

Near-Optimal Dynamical Decoupling of a Qubit

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We present a near-optimal quantum dynamical decoupling scheme that eliminates general decoherence of a qubit to order n using $O(n^2)$ pulses, an exponential decrease in pulses over all previous decoupling methods. Numerical simulations of a qubit coupled to a spin bath demonstrate the superior performance of the new pulse sequences.

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Coherent quantum control such as quantum information processing requires the faithful manipulation and preservation of quantum states. In the course of a quantum evolution, uncontrolled coupling between a quantum system and its environment (or bath) may cause the system state to decohere and deviate from its desired evolution. Here we present a dynamical decoupling (DD) [1] scheme designed to mitigate this effect. DD combats this decoherence by suppressing the system-bath interaction through stroboscopic pulsing of the system, an idea which can be traced to the Hahn spin echo [2]. Recently, various DD schemes have been put to experimental tests, successfully suppressing qubit decoherence in a variety of different systems [3,4]. Our new DD scheme is near optimal, and provides an exponential improvement over all previously known DD protocols utilizing instantaneous pulses. It suppresses arbitrary coupling between a single-qubit and its environment to n th order in a perturbative expansion of the total qubit-bath propagator, using $O(n^2)$ pulses.

The most general interaction between a qubit and a bath can be modeled via a Hamiltonian of the form $H = I \otimes B_I + X \otimes B_X + Y \otimes B_Y + Z \otimes B_Z$, where X , Y , and Z are the Pauli matrices, I denotes the identity matrix, and B_α are arbitrary Hermitian bath operators. The term B_I is the internal bath Hamiltonian. Our scheme builds upon two recent insights. The first is due to Uhrig [5], who—for single-qubit decoherence consisting of pure dephasing errors ($H_Z = I \otimes B_I + Z \otimes B_Z$)—found a scheme (“Uhrig DD”—UDD) which prescribes a sequence of X pulses (π rotations around the \hat{x} axis) at times

$$t_j = T \sin^2\left(\frac{j\pi}{2n+2}\right), \quad (1)$$

where T is the total evolution time and $j = 1, 2, \dots, n$, if n is even, and $j = 1, 2, \dots, n+1$, if n is odd. These times characterize a filter function [4] that removes the qubit-bath coupling to n th order in a perturbative expansion of the total system-bath propagator. UDD is provably optimal in that it achieves the minimum number of pulse intervals, $n+1$, required to accomplish this removal [6,7]. For

single-qubit decoherence consisting of pure spin-flip errors ($H_X = I \otimes B_I + X \otimes B_X$), UDD is still optimal, simply by substituting Z pulses (π pulses around the \hat{z} axis) for X pulses. However, for systems subject to general errors as prescribed by H , DD schemes incorporating both X and Z pulses are required. One such scheme, which provides our second source of insight, is concatenated DD (CDD) [8]: the CDD sequence is capable of eliminating arbitrary qubit-bath coupling to order n at a cost of $O(4^n)$ pulses [9]. CDD works by recursively nesting a pulse sequence found in Ref. [1], capable of canceling arbitrary decoherence to first order. Uhrig recently introduced a hybrid scheme (CUDD) which reduces the pulse count to $O(n2^n)$ for exact order n cancellation [10]. By appropriately concatenating the UDD sequences for H_Z and H_X we show here how arbitrary decoherence due to H can be exactly canceled to order n using only $(n+1)^2$ pulse intervals. A numerical search we conducted found that this is very nearly optimal for small n , differing from the optimal solutions by no more than two pulses.

Near-optimal pulse sequence construction.—The goal of our construction is to integrate an X -type UDD $_n$ sequence, which suppresses pure dephasing error to order n , with a Z -type UDD $_n$ sequence, which suppresses longitudinal relaxation to order n , so that the resulting sequence removes arbitrary error to order n . Since the total time T in Eq. (1) is arbitrary, what matters for error cancellation is not the precise pulse times t_j , but rather the *relative sizes* of the pulse intervals $\tau_j \equiv (t_j - t_{j-1})$, for $j = 1, 2, \dots, n+1$. Thus, the most relevant quantities are normalized pulse intervals,

$$s_j \equiv \frac{t_j - t_{j-1}}{t_1 - t_0} = \sin\left(\frac{(2j-1)\pi}{2n+2}\right) \csc\left(\frac{\pi}{2n+2}\right), \quad (2)$$

