Towards Fault Tolerant Adiabatic Quantum Computation

Daniel A. Lidar

Departments of Chemistry, Electrical Engineering, and Physics, Center for Quantum Information Science & Technology, University of Southern California, Los Angeles, California 90089, USA

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I show how to protect adiabatic quantum computation (AQC) against decoherence and certain control errors, using a hybrid methodology involving dynamical decoupling, subsystem and stabilizer codes, and energy gaps. Corresponding error bounds are derived. As an example, I show how to perform decoherence-protected AQC against local noise using at most two-body interactions.

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Adiabatic quantum computation (AQC), originally developed to solve optimization problems [1], offers a fascinating alternative to the standard circuit model [2] to which it is computationally equivalent [3]. The effects of decoherence on AQC were studied in several works [4–6]. Unlike the circuit model, for which an elaborate theory of fault tolerant QC exists along with a noise threshold for fault tolerance [7], it is not yet known how to make AQC fault tolerant. Here, I show how AQC can be protected against decoherence and certain control errors. To do so, I devise a hybrid method involving dynamical decoupling (DD) [8], subsystem [9–11] and stabilizer codes [12], and energy gaps [13,14].

Viewed as a closed system, AQC proceeds via slow evolution on a time scale set by the system's minimal energy gap Δ from the ground state [1,3]. In the presence of the system-bath interaction H_{SB}, this gap can be significantly reduced because the interaction will cause energy level splittings, or an effective broadening of system energy levels; when these levels overlap, adiabaticity breaks down and so does AQC, even at zero temperature [5]. A bath at finite temperature presents another problem: in the universality proofs [3], the system energy gap scales as an inverse polynomial in the problem size, so that the temperature too must be lowered polynomially to prevent thermal excitations. All of the problems listed above are due to the presence of H_{SB}. Clearly, if H_{SB} can be effectively eliminated or reduced, this will enhance the fidelity of AQC. The main tool I shall use to this end is dynamical decoupling, which involves the application of strong and fast pulses. Perhaps surprisingly, this can be done without interfering with the slow adiabatic evolution.

Distance measure and operator norm.—As a distance measure between states, I use the trace distance $D[\rho_1, \rho_2] \equiv \frac{1}{2}\|\rho_1 - \rho_2\|_1$, where $\|A\|_1 \equiv \mathrm{Tr}|A|$, $|A| \equiv \sqrt{A^\dagger A}$ [2]. When applied to pure states $\rho_i = |\psi_i\rangle\langle\psi_i|$, I shall write $D[\psi_1, \psi_2]$. The operator norm is $\| A \| \equiv \sup_{\||\psi\rangle\|=1}\|A|\psi\rangle\| = \max_i \lambda_i$, where $\lambda_i \in \mathrm{Spec}(|A|)$.

Closed-system adiabatic error.—Let $s = t/T \in [0, 1]$ be the dimensionless time, with T the final time. Let the system Hamiltonian that implements AQC, $H_{\mathrm{ad}}(s)$, act on n qubits. In AQC, the ground state $|\phi_{\mathrm{ad}}(s)\rangle$ of $H_{\mathrm{ad}}(s)$ at the final time $s = 1$ encodes the solution to the computational problem [1]. The actual final state $|\psi(1)\rangle$ is the solution of the Schrödinger equation $d|\psi\rangle/ds = -iTH_{\mathrm{ad}}|\psi\rangle$ ($\hbar = 1$ units are used throughout). In AQC, one is therefore interested in minimizing the error $\delta_{\mathrm{ad}} \equiv D[\psi(1), \phi_{\mathrm{ad}}(1)]$. Most of the known AQC algorithms interpolate between initial and final local Hamiltonians, H_0 and H_1, via $H_{\mathrm{ad}}(s) = [1 - f(s)]H_0 + f(s)H_1$, where $f(0) = 0$ and $f(1) = 1$, and exhibit a final time that scales as a polynomial in the problem/system size n. Locality means that $\| H_{\mathrm{ad}} \| \sim \Delta_0 O(n)$, where Δ_0 is the energy scale. Thus, $\| d^j H_{\mathrm{ad}}/ds^j \| \sim \Delta_0 |d^j f/ds^j| O(n)$. Let $\{E_i(s)\}_{i=0}$ be the eigenvalues of $H_{\mathrm{ad}}(s)$, and let $\Delta \equiv \min_{i,s}|E_i(s) - E_0(s)|$ be the minimum gap from the instantaneous ground state energy $E_0(s)$. Assume that $\Delta(n) \sim \Delta_0 n^{-z}$, where $z > 0$ is the dynamical critical exponent. Depending on the differentiability of H_{ad}, and assuming that $\dot{H}_{\mathrm{ad}}(0) = \dot{H}_{\mathrm{ad}}(1) = 0$, one can prove different versions of the adiabatic theorem. For example, (i) [15]: if $H_{\mathrm{ad}}(s)$ is twice differentiable on [0,1], then provided $T \sim r\|\dot{H}_{\mathrm{ad}}\|^2/\Delta^3$, the error can be made arbitrarily small in the time dilation factor $r > 1$: $\delta_{\mathrm{ad}} < r^{-2}$. Or, (ii) [16]: if $H_{\mathrm{ad}}(s)$ is infinitely differentiable on [0,1], then provided $T \sim rN\|\dot{H}_{\mathrm{ad}}\|/\Delta^2$, the error can be made exponentially small in the order N of an asymptotic expansion: $\delta_{\mathrm{ad}} < r^{-N}$. In both cases,

