# Entropy power inequality for a family of discrete random variables

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***Abstract*—It is known that the Entropy Power Inequality (EPI) always holds if the random variables have density. Not much work has been done to identify discrete distributions for which the inequality holds with the differential entropy replaced by the discrete entropy. Harremoës and Vignat showed that it holds for the pair $(B(m,p), B(n,p))$, $m,n \in \mathbb{N}$, (where $B(n,p)$ is a Binomial distribution with $n$ trials each with success probability $p$) for $p = 0.5$. In this paper, we considerably expand the set of Binomial distributions for which the inequality holds and, in particular, identify $n_0(p)$ such that for all $m,n \geq n_0(p)$, the EPI holds for $(B(m,p), B(n,p))$. We further show that the EPI holds for the discrete random variables that can be expressed as the sum of $n$ independent and identically distributed (IID) discrete random variables for large $n$.**

***Index Terms*—Entropy power inequality, Taylor's theorem, asymptotic series, binomial distribution.**

## I. Introduction

The Entropy Power Inequality

$$e^{2h(X+Y)} \geq e^{2h(X)} + e^{2h(Y)} \quad (1)$$

holds for independent random variables $X$ and $Y$ with densities, where $h(\cdot)$ is the differential entropy. It was first stated by Shannon in Ref. [1], and the proof was given by Stam and Blachman [2]. See also Refs. [3], [4], [5], [6], [7], [8], [9].

This inequality is, in general, not true for discrete distributions where the differential entropy is replaced by the discrete entropy. For some special cases (binary random variables with modulo 2 addition), results have been provided by Shamai and Wyner in Ref. [10].

More recently, Harremoës and Vignat have shown that this inequality will hold if $X$ and $Y$ are $B(n, 1/2)$ and $B(m, 1/2)$ respectively for all $m, n$ [11]. Significantly, the convolution operation to get the distribution of $X+Y$ is performed over the usual addition over reals and not over finite fields.

Recently, another approach has been expounded by Harremoës et. al. [12] and by Johnson and Yu [13], wherein they interpret Rényi's thinning operation on a discrete random variable as a discrete analog of the scaling operation for continuous random variables. They provide inequalities for the convolutions of thinned discrete random variables that can be interpreted as the discrete analogs of the ones for the continuous case.

In this paper, we take a re-look at the result by Harremoës and Vignat [11] for the Binomial family and extend it for all $p \in (0,1)$. We show that there always exists an $n_0(p)$ that is a function of $p$, such that for all $m,n \geq n_0(p)$,

$$e^{2H[B(m+n,p)]} \geq e^{2H[B(m,p)]} + e^{2H[B(n,p)]}, \quad (2)$$

where $H(\cdot)$ is the discrete entropy. The result in Ref. [11] is a special case of our result since we obtain $n_0(0.5) = 7$ and it can be checked numerically by using a sufficient condition that the inequality holds for $1 \leq m,n \leq 6$.

We then extend our results for the family of discrete random variables that can be written as the sum of $n$ IID random variables and show that for large $n$, EPI holds.

We also look at the semi-asymptotic case for the distributions $B(m,p)$ with $m$ small and $B(n,p)$ with $n$ large. However, when $n$ is large, there may exist some $m$ such that EPI does not hold.

The proofs for all the lemmas and theorems except that of Theorem 2 and 3 are omitted due to lack of space, but they are available in Ref. [14].

## II. EPI for the Binomial distribution

Our aim is to have an estimate on the threshold $n_0(p)$ such that

$$e^{2H[B(m+n,p)]} \geq e^{2H[B(m,p)]} + e^{2H[B(n,p)]}, \quad (3)$$

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holds for all $m, n \geq n_0(p)$.

It is observed that $n_0(p)$ depends on the skewness of the associated Bernoulli distribution. Skewness of a probability distribution is defined as $\kappa_3/\sqrt{\kappa_2^3}$ where $\kappa_2$ and $\kappa_3$ are respectively the second and third cumulants of the Bernoulli distribution $B(1,p)$, and it turns out to be $(2p-1)/\sqrt{p(1-p)}$. Let

$$\omega(p) = \frac{(2p-1)^2}{p(1-p)}. \tag{4}$$

We find an expression for $n_0(p)$ that depends on $\omega(p)$. The well known Taylor's theorem will be useful for this purpose (see for example p. 110 in Ref. [15]) and is stated as follows.