again with $j = 1, 2, \dots, n+1$. Here we chose to normalize with respect to the shortest pulse interval $(t_1 - t_0) = t_1$, so that $\tau_j = s_j t_1$, which has the important consequence that the (normalized) total time grows with n , as pulse intervals are added to address higher order error. This fixing of the minimum pulse interval t_1 corresponds to imposing a finite

bandwidth constraint. Since any physical implementation will not be able to shrink the pulse intervals arbitrarily, but will be limited by the fastest pulsing technology available, the change in perspective from fixed total time to fixed minimum interval is appropriate. The total normalized time of a UDD_n sequence is then given by

$$S_n \equiv \sum_{j=1}^{n+1} s_j = \frac{t_{n+1}}{t_1} = \csc^2\left(\frac{\pi}{2n+2}\right), \quad (3)$$

so that the total physical time is $T = S_n t_1$.

Let $U(\tau)$ denote the joint unitary free evolution of a qubit and its bath for a time τ , subject to the Hamiltonian H . A Z-type UDD_n sequence then takes the form

$$Z_n(\tau) \equiv Z^n U(s_{n+1}\tau) Z U(s_n\tau) \cdots Z U(s_2\tau) Z U(s_1\tau). \quad (4)$$

Note that a final Z pulse is required for n odd. Define the X-type UDD_n sequence $X_n(\tau)$ similarly. It follows from the proofs in [6] that the entire UDD sequence $X_n(t_1)$ can be expressed as the propagator $\exp(-iS_n t_1 (I \otimes B'_I + X \otimes B'_X) + O((\lambda S_n t_1)^{n+1}))$ (in units of $\hbar = 1$), where $\lambda \sim \|H\|$, provided $\lambda S_n t_1$ is sufficiently small to ensure convergence of the time-perturbative expansion. The important point here is that the resulting effective Hamiltonian $H' = I \otimes B'_I + X \otimes B'_X$ is a purely spin-flip, analytically time-dependent coupling [11]. The correction term $O((\lambda S_n t_1)^{n+1})$ potentially contains all couplings with a complicated time dependence, but is suppressed to order n . Moreover, subject to the convergence condition, we are free to choose the minimum interval arbitrarily without impacting the validity of the proofs in [6]. This is the key to correctly integrating the X-type and Z-type UDD_n sequences. The desired error cancellation properties only require that the normalized pulse intervals s_j have the specified structure. The precise physical timing is inconsequential. Therefore, to integrate X_n and Z_n sequences properly, without breaking the delicate pulse timing structure required for error cancellation, we must *scale* the pulse intervals of the inner sequences uniformly with respect to the outer pulse sequence structure. Hence, if each $U(s_j\tau)$ in Eq. (4) is replaced by the time-scaled DD sequence $X_n(s_j\tau)$, then the outer Z_n sequence suppresses the purely spin-flip coupling H' remaining after each $X_n(s_j\tau)$ sequence, producing general decoherence suppression to order n with only $(n+1)^2$ pulse intervals. So the combined, near-optimal “quadratic DD” (QDD) sequence, takes the form

$$\text{QDD}_n(\tau) \equiv Z^n X_n(s_{n+1}\tau) Z X_n(s_n\tau) \cdots Z X_n(s_1\tau), \quad (5)$$

abbreviated by the notation $\text{QDD}_n(\tau) = Z_n(X_n(\tau))$. Notice how the relative scales of the pulse intervals are preserved. In each of the inner $X_n(s_j\tau)$ sequences, the ratio between successive intervals remains $(s_{k+1}s_j\tau)/(s_k s_j\tau) = s_{k+1}/s_k$, and for the outer Z_n sequence the ratio is $(S_n s_{j+1}\tau)/(S_n s_j\tau) = s_{j+1}/s_j$, thereby ensuring the error cancellation properties of each sequence are left intact.

Of course, an equivalent QDD_n sequence may be constructed as $X_n(Z_n(\tau))$. Moreover, though the inner DD sequences must have an equal number of intervals, they need not be the same length as the outer sequence, but can instead be adjusted to more efficiently address the dominant sources of error in any particular implementation. In this way, $\text{QDD}_{m,n} = Z_m(X_n(\tau))$ is the more general construction, where the inner sequences suppress one type of error to order n , while the outer sequence suppresses the remaining error to order m . As the simplest explicit example, $\text{QDD}_1 = ZX_1(s_2\tau)ZX_1(s_1\tau) = Z(XU(s_2^2\tau)XU(s_1s_2\tau))Z(XU(s_2s_1\tau)XU(s_1^2\tau)) = YU(\tau) \times XU(\tau)YU(\tau)XU(\tau)$, which we recognize as the so-called “universal decoupler” sequence found in [1] and used as the basis of the CDD sequence in [8].