$$T \sim n^\zeta/\Delta_0, \tag{1}$$

where $\zeta = 3z + 2$ for case (i) and $\zeta = 2z + 1$ for case (ii), and I omitted the n-independent term $|d^j f/ds^j|$. In AQC, the interpolation from $H_{\mathrm{ad}}(0)$ to $H_{\mathrm{ad}}(1)$ can be chosen at will, in particular, so as to satisfy the above conditions on H_{ad}. This shows that closed-system AQC is resilient against control errors which cause $H_{\mathrm{ad}}(s)$ to deviate from its intended path, as long as these do not modify the end point $H_{\mathrm{ad}}(1)$. This is a form of inherent fault tolerance to control errors which is not shared by the circuit model [17].

Open system evolution.—A description in terms of H_{ad} alone neglects the fact that in reality, the adiabatic quantum computer system is never perfectly isolated. The actual

Hamiltonian is $H(t) = H_S(t) \otimes I_B + I_S \otimes H_B + H_{SB}$, where I denotes the identity operator, $H_S = H_{\rm ad} + H_C$ (H_B) acts on the system (bath) alone, $H_C(t)$ is a control Hamiltonian, and $H_{SB} = \sum_\alpha S_\alpha \otimes B_\alpha$, where S_α (B_α) acts on the system (bath). The role of H_C is to implement a DD procedure. The total propagator is $U(t) = \mathcal{T} \exp[-i \int_0^t H(t')dt']$, where $\mathcal{T}$ denotes time ordering. The time evolved system state is $\rho_S(t) = {\rm Tr}_B \rho(t)$, where $\rho(t) = U(t)\rho(0)U(t)^\dagger$ is the joint system-bath state. Below, I explain how to choose $H_C(t)$ so that

$$[H_{\rm ad}(t), H_C(t')] = 0 \quad \forall\ t, t'. \tag{2}$$

It is this condition that will allow application of DD without interfering with the adiabatic evolution. Consider the uncoupled setting $H_{SB} = 0$, to be denoted by the superscript 0. The ideal, noise-free adiabatic system state is $\rho^0_{S,{\rm ad}}(t) = |\phi_{\rm ad}(t)\rangle\langle\phi_{\rm ad}(t)|$. Because the adiabatic, control, and bath Hamiltonians all commute, we have $\rho^0(t) = \rho^0_S(t) \otimes \rho^0_C(t) \otimes \rho^0_B(t)$, where $\rho^0_S(t) = |\psi(t)\rangle\langle\psi(t)|$ [$\rho^0_C(t) = |\psi_C(t)\rangle\langle\psi_C(t)|$] is the actual system evolution under $H_{\rm ad}$ [H_C], and $\rho^0_B(t)$ is the bath state evolved under H_B. Let $\rho^0_{\rm ad}(t) \equiv \rho^0_{S,{\rm ad}}(t) \otimes \rho^0_C(t) \otimes \rho^0_B(t)$ denote the "ideal adiabatic joint state," with purely adiabatic evolution of the first factor. Note that $\rho^0_S(0) = \rho^0_{S,{\rm ad}}(0)$.

