for a fermion particle state i.

In the presence of pulsed control, in analogy with the free evolution case, Eq. (10), we can rewrite the formula for the LE in Eq. (18) in terms of the determinant of a $2N \times 2N$ matrix. Indeed the trace formula (A7) is straightforwardly generalized to products of more than two operators [42] by using the following identity:

$$\langle\psi_0|e^{-i\mathcal{H}_1 t}e^{-i\mathcal{H}_2 t}\cdots e^{-i\mathcal{H}_n t}|\psi_0\rangle$$
$$= \det(\mathbb{1} - \mathbf{r}_0 + \mathbf{r}_0 e^{-i\mathbf{C}_1 t}e^{-i\mathbf{C}_2 t}\cdots e^{-i\mathbf{C}_n t}), \quad \text{(A8)}$$

where we supposed that $\mathcal{H}_k = \sum_{i,j}[\mathbf{C}_k]_{ij}a_i^\dagger a_j$ and $\mathbf{r}_0 = \Gamma(\mathcal{N})$ with $\mathcal{N}$ occupation number operator [i.e., $(\mathbf{r}_0)_{ij} = \langle\psi_0|a_i^\dagger a_j|\psi_0\rangle$].

APPENDIX B

We evaluate here the leading order expansion of the LE in Eq. (18) in terms of the pulse interval Δt, in the limit of short pulses. To simplify the notations, let us define $A \equiv i\mathcal{H}_\downarrow$, $B \equiv i\mathcal{H}_\uparrow$, and $\varepsilon \equiv \Delta t$. We consider Eq. (18) at integer multiples of a spin-flip cycle, i.e., $t = 2M\Delta t$,

$$\mathcal{L}(t) = |\text{Tr}[|G\rangle\langle G|(e^{\varepsilon A}e^{\varepsilon B})^M(e^{-\varepsilon A}e^{-\varepsilon B})^M]|^2. \quad \text{(B1)}$$

Now recall the (approximate) Lie sum and product formulas

$$e^{\varepsilon A}e^{\varepsilon B} = e^{\varepsilon(A+B)} + O(\varepsilon^2), \quad \text{(B2)}$$

$$e^{\varepsilon A}e^{\varepsilon B}e^{-\varepsilon A}e^{-\varepsilon B} = e^{\varepsilon^2[A,B]} + O(\varepsilon^3). \quad \text{(B3)}$$

Using this we have

$$(e^{\varepsilon A}e^{\varepsilon B})^M(e^{-\varepsilon A}e^{-\varepsilon B})^M = (e^{\varepsilon A}e^{\varepsilon B})^{M-1}[e^{\varepsilon^2[A,B]} + O(\varepsilon^3)]$$
$$\times (e^{-\varepsilon A}e^{-\varepsilon B})^{M-1}$$
$$= (e^{\varepsilon A}e^{\varepsilon B})^{M-2}[e^{\varepsilon^2[A,B]}e^{\varepsilon A}e^{\varepsilon B} + O(\varepsilon^2)]$$
$$\times (e^{-\varepsilon A}e^{-\varepsilon B})^{M-1} + O(\varepsilon^3)$$
$$\times (e^{\varepsilon A}e^{\varepsilon B})^{M-1}(e^{-\varepsilon A}e^{-\varepsilon B})^{M-1}. \quad \text{(B4)}$$

Keeping terms only to leading order $O(\varepsilon^2)$ we can neglect the last line, yielding

$$(e^{\varepsilon A}e^{\varepsilon B})^M(e^{-\varepsilon A}e^{-\varepsilon B})^M = (e^{\varepsilon A}e^{\varepsilon B})^{M-2}[e^{\varepsilon^2[A,B]}e^{\varepsilon A}e^{\varepsilon B}e^{-\varepsilon A}e^{-\varepsilon B}$$

$$+ O(\varepsilon^2)e^{-\varepsilon A}e^{-\varepsilon B}](e^{-\varepsilon A}e^{-\varepsilon B})^{M-2}$$

$$= (e^{\varepsilon A}e^{\varepsilon B})^{M-2}[e^{2\varepsilon^2[A,B]} + O(\varepsilon^3) + O(\varepsilon^2)$$

$$\times(\mathbb{1} - \varepsilon(A + B))](e^{-\varepsilon A}e^{-\varepsilon B})^{M-2}$$

$$= (e^{\varepsilon A}e^{\varepsilon B})^{M-2}[e^{2\varepsilon^2[A,B]} + O(\varepsilon^2)\mathbb{1}]$$

$$\times(e^{-\varepsilon A}e^{-\varepsilon B})^{M-2}, \qquad (B5)$$

where in the last line we again neglected $O(\varepsilon^3)$ terms. Continuing in this manner we have

$$(e^{\varepsilon A}e^{\varepsilon B})^M(e^{-\varepsilon A}e^{-\varepsilon B})^M = e^{M\varepsilon^2[A,B]} + MO(\varepsilon^2)\mathbb{1}, \quad (B6)$$

which yields Eqs. (19) and (20).

APPENDIX C

Here we derive Eq. (25). We first notice that, in the spin-star case, both $\mathcal{H}_\uparrow(\lambda) \equiv \mathcal{H}_E(\lambda)$ and $\mathcal{H}_\downarrow(\lambda) \equiv \mathcal{H}_E(\tilde{\lambda})$ can be written in momentum space by using the Jordan-Wigner transformation (A1) followed by a Fourier transform, in this way,

$$\mathcal{H}_E(\lambda) = 2J\sum_{k>0}[\varepsilon_k(\lambda)(c_k^\dagger c_k + c_{-k}^\dagger c_{-k}) - i\Delta_k(c_k^\dagger c_{-k}^\dagger - c_{-k}c_k)],$$
$$(C1)$$

where $\varepsilon_k(\lambda) = \lambda - \cos(2\pi k/N)$ and $\Delta_k = \sin(2\pi k/N)$, and the sum over k runs from 1 to $N/2$.

