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Bang–Bang Operations from a Geometric Perspective

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Strong, fast pulses, called "bang–bang" controls can be used to eliminate the effects of system-environment interactions. This method for preventing errors in quantum information processors is treated here in a geometric setting which leads to an intuitive perspective. Using this geometric description, we clarify the notion of group symmetrization as an averaging technique, provide a geometric picture for evaluating errors due to imperfect bang–bang controls and give conditions for the compatibility of BB operations with other controlling operations. This will provide additional support for the usefulness of such controls as a means for providing more reliable quantum information processing.

KEY WORDS: decoherence; bang–bang controls; quantum error correction; quantum computing.

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1. INTRODUCTION

Recently, controlling the evolution of a system by using strong, short pulses has been introduced as a new means for quantum error correction/prevention.[1–4] These operations have been termed "bang–bang" (BB) pulses[1] (a name derived from classical control theory[5]) "parity kicks",[2, 6] (for the special case of a sign changing operation), decoupling operations[7, 8] (since they can serve to decouple the system from environmental degrees of freedom), and symmetrization procedures[3] (which are associated with a group symmetrization/averaging). The advantage they have over the active and passive error correction procedures associated with quantum error correcting codes (QECCs)[9, 10] (also see Ref. 11 and references therein) and decoherence free subspaces (DFSs)[12–14] (or "noiseless subsystems"[15]; also

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see Ref. 16 and references therein) is the use of external pulses rather than requiring several physical qubits to encode one logical qubit. Since today's experiments use < 10 qubits, this may, for the time being, make BB controls a method of choice for small-scale quantum computer implementations. However, it is clear that the time constraints imposed by bang–bang operations on the system are severe[7] and may not be practical for eliminating noise altogether. Even if this technique cannot completely eliminate the noise, it can still be used to reduce noise.[1, 4, 6] This is important for possibly reducing error rates and thus extending computing time and/or the utility of QECCs and/or DFSs.

In order to take full advantage of the BB technique, the symmetrization operations and their effects must be made clear so that the benefit from the implementation can be readily determined. The work put forth here will aid in the analysis of the results of BB operations by providing an explicit geometric representation for the group-algebraic elements describing such interactions. This geometric picture also has the advantage of describing the error between a desired and a modified evolution. In addition, for two-state systems (qubits), we recover a familiar Bloch sphere representation and this provides us with an intuitive understanding of BB/symmetrization operations.

Section 2 contains a brief survey of quantum BB controls for quantum error suppression. In Sec. 3 we present the general geometric perspective for the BB controls. Section 4 provides an accompanying geometric perspective for evaluating the errors of a set of BB controls and Sec. 5 shows how these techniques may be used in practice to rederive the subgroup condition (see Sec. 2) and extend the viewpoint of this as an averaging technique. A further example (Subsection 5.2) demonstrates a general condition (Subsection 5.3) for the compatibility of BB controls with other controlling interactions (for example gating sequences).

2. DECOUPLING BY SYMMETRIZATION

The process of decoupling by symmetrization, counteracts decoherence by applying sequences of frequent pulses.[1, 2, 4] The time scales are crucial: roughly speaking, one needs to perform a complete cycle of symmetrization operations in a time shorter than the bath correlation time. An elegant group theoretical treatment shows that if the applied pulses are unitary transformations forming a finite-dimensional group, then the application of that series of pulses amounts to an average (symmetrization) over this group.[3, 7, 8, 17, 18] We briefly review this theory.

The general evolution of a system and a bath coupled to it can be written in the form

$$H = H_S \otimes \mathbb{1}_B + \mathbb{1}_S \otimes H_B + \sum_\gamma S_\gamma \otimes B_\gamma \tag{2.1}$$

where H_S acts on the system alone, H_B acts on the bath alone, and $H_I \equiv \sum_\gamma S_\gamma \otimes B_\gamma$ is the interaction part of the Hamiltonian composed of traceless operators S_γ (B_γ) which act on the system (bath). The objective of the BB procedure is to modify this evolution.