General error bound.—Let d (δ) denote distances in the joint (system) Hilbert space. To quantify the deviation of the actual evolution from the desired one, let

$$\delta_S \equiv D[\rho_S(T), \rho^0_{S,{\rm ad}}(T)], \qquad d_D \equiv D[\rho(T), \rho^0(T)]$$
$$d_{\rm ad} \equiv D[\rho^0(T), \rho^0_{\rm ad}(T)] = \delta_{\rm ad}, \qquad d_{\rm tot} \equiv D[\rho(T), \rho^0_{\rm ad}(T)].$$

The overall objective is to minimize the distance δ_S between the actual system state and the ideal, noise-free adiabatic system state. The distance between the uncoupled joint state and the ideal adiabatic joint state is $d_{\rm ad}$, which equals $\delta_{\rm ad}$ since $\|A \otimes B\|_1 = \|A\|_1\|B\|_1$ and $\|\rho^0_B\|_1 = \|\rho^0_C\|_1 = 1$. The "decoupling distance" is d_D: the distance between the joint state in the coupled and uncoupled settings. Minimization of this distance is the target of the DD procedure. Finally, $d_{\rm tot}$ is the distance between the actual and ideal joint states.

Because taking the partial trace can only decrease the distance between states [2], we have $\delta_S \le d_{\rm tot}$. Using the triangle inequality, we have $d_{\rm tot} \le d_D + d_{\rm ad}$. Therefore,

$$\delta_S \le d_D + \delta_{\rm ad}. \tag{3}$$

This key inequality shows that the total system error is bounded above by the sum of two errors: (i) due to the system-bath interaction in the presence of decoupling (d_D); (ii) due to the deviations from adiabaticity in the closed system ($d_{\rm ad}$). I shall present a procedure intended to minimize d_D jointly with $d_{\rm ad}$. This is an optimization problem: generically, decoherence (closed-system adiabaticity) worsens (improves) with increasing T.

Dynamical decoupling.—I now show how to minimize the decoupling error d_D. To do so, I propose to apply strong and fast dynamical decoupling (DD) pulses to the system on top of the adiabatic evolution. It is convenient to first transform to an interaction picture defined by $H_{\rm ad} + H_B$, i.e., $U(t) = U_{\rm ad}(t) \otimes U_B(t)\tilde{U}(t)$, where $U_X(t) = \mathcal{T} \exp[-i \int_0^t H_X(t')dt']$, $X \in \{{\rm ad}, B\}$. Then $\tilde{U}$ satisfies the Schrödinger equation $\partial\tilde{U}/\partial t = -i\tilde{H}\,\tilde{U}$, with $\tilde{H} = U_B^\dagger \otimes U_{\rm ad}^\dagger[H_C + H_{SB}]U_B \otimes U_{\rm ad} = H_C + \tilde{H}_{SB}$, where the second equality required Eq. (2). Define an effective "error Hamiltonian" $H_{\rm eff}(t)$ via $\tilde{U}(t) = e^{-itH_{\rm eff}(t)}$, which can be conveniently evaluated using the Magnus expansion [18]. Now consider a sequence of nonoverlapping control Hamiltonians $H^{(k)}_{\rm DD}(t)$ applied for duration w (pulse width) at pulse intervals τ, i.e., $H_C(t) = 0$ for $t_k \le t < t_{k+1} - w$ and $H_C(t) = H^{(k)}_{\rm DD}$ for $t_{k+1} - w \le t < t_{k+1}$, where $t_k = k(\tau + w)$, $k \in \mathbb{Z}_K$. The sequence $\{H^{(k)}_{\rm DD}\}_{k=0}^{K-1}$ defines a "DD protocol" with cycle time $T_c = K(\tau + w)$ and unitary pulses P_k generated by $\tilde{H}(t) = H^{(k)}_{\rm DD} + \tilde{H}_{SB}$, $t_{k+1} - w \le t < t_{k+1}$. In the "ideal pulse limit" $w = 0$, one defines the "decoupling group" $\mathcal{G} = \{G_k \equiv P_{K-1} \cdots P_{k+1}P_k\}_{k=0}^{K-1}$ such that $G_0 = I_S$. Then, the total propagator becomes $\tilde{U}(T_c) = \prod_{k=0}^{K-1} \exp[-i\tau(G_k^\dagger \tilde{H}_{SB} G_k)] \equiv e^{-iT_c H^{\rm id}_{\rm eff}}$, where $H^{\rm id}_{\rm eff}$ denotes the resulting effective Hamiltonian, with Magnus series $H^{\rm id}_{\rm eff} = \sum_{j=0}^\infty H^{{\rm id}(j)}_{\rm eff}$ [8]. To lowest order

