Wu *et al.* Reply: In our Letter [1] we presented a numerically *exact*, polynomial-time quantum computer algorithm for simulating the class of *general* pairing Hamiltonians (with arbitrary coupling constants V^r_{ml})

$$H_{\mathrm{p}} = \sum_{m=1}^{N} \frac{1}{2}\varepsilon_m \sigma_m^z + \sum_{r=\pm} \sum_{l>m=1}^{N} \frac{1}{2} V^r_{ml} (\sigma_m^x \sigma_l^x + r\sigma_m^y \sigma_l^y)$$

[see Eq. (1) in [1] for definitions]. The Comment by Dukelsky *et al.* [2] reviews an interesting but numerically *approximate* classical algorithm [density matrix renormalization group (DMRG)] which they claim to be applicable for the same problem. It is this difference between numerically approximate and exact algorithms that essentially renders irrelevant the criticism of our Letter [1] in the Comment [2]. There is no known classical algorithm for simulating the class of general pairing models that is both efficient (polynomial) *and* numerically exact. There is such a quantum algorithm: the one proposed in our Letter.

Unfortunately, there is a misleading and confusing statement in the Comment [2]: Dukelsky *et al.* write in the fourth paragraph that (all emphasis ours) "It was shown in this context that only by resorting to the *exact* numerical solution [4], using the DMRG...." However, the DMRG is *not* an exact algorithm. It is an approximation. Indeed, citing from Dukelsky and Sierra's own paper [3] (their Ref. [4], Ref. [14] in our Letter): "We show that the DMRG gives an accurate *approximation* to the exact ground state if the block is taken to be the set of particles while the environment is taken to be the set of holes." And later in the same paper they write: "The DMRG is a *variational* method and in the region under study we expect our results to coincide with the exact ones with a relative *error* less than 10^{-4}." [3].

These statements by the authors of the Comment clearly demonstrate that the DMRG method is not a numerically exact solution of the pairing Hamiltonians. Hence the term "exact" in the the above quote from their Comment is not correct, and the basis for their criticism of our Letter is invalidated.

The authors have provided evidence that the DMRG can give a good approximation to the low-lying energy spectrum of the *reduced* BCS Hamiltonian (with constant pairing interaction $V^r_{ml} \equiv V^r$), and can do so in $O(N)$ steps [3]. We do not dispute this, and made no statement to the contrary in our Letter, since we were concerned only with a numerically exact algorithm. However, in spite of their claim to the contrary in their Comment ("the DMRG approach can easily accommodate arbitrary pairing matrix elements"), there is presently no evidence to support the hope that the quality of the approximation and the efficiency of the algorithm will remain as good for *general* pairing Hamiltonians (i.e., $V^r_{ml} \neq V^r$ for all m, l). The burden of proof is upon the authors of the Comment. Moreover, even if they could demonstrate this, the basic point remains that theirs is a numerically approximate algorithm while ours is exact. Indeed, as we stated in our Letter, for half-filled states the dimension of the Hilbert space grows as $N!/[(N/2)!]^2$ (for even N), which is superexponential in N, and there is no known way to avoid this divergence in a classical algorithm that is numerically exact. In contrast, we have given a quantum computer algorithm that is numerically exact (in the sense that it exactly diagonalizes the pairing Hamiltonian), requiring only $O(N^4)$ steps, for arbitrary N. Any approximate method, including DMRG and projected BCS [3], must eventually be checked and corrected by comparison to numerically exact algorithms if and when we have quantum computers.

In the second part of the Comment, the authors consider a pairing model that includes a monopole interaction [third term in their Eq. (1)], not included in our pairing Hamiltonian, so it does not apply to our Letter. They point out that if the coupling constants V^1_{ij} and V^2_{ij} (in their notation) satisfy certain relations then the model is exactly solvable. This is an elegant result which we were well aware of, but that bears *no* relation to our work, which deals, of course, with the cases for which no analytical solution is possible. However, Dukelsky *et al.* write the following [2]: "We believe that most of the physical problems can be modeled with a pairing Hamiltonian within the integrable subset...," indicating that the cases that are not analytically solvable are somehow typically not physically relevant. This conclusion is invalidated by their own parameter count: the ratio of integrable to nonintegrable models is $[(6N + 3)/(2N^2 - N)] \to 0$ as $N \to \infty$, so the class of integrable models occupies a negligibly small fraction of the parameter space in the interesting regime of $N \gg 1$.

In conclusion, the Comment [2] discusses approximate classical algorithms and analytically solvable instances of the pairing model which are certainly important, but are not directly comparable to our general and numerically exact algorithm.

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