It is worth noting that the construction of QDD_n affords a nice visualization. For $\sigma \in \{I, X, Y, Z\}$, define $\sigma(s_j) \equiv \sigma U(s_j\tau)\sigma$, then consider,

$$\begin{array}{cccc|c} Y(s_{n+1}^2) & Z(s_n s_{n+1}) & \cdots & Y(s_2 s_{n+1}) & Z(s_1 s_{n+1}) & Z(s_{n+1}) \\ X(s_{n+1} s_n) & I(s_n^2) & \cdots & X(s_2 s_n) & I(s_1 s_n) & I(s_n) \\ & & \vdots & & & \vdots \\ Y(s_{n+1} s_2) & Z(s_n s_2) & \cdots & Y(s_2^2) & Z(s_1 s_2) & Z(s_2) \\ X(s_{n+1} s_1) & I(s_n s_1) & \cdots & X(s_2 s_1) & I(s_1^2) & I(s_1) \\ \hline X(s_{n+1}) & I(s_n) & \cdots & X(s_2) & I(s_1) & |\psi\rangle \end{array}$$

with the final QDD_n pulse sequence formed by reading off rows of the large upper square from top to bottom, or columns from left to right. In other words, our construction may be succinctly described as an outer product between X-type and Z-type UDD sequences.

Also, note that the QDD_n sequence requires a total physical time of $S_n^2 \tau$, and just as in the proofs [6] of order n error suppression for the Z_n sequence, the condition for convergence of the perturbative expansion still remains, namely, that $\lambda S_n^2 \tau$ is sufficiently small. An immediate consequence of this constraint is that there exists a maximal order of error n that can be suppressed for a given τ , beyond which the error cancellation properties of the sequence begin to break down. Conversely, if one hopes to suppress error to some fixed order n , then this implies a maximum pulse rate $r = 1/\tau$ which must be attained. Specifically, a sufficient condition for convergence is $\lambda S_n^2 \tau < 1$, hence $r > \lambda \csc^4(\pi/(2n+2))$.

Finally, we translate these results back into a precise physical timing for the individual pulses. If the total time for the sequence is T , then the outer Z_n sequence requires that Z pulses occur at the original Uhrig times prescribed in Eq. (1), while the inner X_n sequences require X pulses executed at the times, $t_{j,k} = \tau_j \sin^2(\frac{k\pi}{2n+2}) + t_{j-1}$, where $j, k = 1, 2, \dots, n$ if n is even and $j, k = 1, 2, \dots, n+1$ if n is odd. When n is odd, X and Z pulses coincide at times t_j , in which case $Y = ZX$ pulses are used.

Numerical results.—We now present numerical results that illustrate the efficiency of our new DD pulse sequences in preserving arbitrary initial quantum states. In the full-state quantum memory simulations that follow, the system

and bath are initialized together as a (generally nonseparable) random pure state $|\psi\rangle$. The results presented involve a single system qubit coupled to four bath qubits. While this is clearly an unrealistically small bath, larger simulations we performed indicate that the relevant error suppression properties of our DD sequences are qualitatively unaffected by bath size. Similarly idealized are the DD pulses themselves, which we take to be infinitely strong zero-width pulses, consistent with the analysis in [6]. The evolution of the coupled system and bath in our simulations is governed by the generic Hamiltonian H , with additional parameters $\{J, \beta\}$ included to control the coupling and bath strengths, respectively:

$$H = \beta(I \otimes B_I) + J(X \otimes B_X + Y \otimes B_Y + Z \otimes B_Z), \quad (6)$$

where each bath operator is given by $B_\alpha = \sum_{i \neq j} \sum_{k,l} r_{kl}^\alpha (\sigma_i^k \otimes \sigma_j^l)$, with $\alpha, k, l \in \{I, X, Y, Z\}$, i, j indexing the bath qubits, and randomly chosen coefficients $r_{kl}^\alpha \in [0, 1]$. Note that these bath operators include all 1- and 2-body terms, so that the system-bath Hamiltonian includes 2- and 3-body terms, a worse-case scenario than pure 2-body terms. The numerical simulations are run with more than 100 digits of precision and results are averaged over 10 random realizations, with each instance randomly generating new bath operators and a new initial state. The vertical axes in each of these plots quantifies the DD sequence performance as $\log_{10}(D)$, where D is the standard trace-norm distance $\frac{1}{2} \|\rho(T) - \rho(0)\|_1$ between the evolved system state $\rho(T) = \text{tr}_B(|\phi\rangle\langle\phi|)$, with $|\phi\rangle = \text{QDD}_n(\tau)|\psi\rangle$, and the initial system state $\rho(0) = \text{tr}_B(|\psi\rangle\langle\psi|)$. Here tr_B is the partial trace over the bath. This distance measure bounds the usual Uhlman fidelity F from above and below, $1 - D \leq F \leq \sqrt{1 - D^2}$, and is itself bound by the norm of a perturbative expansion of the propagator [12]. Indeed, in our simulations, D always varies as the norm of leading order term, as one might expect. Error bars are included at every point indicating the maximum deviation from the average distance measured.