A set of symmetrization or BB operations can be chosen such that they form a discrete (finite order) subgroup of the full unitary group of operations on the Hilbert space of the system. Denote this subgroup $\mathcal{G}$ and its elements g_k, $k = 0, 1, \ldots, |\mathcal{G}| - 1$, where $|\mathcal{G}|$ is the order of the group. The cycle time is $T_c = |\mathcal{G}|\Delta t$, where $|\mathcal{G}|$ is now also the number of symmetrization operations, and Δt is the time that the system evolves freely between operations under $U_0 = \exp(-iHt)$. The symmetrized evolution is given by

$$U(T_c) = \prod_{k=0}^{|\mathcal{G}|-1} g_k^\dagger U_0(\Delta t) g_k \equiv e^{-iH_{\text{eff}}T_c}. \tag{2.2}$$

H_{eff} denotes the resulting effective Hamiltonian. Since the approximation requires very strong, short pulses to be implemented in a sequence, they have been termed bang–bang (BB) operations (we will use symmetrization and BB operations interchangeably). In this (BB) limit

$$H \mapsto H_{\text{eff}} = \frac{1}{|\mathcal{G}|} \sum_{k=0}^{|\mathcal{G}|-1} g_k^\dagger H g_k \equiv \Pi_{\mathcal{G}}(H) \tag{2.3}$$

where H_{eff} is the desired Hamiltonian (without noise). The map $\Pi_{\mathcal{G}}$ is the projector into the centralizer, $Z(\mathcal{G})$, defined as

$$Z(\mathcal{G}) = \{X | \, [X, g_k] = 0, \ \forall g_k \in \mathcal{G}\} \tag{2.4}$$

It is clear that $\Pi_{\mathcal{G}}$ commutes with all g_k so that, if our group is generated by $\{\mathbb{1}, H_S, S_\gamma\}$, the system is effectively decoupled from its environment. The *control algebra* is the algebra generated by the set $\{g_k\}$. Even if the symmetrization is performed under less than ideal conditions, it can still reduce the noise in the system.[1, 4, 6]

3. GEOMETRIC INTERPRETATION OF THE EFFECT OF BB OPERATIONS

Now consider a set of unitary operators $\{U_k\}$, $U_0 \equiv \mathbb{1}_S$, as an explicit realization of the subgroup $\mathcal{G}$ and choice of our set of BB operations. Then the following condition must be satisfied for an evolution generated by the effective Hamiltonian:

$$H_{\text{eff}} = \frac{1}{|\mathcal{G}|} \sum_{k=0}^{|\mathcal{G}|-1} U_k^\dagger H U_k \tag{3.1}$$

Note that $H_{\text{eff}} = 0$ is the case of *storage*. Considering Eq. (2.1), we can always include the terms $H_S \otimes \mathbb{1}_B$ in H_I. We do not include the identity component $\mathbb{1}_S \otimes \mathbb{1}_B$ since it only gives rise to an overall phase. Thus H and H_{eff} are traceless. Let us now introduce $N \equiv n^2 - 1$ traceless, Hermitian generators $\{\lambda_i\}_{i=1}^N$ of $SU(n)$. These generators are closed under commutation and span the space of traceless Hermitian matrices. For $SU(2)$, the Pauli matrices are commonly used; for $SU(3)$, the Gell–Mann matrices, and for higher dimensions, one may use a direct generalization of the Gell–Mann matrices. For dimensions that are a power of two it is often convenient to use the Pauli group (tensor products of Pauli matrices). The $\{\lambda_i\}$ satisfy trace-orthogonality,

$$\text{Tr}(\lambda_i \lambda_j) = M \delta_{ij}, \tag{3.2}$$

where M is a normalization constant (often taken to be 2 for Lie algebras or n for $n \times n$ matrices). Expanding the system operators in terms of the $\{\lambda_i\}$ yields:

$$S_\gamma = \sum_i a_{i\gamma} \lambda_i \tag{3.3}$$

where the expansion coefficients are

$$a_{i\gamma} = \frac{1}{n} \text{Tr}(\lambda_i S_\gamma) \tag{3.4}$$

Using this, H can be written as as follows:

$$H = \sum_\gamma S_\gamma \otimes B_\gamma = \sum_\gamma \sum_{i=1}^N a_{i\gamma} \lambda_i \otimes B_\gamma \equiv \sum_\gamma (\vec{a}_\gamma \cdot \vec{\lambda}) \otimes B_\gamma \tag{3.5}$$

Here $\vec{a}_\gamma$ and $\vec{\lambda}$ are vectors of length N. In this representation, used extensively in,[19] an $n \times n$ Hamiltonian, H, is a vector with coordinates $\vec{a}_\gamma$ for

each error γ in an N-dimensional vector space spanned by the $\{\lambda_i\}$ as basis vectors, with ordinary vector addition and scalar multiplication.

As is well-known, there is a homomorphic mapping between the Lie groups $SU(2)$ and $SO(3)$.[20] This mapping is generalized as follows for $SU(n)$ and a subgroup of the rotation group $SO(N)$:

$$U_k^\dagger \lambda_i U_k = \sum_{j=1}^{N} R_{ij}^{(k)} \lambda_j \tag{3.6}$$

where the matrix $R^{(k)} \in SO(N)$, the adjoint representation of $SU(n)$.