Suppose $f$ is a real function on $[a,b]$, $n \in \mathbb{N}$, the $(n-1)$th derivative of $f$ denoted by $f^{(n-1)}$ is continuous on $[a,b]$, and $f^{(n)}(t)$ exists for all $t \in (a,b)$. Let $\alpha, \beta$ be distinct points of $[a,b]$, then there exists a point $y$ between $\alpha$ and $\beta$ such that

$$\begin{aligned} f(\beta) &= f(\alpha) + \sum_{k=1}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(\beta-\alpha)^k \\ &\quad + \frac{f^{(n)}(y)}{n!}(\beta-\alpha)^n. \end{aligned} \tag{5}$$

For $0 \leq p \leq 1$, let $H(p)$ denote the discrete entropy of a Bernoulli distribution with probability of success $p$, that is, $H(p) \triangleq -p\log(p) - (1-p)\log(1-p)$. We shall use the natural logarithm throughout this paper. Note that we earlier defined $H(\cdot)$ to be the discrete entropy of a discrete random variable. The definition to be used would be amply clear from the context in what follows. Let

$$\hat{H}(x) \triangleq H(p) - H(x), \quad x \in (0,1), \tag{6}$$

$$F^{(k)}(x) \triangleq \frac{\hat{H}^{(k)}(x)}{k!}. \tag{7}$$

Note that $\hat{H}(x)$ satisfies the assumptions in the Taylor's theorem in $x \in (0,1)$. Therefore, we can write

$$\begin{aligned} \hat{H}(x) &= \hat{H}(p) + \sum_{k=1}^{n-1} F^{(k)}(x)(x-p)^k \\ &\quad + F^{(n)}(x_1)(x-p)^n, \end{aligned} \tag{8}$$

for some $x_1 \in (x,p)$. Note that $\hat{H}(p) = 0$.

For even $k$, $F^{(k)}(x) \geq 0$ for all $x \in (0,1)$, and hence,

$$\hat{H}(x) \geq \sum_{k=1}^{2l+1} F^{(k)}(p)(x-p)^k \tag{9}$$

for all $x \in (0,1)$ and any non-negative integer $l$. The following useful identity would be employed at times

$$\log(2) - H(p) = \sum_{\nu=1}^{\infty} \frac{2^{2\nu}}{2\nu(2\nu-1)}\left(p - \frac{1}{2}\right)^{2\nu}. \tag{10}$$

Let $P \triangleq \{p_i\}$ and $Q \triangleq \{q_i\}$ be two probability measures over a finite alphabet $\mathcal{A}$. Let $C^{(p)}(P,Q)$ and $\triangle_\nu^{(p)}(P,Q)$ be measures of discrimination defined as

$$C^{(p)}(P,Q) \triangleq pD(P \parallel M) + qD(Q \parallel M), \tag{11}$$

$$\triangle_\nu^{(p)}(P,Q) \triangleq \sum_{i \in A} \frac{|pp_i - qq_i|^{2\nu}}{(pp_i + qq_i)^{2\nu-1}}, \tag{12}$$

where $M = pP + qQ$, $q = 1-p$ and $D(\cdot \parallel \cdot)$ is the Kullback-Leibler divergence. These quantities are generalized capacitory discrimination and triangular discrimination of order $\nu$ respectively that were introduced by Topsøe [16].

The following theorem relates $C^{(p)}(P,Q)$ with $\triangle_\nu^{(p)}(P,Q)$ and would be used later to derive an expression for $n_0(p)$. It generalizes Theorem 1 in Ref. [16].