Figure 1 shows the performance of various DD schemes plotted against the number of pulses each requires. PDD_n is the n -times repeated universal decoupling sequence found in [1], requiring $4n$ pulses (this sequence is capable only of first order decoupling); CDD_n is the n -times concatenated universal decoupling sequence, requiring 4^n pulses [8]; CUDD_n is concatenated Uhrig DD, combining an n th order X -type Uhrig sequence with n concatenation levels of Z pulses, requiring $n2^n$ pulses [10]. The coupling parameters J and β are fixed so that $J\tau = \beta\tau = 10^{-6}$, given a shortest pulse interval τ . Recall that since the shortest pulse interval is held constant, the total evolution time grows with increasing n . Specifically, for QDD_n the total evolution time is $S_n^2\tau$. The evolution times of the other DD sequences also increase with n , but at different rates depending on how the number of required pulse intervals scales with n . Notice the two most visible consequences of increasing total time in Fig. 1: (1) at $n = 0$ the evolution time is only $S_0^2\tau = \tau$, so

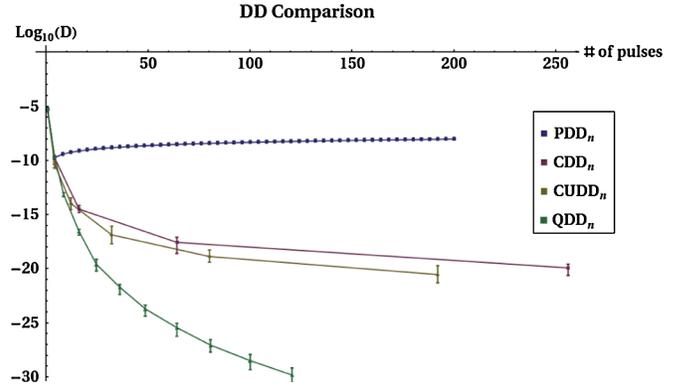


FIG. 1 (color online). Performance comparison of various DD schemes vs number of pulses required. QDD_n dramatically outperforms all known alternatives in both state preservation and number of pulses. In this and the next two figures labels in the legends are positioned in correspondence with the positions of the curves they label.

the dominant contribution to the overall fidelity is $J\tau = 10^{-6}$, which explains why the uncoupled evolution still achieves excellent agreement with the initial state in this plot; and (2) as the total time increases so does decoherence, explaining why the performance of PDD_n actually degrades with large n .

Figure 2 shows QDD_n performance as a function of J , the system-bath coupling strength. As $JS_n^2\tau$ approaches 1, the qubit decoheres so rapidly that DD has essentially no effect, and the distance between the initial and final states approaches its maximum of 1. Again, the $n = 0$ line represents uncoupled free evolution, for which $JS_n^2\tau = 1$ when $J\tau = 1$, corresponding to the point in this plot where the $\log_{10}(D)$ becomes zero and subsequently stays there. State preservation improves as n increases and another order of error is suppressed, with the effect magnified as $J\tau$ decreases, though increasing total time counteracts the overall performance gain. This is evidenced in how, at each fixed $J\tau$, the magnitude of the improvement decreases as n increases, or in other words, in how the gap between lines narrows as n increases. The dotted line in both Figs. 2 and

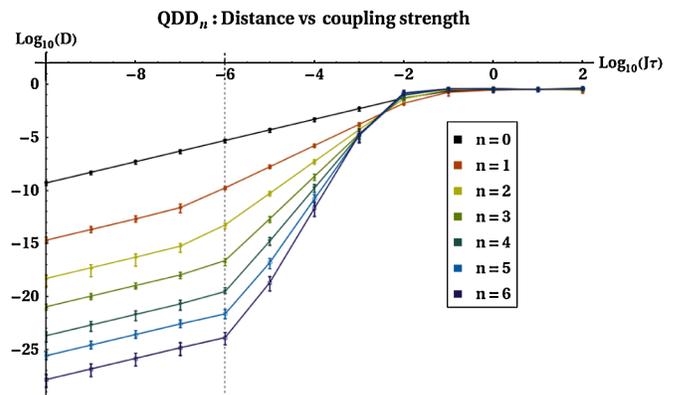


FIG. 2 (color online). QDD_n performance dependence on coupling strength J , with fixed $\beta\tau = 10^{-6}$.