The BB operation [Eq. (3.1)] may now be viewed as a weighted sum of rotations of the (adjoint) vectors $\vec{a}_\gamma$. To see this, first let

$$\vec{a}_\gamma^{(k)} = R^{(k)} \vec{a}_\gamma$$

This represents the rotation by $R^{(k)}$ of the coordinate vector $\vec{a}_\gamma$. Next average over all rotations:

$$\vec{a}'_\gamma = \frac{1}{|\mathcal{G}|} \sum_{k=0}^{|\mathcal{G}|-1} \vec{a}_\gamma^{(k)} \tag{3.7}$$

Finally, note that the effective Hamiltonian, after the BB operations, can be rewritten as:

$$H_{\text{eff}} = \frac{1}{|\mathcal{G}|} \sum_{k=0}^{|\mathcal{G}|-1} U_k^\dagger H U_k = \sum_\gamma (\vec{a}'_\gamma \cdot \vec{\lambda}) \otimes B_\gamma \tag{3.8}$$

Equation (3.9) [compare to Eq. (3.5)] is our desired geometric representation of BB operations. Their effect is to simply transform, for each error γ, the coordinates $\vec{a}_\gamma$ to $\vec{a}'_\gamma$. It is simplest to interpret this in the case of storage, where we seek BB operations such that $H_{\text{eff}} = 0$. Since the errors can be decomposed in the linearly independent basis set indexed by γ, each term $\vec{a}'_\gamma \cdot \vec{\lambda}$ must vanish separately. Furthermore, since the λ_i are independent this can only be satisfied if $\vec{a}'_\gamma = \vec{0}$ for each γ. This means that

$$\vec{a}'_\gamma = \left(\frac{1}{|\mathcal{G}|} \sum_k R^{(k)} \right) \vec{a}_\gamma = \vec{0} \tag{3.10}$$

i.e., the sum of all rotations applied to the original coordinate vector $\vec{a}_\gamma$ must vanish.

Similarly, to obtain a modified evolution corresponding to a target Hamiltonian $H'_{\text{eff}} = \sum_\gamma (\vec{a}'^t_\gamma \cdot \vec{\lambda}) \otimes B_\gamma$, we require the weighted sum of rotations applied to the original coordinate vector to be equal to the corresponding target coordinate vector $\vec{a}'^t_\gamma$. I.e., for $H_{\text{eff}} \neq 0$, the following

condition should be satisfied to obtain the desired evolution:

$$\vec{a}'_\gamma = \vec{a}^t_\gamma \tag{3.11}$$

This may require a combination of switching strategies for the BB pulses.[8]

It should be noted that our geometrical picture is an explicit representation of a subset of the group algebra $\mathbb{C}\mathcal{G}$ using the set of traceless Hermitian matrices and the identity as the basis. When the coefficients of the adjoint vector are real, the resulting matrix H_{eff} is Hermitian. When they are complex, the resulting matrix is not Hermitian and the evolution is not unitary but may still be treated empirically.[21]

4. ERRORS

The geometric setting provides a natural way in which to evaluate the error that remains in the evolution after the application of the BB pulses. Here, the appropriate evaluation and interpretation of these errors is detailed and a comparison with other measures of errors is made.

Let $\vec{a}^t_\gamma$ be the coordinates vector corresponding to the desired or target Hamiltonian evolution and $\vec{a}'_\gamma$ the actual vector after BB operations. Then $\vec{a}'_\gamma$ corresponds to the effective Hamiltonian, Eq. (3.9) (and may be determined using quantum process tomography, see Ref. 21 and references therein). The error vector $\vec{e}$ is given by their difference in the n^2-dimensional vector space where our geometric picture holds:

$$\vec{e} = \vec{a}'_\gamma - \vec{a}^t_\gamma \tag{4.1}$$

The vector $\vec{e}$ gives us the magnitude and direction of the error (i.e., the basis elements λ_i. This error is to be interpreted as the effective undesirable evolution of the system.

Now consider the magnitude of this error,

$$d(\vec{a}'_\gamma, \vec{a}^t_\gamma) = (\vec{e}^* \cdot \vec{e})^{1/2} \tag{4.2}$$

(in the case of Hamiltonian evolution there is no need for complex conjugation). This is the Euclidean distance between the two vectors in the adjoint representation space. For two two-state density matrices, it is proportional to the Euclidean distance between the two Bloch vectors, as is the trace distance.