**Theorem 1.** *Let $P$ and $Q$ be two distributions over the alphabet $\mathcal{A}$ and $0 < p < 1$. Then*

$$C^{(p)}(P,Q) = \sum_{\nu=1}^{\infty} \frac{\triangle_\nu^{(p)}(P,Q)}{2\nu(2\nu-1)} - [\log(2) - H(p)].$$

Let $X^{(n)}$ be a discrete random variable that can be written as $X^{(n)} = Z_1 + Z_2 + \cdots + Z_n$, where $Z_i$'s are IID random variables. We note that when $X^{(n)}$ is defined as above, we have $X^{(n)} + X^{(m)} = X^{(n+m)}$. Let $Y_n \triangleq \mathrm{e}^{2[H(X^{(n)})]}$. We first use a lemma due to Harremoës and Vignat [11].

**Lemma 1** (Harremoës and Vignat [11])**.** *If $Y_n/n$ is increasing, then $Y_n$ is super-additive, i.e., $Y_{m+n} \geq Y_m + Y_n$.*

It is not difficult to show that this is a sufficient condition for the EPI to hold [11]. By the above lemma, the inequality

$$H(X^{(n+1)}) - H(X^{(n)}) \geq \frac{1}{2}\log\left(\frac{n+1}{n}\right) \tag{13}$$

is sufficient for EPI to hold. Let $X^{(n)} = B(n,p)$. We have

$$P_{X^{(n+1)}}(k+1) = pP_{X^{(n)}}(k) + qP_{X^{(n)}}(k+1). \tag{14}$$

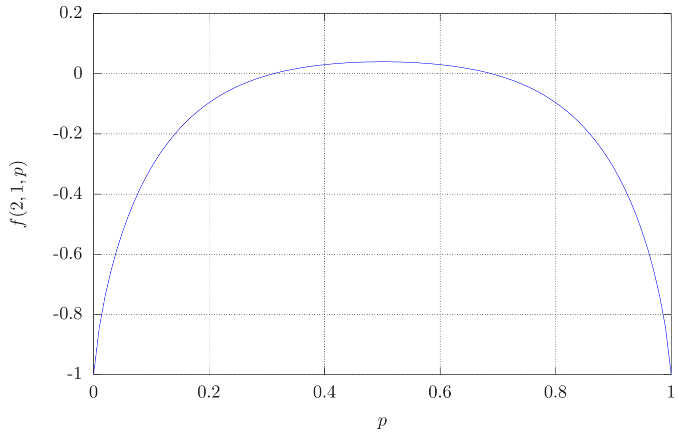


Fig. 1. Plot of $e^{2H[B(3,p)]} - \left\{e^{2H[B(2,p)]} + e^{2H[B(1,p)]}\right\}$ versus $p$.

Define a random variable $X^{(n)}+1$ as $P_{X^{(n)}+1}(k+1) = P_{X^{(n)}}(k)$ for all $k \in \{0,1,\cdots,n\}$.

Hence, using $H(X^{(n)}+1) = H(X^{(n)})$, $P_{X^{(n+1)}} = pP_{X^{(n)}+1} + qP_{X^{(n)}}$ and (11), we generalize Eq. (3.7) in Ref. [11] as

$$H(X^{(n+1)}) = H(X^{(n)}) + C^{(p)}(P_{X^{(n)}+1}, P_{X^{(n)}}). \quad (15)$$

We now derive the lower bound for $C^{(p)}(P_{X^{(n)}+1}, P_{X^{(n)}})$.

**Lemma 2.** *For $l \in \mathbb{N}$,*

$$C^{(p)}(P_{X^{(n)}}, P_{X^{(n)}+1}) \geq \sum_{k=1}^{2l+1} F^{(k)}(p)(n+1)^{-k}\mu_k^{(n+1)},$$

*where $\mu_k^{(n)}$ is the $k$-th central moment of $B(n,p)$, i.e.,*

$$\mu_k^{(n)} = \sum_{i=0}^{n}(i-np)^k P_{X^{(n)}}(i). \quad (16)$$

We note that there exist $m,n$ for $p \neq 0.5$ for which EPI does not hold, that is,

$$e^{2H[B(m+n,p)]} < e^{2H[B(m,p)]} + e^{2H[B(n,p)]}.$$

In fact, one can easily see that

$$e^{2H[B(2,p)]} \leq e^{2H[B(1,p)]} + e^{2H[B(1,p)]} \quad \forall \;\; p,$$

with equality if and only if $p = 0.5$.