$$H^{{\rm id}(0)}_{\rm eff} = \frac{1}{K}\sum_{k=0}^{K-1} G_k^\dagger \tilde{H}_{SB} G_k \equiv \Pi_{\mathcal{G}}(\tilde{H}_{SB}). \tag{4}$$

In the limit $\tau \to 0$, one has $H^{\rm id}_{\rm eff} = H^{{\rm id}(0)}_{\rm eff}$, so that by properly choosing $\mathcal{G}$, one can effectively eliminate H_{SB}.

Returning to nonideal ($w > 0$) pulses, we have shown by use of $\|[A, B]\|_1 \le 2\|A\|\|B\|_1$ and the Dyson expansion that minimization of the "error phase" $\Phi(T) \equiv T \parallel H_{\rm eff}(T) \parallel$ implies minimization of the decoupling distance d_D [19]:

$$d_D \le \min[1, (e^\Phi - 1)/2]$$
$$\le \Phi \quad \text{if } \Phi \le 1. \tag{5}$$

For single-qubit systems, we and others have shown that concatenated DD pulse sequences can decrease Φ exponentially in the number of concatenation levels [20]. Here, I focus on periodic pulse sequences for simplicity. In periodic DD (PDD), one repeatedly applies the DD protocol $\{H^{(k)}_{\rm DD}\}_{k=0}^{K-1}$ to the system, i.e., $H_C(t + lK) = H_C(t)$, $l \in \mathbb{Z}_L$. The total time is thus $T = L(\tau + w)$, where the total number of pulses is L and the number of cycles is L/K. A calculation of the total error phase $\Phi(T)$ proceeds in two steps. First, we find an upper bound Θ_l on $\Phi_l(T_c)$ for the lth cycle, using the Magnus expansion. Then, we upper bound $\Phi(T)$ by $(L/K)\max_l\Theta_l$. Let $J \equiv \parallel H_{SB} \parallel$ (system-bath coupling strength), $\beta \equiv \parallel H_{\rm ad} + H_B \parallel \le \beta_S + \beta_B$, where $\beta_S = \parallel H_{\rm ad} \parallel$ and $\beta_B = \parallel H_B \parallel$, and $\alpha = O(1)$ a constant. A worst case analysis yields [21]

$$\Phi(T) \leq \frac{\alpha(JT)^2}{L/K} + \frac{JTw}{\tau + w} + JT\left(\frac{\exp(2\beta T_c) - 1}{2\beta T_c} - 1\right). \quad (6)$$

This bound is valid as long as the third term is $\leq JT$ and the Magnus series is absolutely convergent over each cycle, a sufficient condition for which is $JT_c < \pi$ [18,21].

Joint AQC-DD optimization.—Recall Eq. (1) for closed-system adiabaticity. The given and fixed parameters of the problem are J, Δ_0, and z (or ζ). The task is to ensure that each of the terms in Eq. (6) vanishes as a function of n. I show in [22] that if τ and w scale as

$$\tau \sim n^{-(\zeta+\epsilon_1)}/\Delta_0, \qquad w \sim n^{-(2\zeta+\epsilon_1+\epsilon_2)}/J, \quad (7)$$

with $\epsilon_1 > 1$ and $\epsilon_2 > 0$, then

$$d_D \lesssim (J/\Delta_0)^2 n^{-\epsilon_1} + n^{-\epsilon_2} + (J/\Delta_0) n^{1-\epsilon_1}, \quad (8)$$

which is arbitrarily small in the large n limit. Combining this with the bounds above ($\delta_{\rm ad} < r^{-2}$ or $\delta_{\rm ad} < r^{-N}$) and inequality (3), it follows that for an AQC algorithm with time scaling as $T = L(\tau + w) \sim \Delta_0^{-1} n^{\zeta}$, the total error δ_S can be made arbitrarily small. This is the first main result of this work: using PDD with properly chosen parameters, we can obtain arbitrarily accurate AQC.