In the case of imperfect BB operations, the goal is to minimize the distance d. For the purposes of optimization, note that

$$d(\vec{a}'_\gamma, \vec{a}^t_\gamma) = (\vec{e} \cdot \vec{e})^{1/2} = ((a'_\gamma)^2 + (a^t_\gamma)^2 - 2M\vec{a}_\gamma \cdot \vec{a}^t_\gamma)^{1/2} \tag{4.3}$$

whereas the ordinary trace distance (Hilbert–Schmidt norm) between $(\vec{a}'_\gamma, \vec{a}^t_\gamma)$ gives [using Eq. (3.2)]

$$\mathrm{Tr}[(\vec{a}_\gamma \cdot \vec{\lambda})(\vec{a}^t_\gamma \cdot \vec{\lambda})] = M\vec{a}_\gamma \cdot \vec{a}^t_\gamma \tag{4.4}$$

So minimizing d is equivalent to maximizing $(\vec{a}'_\gamma \cdot \vec{a}^t_\gamma)$. The advantage of using d is interpretational since the error vector describes a Euclidean vector (in the adjoint representation space) corresponding to an undesired evolution.

For obtaining a desired evolution, we note that the measure given in Eq. (4.2) is related to the common trace-norm distance for unitary matrices. For unitary evolutions U and V, this is defined by

$$d_u(U, V) = \sqrt{1 - (1/n)\mathrm{Re}[\mathrm{Tr}(U^\dagger V)]} \tag{4.5}$$

where U, and V are $n \times n$ matrices. For BB controls a short-time approximation is relevant. For the case of unitary evolution, the two measures, Euclidean distance and trace-norm for matrices, are related. Approximating U (desired evolution) and V (actual evolution) by $\mathbb{1} - iHt = \mathbb{1} - it\sum_i a^t_{i\gamma}\lambda_i$ and $\mathbb{1} - iH't = \mathbb{1} - it\sum_i a'_{i\gamma_i}\lambda_i$ respectively,

$$d_u(U, V) \approx \sqrt{1 - (1/n)\mathrm{Re}[\mathrm{Tr}(\mathbb{1} + HH't^2)]} \propto \sqrt{\vec{a}'_\gamma \cdot \vec{a}^t_\gamma} \tag{4.6}$$

is an $O(t)$ approximation to the unitary evolution. Equations (4.2), (4.4), and (4.6) provide targets which, for the purposes of optimization are completely equivalent. In conclusion, the erroneous Hamiltonian evolution is appropriately defined as the error vector in Eq. (4.1) and is directly related to the more commonly used distance measures of the trace-distance and Eq. (4.5).

At this point it should be emphasized that the measure given in this section is particularly well suited to the geometric picture. Equation (4.2) is not a replacement for other distance, or distinguishability measures, but rather a quantity which is more related to the work presented here, namely, the geometric picture. Since it gives a vector-valued quantity for the distance (corresponding to the effective and erroneous evolution) it is clearly appropriate for the comparison of noiseless and noisy evolutions. The corresponding scalar quantity, the Euclidean distance in the adjoint vector space [Eq. (4.2)], gives a less detailed quantity which may be desirable for optimization procedures.

5. EXAMPLES

We now discuss several examples illustrating the geometric picture developed above. We note that in the case of a single qubit we can resort to the familiar Bloch sphere representation.

Consider the noisy evolution of a stored qubit. Suppose the evolution of the qubit is governed by the Hamiltonian

$$H = \sum_{\gamma} \sum_{i=1}^{3} a_{i\gamma} \sigma_i \otimes B_\gamma \tag{5.1}$$

where the $a_{i\gamma}$ are real coefficients, B_γ are bath operators, and the σ_i are the Pauli matrices with the usual identification $\sigma_1 = \sigma_x$, $\sigma_2 = \sigma_y$, $\sigma_3 = \sigma_z$. (As above, the identity component is neglected.) For faithful storage, a set of BB operations $\{U_k\}$ should serve to eliminate this Hamiltonian. Under such controls, the evolution is described by

$$H_{\text{eff}} = \frac{1}{|\mathcal{G}|} \sum_{ik\gamma} U_k^\dagger [(\vec{a}_\gamma \cdot \vec{\sigma}) \otimes B_\gamma] U_k = \sum_{\gamma} (\vec{a}'_\gamma \cdot \vec{\sigma}) \otimes B_\gamma \tag{5.2}$$

where $U_k \in SU(2)$ and $R^{(k)} \in SO(3)$, and

$$\vec{a}'_\gamma = \frac{1}{|\mathcal{G}|} \sum_{k=0}^{|\mathcal{G}|-1} R^{(k)} \vec{a}_\gamma \tag{5.3}$$