The ground state of the Hamiltonian in Eq. (C1) is

$$|G(\lambda)\rangle = \underset{k>0}{\otimes}\left[\cos\left(\frac{\theta_k}{2}\right)|00\rangle_{k,-k} + i\sin\left(\frac{\theta_k}{2}\right)|11\rangle_{k,-k}\right],$$
$$(C2)$$

where $\theta_k = \arctan[\Delta_k/\varepsilon_k(\lambda)]$, and the kets refer to fermion occupation numbers in the two modes k and $-k$. Consider now the space

$$\mathbb{H}_k \otimes \mathbb{H}_{-k} = \mathrm{Sp}\{|00\rangle_{k,-k}, |01\rangle_{k,-k}, |10\rangle_{k,-k}, |11\rangle_{k,-k}\}.$$

Since the subspaces $\mathrm{Sp}\{|00\rangle_{k,-k}, |11\rangle_{k,-k}\}$ and $\mathrm{Sp}\{|01\rangle_{k,-k}, |10\rangle_{k,-k}\}$ are not coupled by $\mathcal{H}_E$, and since $|G(\lambda)\rangle$ lives in the former two-dimensional subspace, we can rewrite the Hamiltonian over the $\mathrm{Sp}\{|00\rangle_{k,-k}, |11\rangle_{k,-k}\}$ subspace, up to a constant, as

$$\mathcal{H}_E(\lambda) = 2J\sum_{k>0}[\varepsilon_k(\lambda)\Sigma_k^z + \Delta_k\Sigma_k^y] \equiv \sum_{k>0}\mathcal{H}_{E,k}(\lambda), \quad (C3)$$

where Σ_k^z and Σ_k^x generate an SU(2) algebra and are defined as

$$\Sigma_k^x = c_k^\dagger c_{-k}^\dagger + c_{-k}c_k, \qquad (C4)$$

$$\Sigma_k^y = -i(c_k^\dagger c_{-k}^\dagger - c_{-k}c_k), \qquad (C5)$$

$$\Sigma_k^z = c_k^\dagger c_k + c_{-k}^\dagger c_{-k} - 1. \qquad (C6)$$

The problem of evaluating $\langle G|e^{it\mathcal{H}_{\mathrm{eff}}}|G\rangle$ is now reduced to computing the action of the 2×2 matrix $[\mathcal{H}_\downarrow, \mathcal{H}_\uparrow]$ over the subspace $\mathrm{Sp}\{|00\rangle_{k,-k}, |11\rangle_{k,-k}\}$. We can rewrite the ground state as

$$|G(\lambda)\rangle = \underset{k>0}{\otimes}\left[\cos\left(\frac{\theta_k}{2}\right)|0\rangle_k + i\sin\left(\frac{\theta_k}{2}\right)|1\rangle_k\right] \equiv \otimes_{k>0}|G_k(\lambda)\rangle,$$
$$(C7)$$

where now $|0\rangle_k$ and $|1\rangle_k$ are the standard ± 1 eigenstates of Σ_k^z. Over this subspace, using the fact that $\mathcal{H}_\downarrow(\lambda) = \mathcal{H}_\uparrow(\lambda) + \epsilon\Sigma^z$, with $\Sigma^z = \sum_{j=1}^N \sigma_j^z$, we have that

$$\mathcal{H}_{\mathrm{eff}} = i\epsilon\frac{\Delta t}{2}[\Sigma^z, \mathcal{H}_\uparrow] = 4i\epsilon_{\mathrm{eff}}\sum_{k>0}\Delta_k[\Sigma_k^z, \Sigma_k^y]$$

$$= 8\epsilon_{\mathrm{eff}}\sum_{k>0}\Delta_k\Sigma_k^x \equiv \sum_{k>0}\mathcal{H}_{\mathrm{eff},k}. \qquad (C8)$$

Now,

$$\Sigma_k^x|G_k(\lambda)\rangle = \left[\cos\left(\frac{\theta_k}{2}\right)|1\rangle_k + i\sin\left(\frac{\theta_k}{2}\right)|0\rangle_k\right], \quad (C9)$$

so that

$$\langle G_k|\Sigma_k^x|G_k\rangle = 0, \qquad (C10)$$

and

$$\langle G_k|e^{it\mathcal{H}_{\mathrm{eff},k}}|G_k\rangle = \langle G_k|e^{8it\epsilon_{\mathrm{eff}}\Delta_k\Sigma_k^x}|G_k\rangle = \langle G_k|\cos(8t\epsilon_{\mathrm{eff}}\Delta_k)\mathbb{1}$$

$$- i\sin(8t\epsilon_{\mathrm{eff}}\Delta_k)\Sigma_k^x|G_k\rangle = \cos(8t\epsilon_{\mathrm{eff}}\Delta_k).$$
$$(C11)$$

Therefore

$$\langle G|e^{it\mathcal{H}_{\mathrm{eff}}}|G\rangle = \Pi_{k>0}\langle G_k|e^{it\mathcal{H}_{\mathrm{eff},k}}|G_k\rangle = \Pi_{k>0}\cos(8t\epsilon_{\mathrm{eff}}\Delta_k),$$
$$(C12)$$

which is Eq. (25).

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