However, there is a shortcoming: the pulse intervals and widths must shrink with n as a power law, with an exponent dictated by the dynamical critical exponent z of the model [Eq. (7)]. I expect that this can be remedied by employing concatenated DD [20,21].

Seamless AQC-DD.—The entire analysis relies so far on the "noninterference" condition (2). When can it be satisfied? Fortunately, the general background theory was worked out in [9,10], though without any reference to AQC. I review this theory and make the connection to AQC explicit. The decoupling group $\mathcal{G}$ induces a decomposition of the system Hilbert space $\mathcal{H}_S$ via its group algebra $\mathbb{C}\mathcal{G}$ and its commutant $\mathbb{C}\mathcal{G}'$, as follows:

$$\mathcal{H}_S \cong \bigoplus_J \mathbb{C}^{n_J} \otimes \mathbb{C}^{d_J}, \quad (9)$$

$$\mathbb{C}\mathcal{G} \cong \bigoplus_J I_{n_J} \otimes M_{d_J}, \qquad \mathbb{C}\mathcal{G}' \cong \bigoplus_J M_{n_J} \otimes I_{d_J}. \quad (10)$$

Here, n_J and d_J are, respectively, the multiplicity and dimension of the Jth irreducible representation (irrep) of the unitary representation chosen for $\mathcal{G}$, while I_N and M_N are, respectively, the $N \times N$ identity matrix and unspecified complex-valued $N \times N$ matrices. The adiabatic state is encoded into (one of) the left factors $C_J \equiv \mathbb{C}^{n_J}$; i.e., each such factor (with J fixed) represents an n_J-dimensional code C_J storing $\log_d n_J$ qudits. The DD pulses act on the right factors. As shown in [9], the dynamically decoupled evolution on each factor (code) C_J will be noiseless in the ideal limit $w, \tau \to 0$ if $\Pi_{\mathcal{G}}(S_\alpha) = \bigoplus_J \lambda_{J,\alpha} I_{n_J} \otimes I_{d_J}$ for all system operators S_α in H_{SB}, whence $H_{\rm eff}^{\rm id(0)} = \bigoplus_J [(I_{n_J} \otimes I_{d_J})]_S \otimes [\sum_\alpha \lambda_{J,\alpha} B_\alpha]_B$. Thus, assuming the latter condition is met, under the action of DD, the action of $H_{\rm eff}^{\rm id(0)}$ on the code C_J is proportional to I_{n_J}, i.e., is harmless. Quantum logic, or AQC, is enacted by the elements of $\mathbb{C}\mathcal{G}'$. Dynamical decoupling operations are enacted via the elements of $\mathbb{C}\mathcal{G}$. Condition (2) is satisfied because $[\mathbb{C}\mathcal{G}, \mathbb{C}\mathcal{G}'] = 0$.

Stabilizer decoupling.—An important example of the general $\mathbb{C}\mathcal{G}/\mathbb{C}\mathcal{G}'$ construction is when $\mathcal{G}$ is the stabilizer of a quantum error correcting code and the commutant is the normalizer $\mathcal{N}$ of the code [12]. Because a stabilizer group is Abelian, its irreps are all one-dimensional. A stabilizer code encoding n qubits into $n_J = k$ has $n - k$ generators, each of which has eigenvalues ± 1. Then, J runs over the 2^{n-k} different binary vectors of eigenvalues, meaning that $\mathcal{H}_S \cong \bigoplus_{J=\{\pm 1,\ldots,\pm 1\}} \mathbb{C}^{2^k}$, and each of the subspaces in the sum is a valid code C_J. Here, the elements of $\mathcal{N}$ are viewed as Hamiltonians. For this reason, only the encoded single-qubit normalizer operations are required; encoded two-body interactions are constructed as tensor products of single-qubit ones.