In the subsections below we consider different choices of the subgroup $\mathcal{G}$. These equations then describe a sum of vectors on the Bloch sphere. The mapping from the unitary matrices U_k to the rotation matrices $R^{(k)}$ is given by

$$U^\dagger \sigma_i U = e^{i\sigma_3\alpha/2} e^{i\sigma_2\beta/2} e^{i\sigma_3\gamma/2} \sigma_i e^{-i\sigma_3\gamma/2} e^{-i\sigma_2\beta/2} e^{-i\sigma_3\alpha/2} = R_{ij}\sigma_j. \tag{5.4}$$

Explicitly, the rotation matrix is given by

$$R = \begin{bmatrix} \cos(\alpha)\cos(\beta)\cos(\gamma) - \sin(\alpha)\sin(\gamma) & -\sin(\alpha)\cos(\beta)\cos(\gamma) - \cos(\alpha)\sin(\gamma) & \sin(\beta)\cos(\gamma) \\ \cos(\alpha)\cos(\beta)\sin(\gamma) + \sin(\alpha)\cos(\gamma) & -\sin(\alpha)\cos(\beta)\sin(\gamma) + \cos(\alpha)\cos(\gamma) & \sin(\beta)\sin(\gamma) \\ -\cos(\alpha)\sin(\beta) & \sin(\alpha)\sin(\beta) & \cos(\beta) \end{bmatrix} \tag{5.5}.$$

Alternatively, one may use

$$e^{i(\theta/2)\hat{n}\cdot\vec{\sigma}} \vec{x} \cdot \vec{\sigma} e^{-i(\theta/2)\hat{n}\cdot\vec{\sigma}} = \vec{x}' \cdot \vec{\sigma} \tag{5.6}$$

where

$$\vec{x}' = (\hat{n} \cdot \vec{x})\hat{n} + [(\hat{n} \times \vec{x}) \times \hat{n}]\cos(\theta) + [\hat{n} \times \vec{x}]\sin(\theta) \tag{5.7}$$

The correspondence between the unitary and orthogonal groups is made by

$$\sum_j x'_j \sigma_j = \sum_{i,j} x_i [R_{\hat{n}}(\theta)]_{ij} \sigma_j = U(\hat{n}, \theta/2) \left(\sum_i x_i \sigma_i \right) U^\dagger(\hat{n}, \theta/2) \tag{5.8}$$

where $R_{\hat{n}}(\theta)$ is a rotation by θ about the axis $\hat{n}$. Although this notation is more compact, the Euler angle parameterization of $SU(3)$ and $SU(4)$ have been given.[22–24]

5.1. Storing One Qubit

To be specific, consider an unwanted pure dephasing interaction described by the Hamiltonian

$$H = g\sigma_3 \otimes B \tag{5.9}$$

Using Eq. (3.4) we find that the coordinate (adjoint) representation of this Hamiltonian is

$$a_i = \frac{g}{2} \mathrm{Tr}(\sigma_i \sigma_3) = g\delta_{i3} \tag{5.10}$$

i.e., $\vec{a} = (0, 0, g)$. This dephasing could be corrected through the use of a single BB operation $U_1 = \exp(-i\sigma_1 \pi/2) = -i\sigma_1$, so that

$$U_1^\dagger \sigma_3 U_1 = -\sigma_3 \tag{5.11}$$

For the geometric picture, using $\beta = \pi,\ \alpha = \gamma = 0$ in Eq. (5.5):

$$R^{(1)} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{5.12}$$

which inverts the adjoint vector $\vec{a}$. This is shown schematically in Fig. 1. It is simple to check that, as required, $H_{\mathrm{eff}} = 0$. This example uses the lowest dimensional finite order group C_2.

Now, let us use our geometric picture to derive another class of BB operations for pure dephasing on a single qubit. Clearly, the point is to find a set of rotations of $\vec{a}$ which when added sum up to zero. The next example is the group C_3, which consists of rotating $\vec{a}$ by $2\pi/3$ and $-2\pi/3$ about a fixed axis, i.e., uses two non-trivial BB operations. This is depicted in Fig. 2 where we have chosen σ_1 as the fixed axis. The set of rotation matrices that

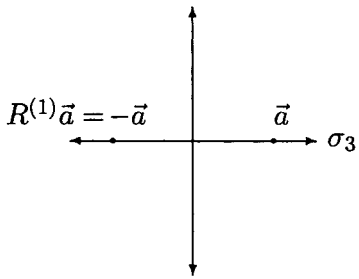

Fig. 1. Addition of adjoint vectors on the Bloch sphere corresponding to a single pulse application. The vertical line represents zero σ_3 component.