For the case $m=1$ and $n=2$, it is clear from th Fig. 1 that EPI holds when $p$ is close to $0.5$, while EPI does not hold if $p$ is close to $0$ or $1$. This leads us to the question that for a given $p$, what should $m,n$ be such that the EPI would hold. The main theorem of this section answers this question.

**Theorem 2.**

$$H[B(n+1,p)] - H[B(n,p)] \geq \frac{1}{2}\log\left(\frac{n+1}{n}\right) \quad \forall \;\; n \geq n_0(p).$$

*Several candidates of $n_0(p)$ are possible such as $n_0(p) = 4.44\ \omega(p) + 7$ and $n_0(p) = \omega(p)^2 + 2.34\ \omega(p) + 7$.*

*Proof:* See the Appendix. ∎

Using Lemma 2, we can obtain a non-asymptotic lower bound to the entropy of the binomial distribution unlike the asymptotic expansion of $H[B(n,p)]$ given in Ref. [17]. For details, see Ref. [14].

## III. EPI FOR THE SUM OF IID

We showed in the previous section that EPI holds for the pair $(B(n,p), B(m,p))$ for all $m,n \geq n_0(p)$. The question naturally arises whether EPI holds for all such discrete random variables that can be expressed as sum of IID random variables. Let $X^{(n)}$ be a discrete random variable such that

$$X^{(n)} \triangleq X_1 + X_2 + \cdots + X_n, \quad (17)$$

where $X_i$'s are IID random variables and $\sigma^2$ is the variance of $X_1$. The series $\sum_{l=1}^{\infty} a_l\phi_l(n)$ is said to be an *asymptotic expansion* for $f(n)$ as $n \to \infty$ if

$$f(n) = \sum_{l=1}^{N} a_l\phi_l(n) + o[\phi_N(n)] \quad \forall \; N, \quad (18)$$

and is written as $f(n) \sim \sum_l a_l\phi_l(n)$, where if $\gamma(n) \in o[\phi(n)]$, then $\lim_{n\to\infty}\gamma(n)/\phi(n) = 0$. We shall use the asymptotic expansion due to Knessl [17].

**Lemma 3** (Knessl [17])**.** *For a random variable $X^{(n)}$, as defined above, having finite moments, we have as $n \to \infty$,*

$$\begin{aligned} g(n) &\triangleq H(X^{(n)}) - \frac{1}{2}\log(2\pi e n\sigma^2) \\ &\sim -\frac{\kappa_3^2}{12\sigma^6}\frac{1}{n} + \sum_{l=1}^{\infty}\frac{\beta_l}{n^{l+1}}, \end{aligned} \tag{19}$$

*where $\kappa_j$ is the $j$th cumulant of of $X_1$. If $\kappa_3 = \kappa_4 = \cdots = \kappa_N = 0$ but $\kappa_{N+1} \neq 0$, then*

$$g(n) \sim -\frac{\kappa_{N+1}^2}{2(N+1)!\sigma^{2N+2}} n^{1-N} + \sum_{l=N-1}^{\infty}\frac{\beta_l}{n^{l+1}}.$$

Note that the leading term in the asymptotic expansion is always negative. We also note using Lemma 3 that as $n \to \infty$,

$$H(X^{(n)}) < \frac{1}{2}\log(2\pi e n\sigma^2). \tag{20}$$

To see this, we invoke the definition of the asymptotic series to get

$$g(n) = -\frac{\kappa_{N+1}^2}{2(N+1)!\sigma^{2N+2}} n^{1-N} + \frac{\beta_{N-1}}{n^N} + o\left(\frac{1}{n^N}\right).$$

From the definition of the "little-oh" notation, we know that given any $\epsilon > 0$, there exists a $L(\epsilon) > 0$ such that for all $n > L(\epsilon)$,

$$g(n) = -\frac{\kappa_{N+1}^2}{2(N+1)!\sigma^{2N+2}} n^{1-N} + \frac{\beta_{N-1} + \epsilon}{n^N}.$$

Choosing $n$ large enough, we get the desired result. We consider the case of the pair $(X^{(n)}, X^{(m)})$ when both $m, n$ are large and have the following result.