Energy-gap protection.—Application of DD pulses is the main mechanism I propose for protection of AQC, but it has a shortcoming as noted above. Fortunately, the formulation presented here easily accommodates the AQC energy-gap protection strategy proposed in [13], which can be viewed as adding another layer of protection for dealing with finite-resource-DD. Namely, if the decoupling group $\mathcal{G}$ is also a stabilizer group for code C_J, then for each Pauli error S_α in H_{SB}, there is at least one element $P_j \in \mathcal{G}$ such that $\{P_j, S_\alpha\} = 0$, and otherwise $[P_j, S_\alpha] = 0$ [12]. We can then add an energy penalty term $H_P = -E_P \sum_{j=1}^{|\mathcal{G}|-1} P_j \in \mathbb{C}\mathcal{G}$ to H_S, where $E_P > 0$ is the penalty. Imperfect decoupling means that $H_{\rm eff}^{\rm id(j\geq 1)} \neq 0$. To lowest order, $H_{\rm eff}^{\rm id(1)} = \sum_\alpha S_\alpha \otimes B_\alpha^{(1)}$, and an "erred state" will be of the form $|\psi_\alpha^\perp\rangle = S_\alpha|\psi\rangle$, where $|\psi\rangle = P_j|\psi\rangle \in C_J \quad \forall\ j$. Then $H_P|\psi_\alpha^\perp\rangle = \{[a - (K-1)](K-1)E_P\}|\psi_\alpha^\perp\rangle$, where a is the number of stabilizer elements that anticommute with S_α. Thus, $|\psi_\alpha^\perp\rangle$ is an eigenstate of H_P and has $a(K-1)E_P$ more energy than any state in the code space. Reference [13] showed, using a Markovian model of qubits coupled to a photon bath, the important result that this energy gap for erred states implies that the temperature need only shrink logarithmically rather than polynomially in the problem size. However, note that to deal with generic system-bath interactions both the stabilizer and normalizer elements must involve k-local interactions, with $k > 2$ [13].

2-local decoherence-resistant universal AQC.—First, recall a recent universality result. The following simple 2-local Hamiltonian allows for universal AQC [23]: $H_{\rm ad}^{\rm univ}(t) = \sum_{i;\alpha\in\{x,z\}} h_i^\alpha(t)\sigma_i^\alpha + \sum_{i,j;\alpha\in\{x,z\}} J_{ij}^\alpha(t)\sigma_i^\alpha\sigma_j^\alpha$. With this, all the tools have been assembled to demonstrate the second main result of this work: a stabilizer decoupling procedure against 1-local noise that uses only 2-local interactions. By 1-local noise, I mean the main nemesis