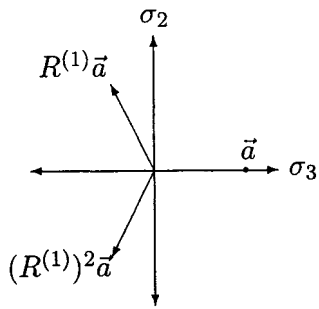

Fig. 2. Addition of adjoint vectors on the Bloch sphere corresponding to BB operations.

accomplish this are $R^{(1)} = R_{\hat{x}}(2\pi/3)$ and $R^{(2)} = R_{\hat{x}}(-2\pi/3) = (R^{(1)})^2$, where

$$R_{\hat{x}}(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix} \tag{5.13}$$

and the corresponding unitary transformations are $U_k = \exp(-i\sigma_1(\pm\pi/3))$, $k = 1, 2$.

A large number of BB operations is undesirable due to time constraints so that large sets become successively more difficult to implement effectively. However, experimental constraints regarding the available set of BB operations as well as considerations regarding the symmmetries of the Hamiltonian, may sometimes render a larger set preferable to a smaller set that is more difficult to implement. One situation exemplifying the former case is when it is difficult to implement large-angle rotations. E.g., using C_2 BB operations involves π rotations, but using C_3 rotations only involves $2\pi/3$ rotations. An inherent limitation on the magnitude of a control parameter and the time that it can be turned on (e.g., the Heisenberg exchange coupling in quantum dots[25]), may then favor the C_3 BB group. An example of Hamiltonian symmetries which may affect the size of the BB set is the following. Consider a case in which the interaction Hamiltonian is of the form $H_I = (\sigma_1 + \sigma_3) \otimes B$. If one is able to reliably and quickly perform a

σ_2 rotation, then a simple parity kick sequence will suffice. However, if these rotations are not available, or are difficult to implement under the BB constraints, then a sequence of σ_1 and σ_3 rotations may be more prudent.

Let us consider the next higher order set of BB operations. This will be a set with 4 elements. The subgroup condition requires a set forming either the cyclic group of order 4 or the so-called Vierergruppe since these are the only two groups of order 4.[20] An example of the cyclic group of order 4 would be the four-fold rotations about a single axis (e.g., $\pi/2$ around the σ_1 axis, as in Fig. 3). An example of the the other fourth order group is the set of rotations by π about three orthogonal symmetry axes.

Note that the set of vectors pointing to the vertices of a tetrahedron also will sum to zero and thus form a set of adjoint vectors, representing BB modified Hamiltonians, that will produce the desired decoupling effect, the elimination of the interaction Hamiltonian Eq. (5.9). This set is determined by

$$\sum_{k=1}^{4} \vec{a}^{(k)} = 0, \quad \text{and} \quad \vec{a}^{(k)} \cdot \vec{a}^{(k')} = \text{const} \equiv \cos(\theta). \tag{5.14}$$

This implies $\theta = \cos^{-1}(-1/3)$ so that for Eq. (5.9) the set of rotations can be $\mathbb{1}$, $\mathrm{R}_{\hat{y}}(\theta)$, $\mathrm{R}_{\hat{a}_2}(2\pi/3)$, $\mathrm{R}_{\hat{a}_2}(-2\pi/3)$ acting on the initial vector $\vec{a}_1 = (0, 0, g)$, and $\hat{a}_2$ is the direction of $\vec{a}_2 = R_{\hat{y}}(\theta)\vec{a}_1$. These rotations will take $\vec{a}$ to different positions which correspond to the vertices of a tetrahedron (see Fig. 4). The corresponding U_k are found using Eq. (5.8).

Note that in this last example we used a set of rotations that *does not satisfy the subgroup condition*, showing that the subgroup condition is sufficient, but not necessary.

5.2. Two Qubits

Now let us suppose a computation is to be performed on two-qubits using the Heisenberg exchange coupling in the presence of an independent-dephasing mechanism. We will assume that the Heisenberg exchange interaction provides qubit–qubit interactions and that single qubit operations are also available. This is a situation one would expect to find in one of

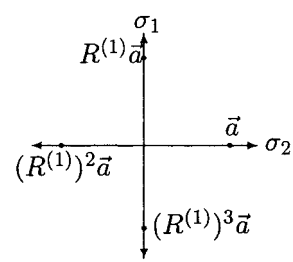

Fig. 3. The application of the cyclic group of order 4. Here $R^{(1)} = R_{\hat{x}}(\pi/4)$.