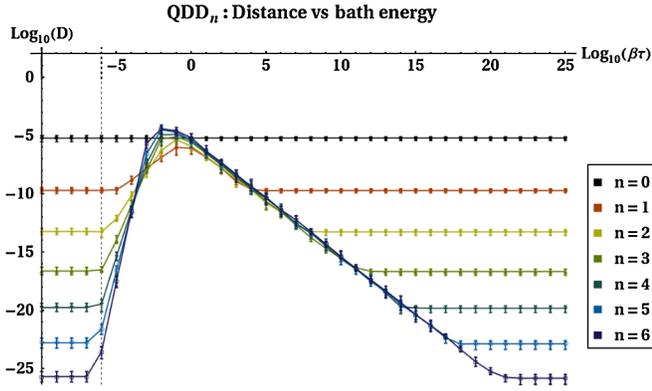


FIG. 3 (color online). QDD_n performance dependence on bath strength, with fixed $J\tau = 10^{-6}$.

3, indicates when $J\tau = \beta\tau$. Across this transition point, the slope increases as the leading order error term changes from β to J dominated in Fig. 2, and vice versa in Fig. 3. Indeed, when $J < \beta$, the leading order term contributes to distance as $\frac{(nJ\beta^n)(S_n^2\tau)^{n+1}}{(n+1)!}$, which gives a slope of 1 as a function of J for every n in Fig. 2. On the other hand, when $J > \beta$, the dominant contribution to performance becomes $\frac{(JS_n^2\tau)^{n+1}}{(n+1)!}$, explaining the observed slopes of $n + 1$ in this plot.

Figure 3 shows the performance dependence of QDD_n vs β , the parameter which quantifies the pure bath energy scale. We again see the improvement with increasing n . The obviously significant feature of this plot, however, is the deviation, and then eventual returning, of the evolved state relative to the initial state as $\beta\tau$ sweeps through the regimes: (1) $\beta\tau < J\tau < S_n^{-2}$, (2) $J\tau < \beta\tau < S_n^{-2}$, and (3) $\beta\tau > S_n^{-2}$. This behavior is similar to motional narrowing and can be understood mathematically by considering an interaction picture with respect to the pure bath dynamics, governed by $H_B = \beta(I \otimes B_I)$, then expanding the other bath operators in the B_I eigenbasis. Using a Dyson series expansion with $JS_n^2\tau < 1$, we find that when $\beta\tau < J\tau$, the distance varies as $\frac{(JS_n^2\tau)^{n+1}}{(n+1)!}$, as in the previous plot, independent of β ; when $J\tau < \beta\tau < S_n^{-2}$, the distance varies as $\frac{(nJ\beta^n)(S_n^2\tau)^{n+1}}{(n+1)!}$, giving a slope of n as a function of β ; and when $\beta\tau > S_n^{-2}$, the distance varies as $[(C\frac{J}{\beta} + 1)\frac{(JS_n^2\tau)^{n+1}}{(n+1)!}]$ (C is a constant), which gives a slope of -1 until β is sufficiently large so as to eliminate the contribution of the first term, leaving a leading order term that goes again like the small β regime.

Conclusions.—We have presented a DD pulse sequence construction that guarantees simultaneous cancellation of both transverse dephasing and longitudinal relaxation to order n using only $(n+1)^2$ pulse intervals, or $O(n^2)$ pulses, an exponential improvement over all previous zero-width pulse DD methods. An exhaustive numerical search we performed produced optimal pulse sequences of lengths 7, 14, and 23, for $n = 2, 3$, and 4, respectively, i.e.,

$(n+1)^2 - 2$ pulses. In view of this, we believe our construction is very nearly optimal, in addition to having the significant conceptual benefit of a simple algorithmic description.

Finally, of particular importance will be generalizing this work to incorporate finite width pulses, as it is well known that the action of the system-bath interaction during the pulse can have substantial modifying effects [13]. Proposals to systematically mitigate this effect in DD have recently been put forth [14]. We look forward to experimental tests of QDD.

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