of quantum computing, namely, the linear decoherence model: $H_{SB}^{\rm lin} = \sum_{\alpha=x,y,z}\sum_{j=1}^{n}\sigma_j^\alpha \otimes B_j^\alpha$, where $\{B_j^\alpha\}$ are arbitrary bath operators. To beat $H_{SB}^{\rm lin}$, use the Abelian "universal decoupling group" [8] $\mathcal{G}_{\rm uni} = \{I, X, Y, Z\}$, where $X(Y, Z) = \bigotimes_{j=1}^{n}\sigma_j^{x(y,z)}$. It is simple to verify that $\Pi_{\mathcal{G}_{\rm uni}}(H_{SB}^{\rm lin}) = 0$. As noted in Ref. [9], $\mathcal{G}_{\rm uni}$ is the stabilizer of an $[[n, n-2, 2]]$ stabilizer code $\mathcal{C}$, whose code words are $\{|\psi_x\rangle = (|x\rangle + |\text{not } x\rangle)/\sqrt{2}\}$, where x is an even-weight binary string of length n, with n even. For example, for $n = 4$, we find: $|00\rangle_L = (|0000\rangle + |1111\rangle)/\sqrt{2}$, $|10\rangle_L = (|0011\rangle + |1100\rangle)/\sqrt{2}$, $|01\rangle_L = (|0101\rangle + |1010\rangle)/\sqrt{2}$, $|11\rangle_L = (|1001\rangle + |0110\rangle)/\sqrt{2}$. Now universal AQC over $\mathcal{C}$ can be implemented using 2-local Hamiltonians. To compute over $\mathcal{C}$, we replace each Pauli matrix in $H_{\rm ad}^{\rm univ}$ by its encoded partner. Encoded single-qubit operations for $\mathcal{C}$ are the 2-local $\bar{X}_j = \sigma_1^x\sigma_{j+1}^x$ and $\bar{Z}_j = \sigma_{j+1}^z\sigma_n^z$, where $j = 1, \ldots, n-2$. The 2-local interactions $\sigma_i^x\sigma_j^x$ and $\sigma_i^z\sigma_j^z$ appearing in $H_{\rm ad}$ are replaced by the 2-local $\bar{X}_i\bar{X}_j = \sigma_{i+1}^x\sigma_{j+1}^x$ and $\bar{Z}_i\bar{Z}_j = \sigma_{i+1}^z\sigma_{j+1}^z$. Thus, we see that universal AQC can be combined with DD using only 2-local $\sigma_i^x\sigma_j^x$ and $\sigma_i^z\sigma_j^z$ interactions over $\mathcal{C}$.

Examples of promising QC implementations where X, Z (as pulses for DD) and $\sigma_i^x\sigma_j^x$, $\sigma_i^z\sigma_j^z$ (as Hamiltonians for AQC) are available and controllable, are systems including capacitive coupling of flux qubits [24] and spin models implemented with polar molecules [25]. Also note that in principle, as discussed above, we can create an additional energy gap [13] against single-qubit errors by adding a penalty term $H_P = -E_P(X + Y + Z)$ to the system Hamiltonian. However, H_P is an n-local interaction.

Conclusions and outlook.—Using a combination of various tools in the arsenal of decoherence control, I have shown how to protect AQC against decoherence. While I believe that the methods proposed here should significantly contribute towards the viability and robustness of AQC, what is still missing is a threshold theorem for fault tolerant AQC. This will most likely require the incorporation of feedback, in order to correct DD pulse imperfections and other control noise [17]. One possibility for doing so might be to perform syndrome measurements on the commutant factor [$\mathbb{C}^{d_J}$ in Eq. (9)] as in recent circuit-model fault tolerance work using subsystems codes [7].

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[22] To derive Eq. (8), let $L = Kn^{2\zeta+\epsilon_1}$, and absorb α into T. The first term in Eq. (6) thus yields $n^{-\epsilon_1} < (\Delta_0/J)^2$, which is satisfied for large enough n if (i) $\epsilon_1 > 0$. Using $T = L(\tau + w)$, the second term yields $w < 1/(JL)$, which we can satisfy with (ii) $w = n^{-(2\zeta+\epsilon_1+\epsilon_2)}/(JK)$, $\epsilon_2 > 0$, and the first term yields (iii) $w < 1/(J\sqrt{LK}) - \tau$; but $w > 0$ so that together these imply (iv) $\tau < 1/(J\sqrt{LK})$. We can satisfy (iii) and (iv) by choosing (v) $\tau = n^{-(\zeta+\epsilon_1)}/(K\Delta_0)$, since then $T = L(\tau + w) \overset{n\gg 1}{\to} \Delta_0^{-1}n^{\zeta}$ as it should. As for the third term in Eq. (6), note that $\frac{e^x-1}{x} - 1 \le (e-2)x$ if $x \le 1$. Indeed, $x = 2\beta T_c \sim 2\beta(n^{-(\zeta+\epsilon_1)}/\Delta_0 + n^{-(2\zeta+\epsilon_1+\epsilon_2)}/J) \ll 1$ for β_S and $\beta_B \sim O(n)$ [local Hamiltonians, e.g., a scenario where each system particle is in contact with a local (spin) bath]. Therefore, the third term is bounded above by $(e-2)2JT\beta T_c \overset{n\gg 1}{\to} O(n) \times (J/\Delta_0)n^{-\epsilon_1}$, so that we require $\epsilon_1 > 1$.
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