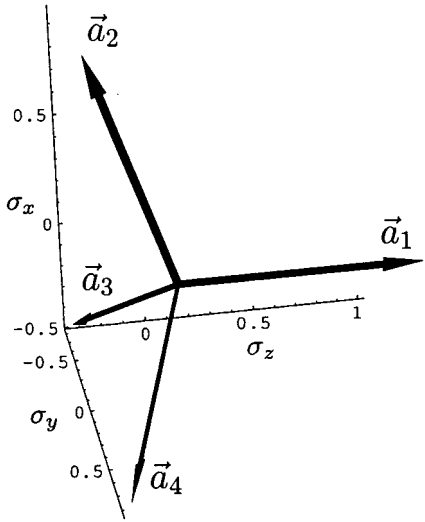

Fig. 4. Rotations of the Hamiltonian to vertices of the Tetrahedron. The rotated vectors are $\vec{a}_2 = R_{\hat{y}}(\theta)$, $\vec{a}_3 = R_{\hat{a}2}(2\pi/3)$, $\vec{a}_4 = R_{\hat{a}2}(-2\pi/3)$.

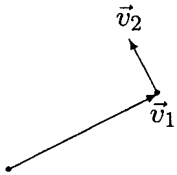

Fig. 5. Two qubit computation.

the proposals for a quantum dot quantum computer[25] where single and two-qubit gates might be available. The single qubit gates can be implemented via localized magnetic fields or g-factor engineering, although we note that this is difficult in practice and can be avoided through the use of encoding, which allows universal quantum computation to be performed using the Heisenberg interaction together with Zeeman splitting.[26]

Let a basis for the Lie algebra be given by

$$\{\lambda_i\}_{i=0}^{15} = \{\sigma_i \otimes \sigma_j\}_{i,j=0}^{3} \tag{5.15}$$

and labels correspond to

$$\begin{aligned}
&\lambda_j, \quad j = 0, 1, 2, 3 && \leftrightarrow \quad \sigma_i \otimes \mathbb{1}, i = 0, 1, 2, 3, \\
&\lambda_j, \quad j = 4, 5, 6 && \leftrightarrow \quad \mathbb{1} \otimes \sigma_i, \quad i = 1, 2, 3, \\
&\lambda_j, \quad j = 7, 8, 9 && \leftrightarrow \quad \sigma_1 \otimes \sigma_i, \ i = 1, 2, 3, \\
&\lambda_j, j = 10, 11, 12 && \leftrightarrow \quad \sigma_2 \otimes \sigma_i, \ i = 1, 2, 3, \\
&\lambda_j, \quad j = 13, 14, 15 && \leftrightarrow \quad \sigma_3 \otimes \sigma_i, \ i = 1, 2, 3
\end{aligned} \tag{5.16}$$

This forms an orthogonal basis with respect to the trace and has normalization given by

$$\mathrm{Tr}(\lambda_i \lambda_j) = 4\delta_{ij} \tag{5.17}$$

The Heisenberg interaction can be written as:

$$H_{ex} = J\vec{\sigma}_1 \cdot \vec{\sigma}_2 \equiv \vec{v}_1 \cdot \vec{\lambda} \tag{5.18}$$

where $\vec{v}_1 = (0, 0, 0, 0, 0, 0, J, 0, 0, 0, J, 0, 0, 0, J)$, so that

$$H_{ex} = J(\lambda_7 + \lambda_{11} + \lambda_{15}) \tag{5.19}$$

The independent dephasing is given by:

$$H_I = (g_1\sigma_3 \otimes \mathbb{1} \otimes + g_2 \mathbb{1} \otimes \sigma_3) \otimes B \equiv \vec{v}_2 \cdot \vec{\lambda} \otimes B$$

where $\vec{v}_2 = (0, 0, g_1, 0, 0, g_2, 0, 0, 0, 0, 0, 0, 0, 0, 0)$. So

$$H_I = (g_1\lambda_3 + g_2\lambda_6) \otimes B \tag{5.20}$$

Note that trace of the product of two Hamiltonians is the dot product of their corresponding adjoint vectors. In this case the two are orthogonal so that, $\mathrm{Tr}(H_{ex}H_I) = 4\vec{v}_1 \cdot \vec{v}_2 = 0$, therefore $\vec{v}_1 \perp \vec{v}_2$. A method for achieving the required decoupling without affecting the desired Heisenberg interaction is to consider the little group of $\vec{v}_1$. (The 1D subgroup of rotations about the axis $\vec{v}_1$.) From that set of rotations, a subset of rotations exists which will rotate the interaction Hamiltonian since the two vectors lie in orthogonal subspaces. These rotations clearly must be about the axis defined by $\vec{v}_1$. Thus we may express this as $R_{\hat{v}_1}(\theta)$. To limit the number of pulses in the sequence of BB operations, a parity-kick operation is desired. This further limits our choices to those operations that rotate $\vec{v}_2$ by an angle π. More specifically, we seek a rotation that inverts the components in the directions λ_3 and λ_6, since $\vec{v}_2 = g_1\lambda_3 + g_2\lambda_6$. The directions λ_3 and λ_6 define a plane perpendicular to $\hat{v}_1$, so that the desired rotation matrix is effectively an $SO(3)$ rotation matrix with a non-trivial component in this plane, i.e., $R_{\hat{v}_1}(\pi)$. It is then simple to check that the corresponding unitary operation satisfying the parity kick condition $U^\dagger H_I U = R_{\hat{v}_1}(\pi)H_I = -H_I$, is