**Theorem 3.** *There exists a $n_0 \in \mathbb{N}$ such that*

$$e^{2H(X^{(m)}+X^{(n)})} \geq e^{2H(X^{(m)})} + e^{2H(X^{(n)})} \tag{21}$$

*for all $m, n \geq n_0$.*

*Proof:* We shall prove the sufficient condition for the EPI to hold (as per Lemma 1) and show that

$$H(X^{(n+1)}) - H(X^{(n)}) \geq \frac{1}{2}\log\left(\frac{n+1}{n}\right) \tag{22}$$

for $n \geq n_0$ for some $n_0 \in \mathbb{N}$.

Let us take the first three terms in the above asymptotic series as

$$g(n) \sim -\frac{C_1}{n^{k_1}} + \frac{C_2}{n^{k_2}} + \frac{C_3}{n^{k_3}} \tag{23}$$

where $0 < k_1 < k_2 < k_3$ and $C_1$ is some non-zero positive constant, and hence,

$$g(n) + \frac{C_1}{n^{k_1}} - \frac{C_2}{n^{k_2}} - \frac{C_3}{n^{k_3}} = o\left(\frac{1}{n^{k_3}}\right). \tag{24}$$

and given any $\epsilon > 0$, there exists a $L(\epsilon) > 0$ such that for all $n > L(\epsilon)$,

$$\left| g(n) + \frac{C_1}{n^{k_1}} - \frac{C_2}{n^{k_2}} - \frac{C_3}{n^{k_3}} \right| \leq \epsilon \left| \frac{1}{n^{k_3}} \right|. \tag{25}$$

From the above inequality, we have, by using the lower and upper bounds respectively for $g(n+1)$ and $g(n)$, that

$$g(n+1) - g(n) \geq C_1 \left[\frac{1}{n^{k_1}} - \frac{1}{(n+1)^{k_1}}\right] + C_2 \left[\frac{1}{(n+1)^{k_2}} - \frac{1}{n^{k_2}}\right] + \left[\frac{C_3 - \epsilon}{(n+1)^{k_3}} - \frac{C_3 + \epsilon}{n^{k_3}}\right].$$

From the above expression, we can clearly see that the first term is strictly positive and is $\Theta\left(1/n^{k_1+1}\right)$. The second and third terms (their signs are irrelevant) are of the order $\Theta(1/n^{k_2+1})$ and $\Theta(1/n^{k_3})$ respectively. The $\Theta(\cdot)$ means both asymptotic lower and asymptotic upper bounding. It is clear that there exists some positive integer $n_0$ such that for all $n \geq n_0$, first (positive) term will dominate and the other two terms will be negligible compared to the first and hence $g(n+1) - g(n) \geq 0$. ■

It must be noted that if one considers the pair $(X^{(n)}, X^{(m)})$ where $n \to \infty$ and $m$ is fixed, then the EPI need not hold. As can can be seen in our paper [14], it can be shown that

$$e^{2H(X^{(m)}+X^{(n)})} \geq e^{2H(X^{(m)})} + e^{2H(X^{(n)})}$$

if $n \to \infty$ and $H(X^{(m)}) < \frac{1}{2}\log[2\pi e m\sigma^2]$.

## IV. Conclusions

We have expanded the set of pairs of Binomial distributions for which the EPI holds. We identified a threshold that is a function of the probability of success beyond which the EPI holds. We further show that EPI would hold for discrete random variables that can be written as sum of IID random variables. We also obtain an improvement of a bound given by Artstein et. al. [6] (see Ref. [14]).