$$U \equiv U_1 U_2 = \exp(-i(\sigma_1^{(1)} + \sigma_1^{(2)})\pi/2) = -\sigma_1^{(1)}\sigma_1^{(2)} \tag{5.21}$$

where the superscript indicates the qubit on which the operator acts. This interaction leaves H_{ex} unaffected and provides decoupling equivalent to Eq. (5.2). Note that this is a useful means for achieving the desired decoupling, because exchange interactions can be used to turn on a gate operation in a solid state device, and the decoupling can be achieved during the process without interruption of the desired interaction. The geometric picture shows

that the above U is by no means unique: any set of $SO(3)$ rotations acting in the (λ_3, λ_6) plane, and whose elements add up to zero, will eliminate the undesired interaction. To our knowledge this is a new example of an explicit BB sequence that is generally compatible with the Heisenberg exchange interaction. Another example employs the Heisenberg interaction itself to generate a BB sequence.[31]

5.3. Discussion

The method we have just presented is quite general and provides an immediate way in which to determine if a decoupled evolution is possible. That is, *one evolution may be eliminated without affecting another if their adjoint vectors lie in orthogonal subspaces*. Since this is equivalent to having the trace of the product of two Hamiltonians vanish, we have provided a necessary and sufficient condition, which is very easily calculated, to determine when one may eliminate one evolution without affecting another. Such conditions are very important since BB operations may be performed imperfectly. If this is so, one would prefer they not adversely affect the desired gating operations while reducing noise (even if it is not perfectly eliminated).

Clearly the number of BB operations is dependent upon the experimentally available BB operations, the interaction one wishes to suppress and the interaction(s) one wishes to retain. With imperfect BB operations, one could introduce extra noise into the system. However, if the orthogonality condition (above) is satisfied, then it is ensured that our BB operations will not interfere with other control operations.

6. CONCLUSION

A geometric treatment of bang–bang (BB) operations has been provided. This perspective provides an intuitive picture for BB operations and their imperfections. The group averaging is made explicit through the corresponding average over a set of coordinate vectors representing rotations of the Hamiltonian; the resultant vector is the sum over all the BB modified Hamiltonians. These quantities are useful for computations, complementing the somewhat more abstract approaches of previous treatments.[8, 18] Since after the application of an imperfect set of decoupling operations, one is concerned with remaining error(s), we have presented and interpreted, a natural error measure for the remaining erroneous evolution which is related to more commonly used measures. In addition, the often promoted Bloch sphere representation used in some of the examples treated here provides a means for extending intuition beyond the the low dimensional cases.

The usual group-theoretic symmetrization description of BB operations assumes that the set of pulses forms a discrete subgroup.[8, 18] We showed here that this is not a necessary condition, through the example of symmetrizing by the vertices of a tetrahedron.

The two-qubit example in Sec. 5.2 provides a way in which this geometric analysis aids in the problem of finding decoupling interactions. The similarity between this example and recoupling techniques in NMR and other systems is no coincidence. The BB operations were, afterall, related to NMR techniques in the earliest papers describing such interactions.[1] The geometric viewpoint is quite general and provides an instructive way in which to decompose such problems and a general compatibility condition. They may be particularly useful for the types of recoupling techniques one requires for reducing constraints on quantum computer proposals[26] and for combining BB techniques with DFSs.[27–31]

In the subgroup framework, our geometric picture uses a homomorphic mapping between the fundamental representation and the corresponding adjoint representation. The problem of inverting this map from the adjoint to the fundamental representation may well be difficult for groups of higher dimension than $SU(4)$. However, for universal quantum computation, one and two qubit gates are sufficient and fortunately the discrete subgroups of unitary groups have been classified up to $SU(4)$. (See Ref. 32–34 and references therein). The determination of the appropriate subgroup could consist of searching a discrete solution space. This appears feasible since the lower order subgroups are more relevant given the strict time constraints of the BB assumptions.

Given the scarcity of qubits in current quantum computing systems, we believe that the BB method is an important tool. We hope that the work presented here will be helpful in constructing sequences of BB pulses and analyzing their imperfections.

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