It would be interesting to know if $C^{(p)}(P_{X^{(n)}+1}, P_{X^{(n)}})$ for $X^{(n)} = B(n,p)$ is a concave function in $p$. It would also be of interest to know that for a given $p \in (0, 0.5)$ if $H[B(n+1,p)] - H[B(n,p)] - 0.5\log(1+1/n)$ would have a single zero crossing as a function of $n$ when $n$ increases from 1 to $\infty$.

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## Appendix

### Proof of Theorem 2

We prove that for all $n \geq n_0(p)$,

$$H[B(n+1,p)] - H[B(n,p)] \geq \frac{1}{2}\log\left(\frac{n+1}{n}\right) \quad (26)$$

Using Lemma 2, we have

$$H[B(n+1,p)] - H[B(n,p)] \geq \sum_{k=1}^{2l+1} F^{(k)}(p)(n+1)^{-k}\mu_k^{(n+1)}. \quad (27)$$

Let $r = p - 1/2$ and $t = \omega(p) = 16r^2/(1-4r^2)$. We have $r^2 = t/[4(t+4)]$.

Note that $r^2 \in [0, 1/4)$ and $t \in [0, \infty)$. We will use the first seven central moments of $B(n,p)$ and they contain only even powers of $r$ and hence, can be written as a function of $t$.

We upper bound the right hand side of (26) as

$$\log\left(\frac{n+1}{n}\right) \leq \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3}. \quad (28)$$

Define

$$f(n,t) \triangleq \sum_{k=1}^{7} F^{(k)}\left[\sqrt{\frac{t}{4(t+4)}} + \frac{1}{2}\right](n+1)^{-k}\mu_k^{(n+1)} - \frac{1}{2}\left(\frac{1}{n} + \frac{1}{2n^2} - \frac{1}{3n^3}\right).$$

Proving (26) is equivalent to showing that $f(n,t) \geq 0 \ \forall\ n > n_0(p)$. Simplifying

$$\begin{aligned} f(n,t) = \frac{1}{420(n+1)^6 n^3}\Big[&35n^7 t + (70 + 315t + 35t^2)n^6 \\ &- (315 + 2989t + 3339t^2 + 721t^3)n^5 \\ &- (826 - 721t - 371t^2 + 1568t^3 + 546t^4)n^4 \\ &- (826 + 90t + 135t^2 + 157t^3 + 10t^5 + 66t^4)n^3 \\ &- 630n^2 - 315n - 70\Big]. \end{aligned} \quad (29)$$

Define $g(n,t) \triangleq 420(n+1)^6 n^3 f(n,t)$.

A simple but elaborate calculation yields $g(4.44t + 7 + m, t) \approx 35tm^7 + (1122.8t^2 + 2030t + 70)m^6 + (14700.90t^3 + 52210.20t^2 + 48120.80t + 2625)m^5 + (1.01t^4 + 5.32t^3 + 9.57t^2 + 6.06t + 0.40)10^5 m^4 + (3.85t^5 + 26.94t^4 + 72.32t^3 + 88.61t^2 + 43.61t + 3.02)10^5 m^3 + (7.76t^6 + 68.23t^5 + 247.042t^4 + 456.97t^3 + 433.17t^2 + 176.77t + 11.80)10^5 m^2 + (6.47t^7 + 70.91t^6 + 338.88t^5 + 880.98t^4 + 1297.85t^3 + 1030.51t^2 + 361.59t + 20.14)10^5 m + (0.15t^8 + 56.29t^7 + 709.80t^6 + 3485.03t^5 + 8728.40t^4 + 11955.74t^3 + 8613.06t^2 + 2628.77t + 64.15)10^4$.

Note that all the coefficients are positive and hence, $f(4.44t + 7 + m, t) \geq 0$ for all $m \geq 0$ or $f(n,t) \geq 0$ for all $n \geq 4.44t + 7$. A more careful choice would yield all coefficients to be positive for $n \geq 4.438t + 7$. Yet another choice that would yield all coefficients as positive would be $n \geq t^2 + 2.34t + 7$. Note that this choice would be better for $0 < t < 2.1$ and, in particular, for $t = 1$, the first choice yields (after constraining $n$ to be a natural number) $n \geq 12$ while the second one yields $n \geq 11$. Further refinements are also possible.