Resilient Quantum Computation in Correlated Environments

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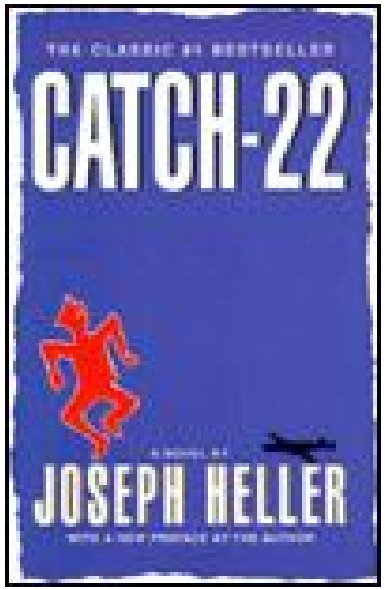
the *"catch-22"* in quantum computation

- catch-22: the solution creates the problem.



low decoherence X interaction and control.

" Okay, let me see if I've got this straight. In order to be grounded, I've got to be crazy, and i must be crazy to be flying, but if i ask to be grounded, that means I'm not crazy anymore and have to keep flying."

Quantum Error Correction

Usual assuptions in the traditional QEC theory

- fast measurements (not fundamental - Aliferes-DiVincenzo 07);
- fast/slow gates;
- error models.

Quantum Error Correction

"threshold theorem"

Provided the noise strength is below a critical value, quantum information can be protected for arbitrarily long times.

Hence, the computation is said to be *fault tolerant* or *resilient*.

Dorit Aharonov, Phys. Rev. A 062311 (2000).

Quantum to classical phase transition in a noisy QC.

threshold theorem

resilient quantum computation	noise quantum computer
	simulated efficiently by a Turing machine
strong entanglement between qubits	weak entanglement between qubits
''low temperature regime"	''high temperature regime"

error probability of a qubit (ε)

What if we would like to start from a microscopic model?

The standard prescription:

- quantum master equation and
- dynamical semi-groups.

This is a natural approach:

The computer is the object of interest; hence one starts the discussion by integrating out the environmental degrees of freedom.

The price that we usually pay:

1. Born approximation (2nd order perturbation theory)
2. Markov approximation.

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What is the problem with the B-M approximation?

R. Alicki, Daniel A. Lidar and Paolo Zanardi, PRA **73** 052311 (2006).

Internal Consistency of Fault-Tolerant Quantum Error Correction in Light of Rigorous Derivations of the Quantum Markovian Limit.

... These assuptions are: fast gates, a constant supply of fresh cold ancillas, and a Markovian bath. We point out that these assuptions may not be mutually consistent in light of rigorous formulations of the Markovian approximation. ...

- QEC is a perturbative method!

What is the problem with the B-M approximation?

Ulrich Weiss in *Quantum Dissipative Systems*

While the Markov assuption can easily be dropped, the more severe limitation of this method is the Born approximation for the kernel.

In conclusion, the Born-Markov quantum master equation method provides a reasonable description in many cases, such as in NMR, in laser physics, and in a variety of chemical reactions. However, this method turned out to be not useful in most problems of solid state physics at low temperatures for which neither the Born approximation is valid nor the Markov assumption holds.

- QEC is a perturbative method!

Correlated environments

We do not want to assume the Born approximation.

Some references that also do not assume B-M:

- Knill, Laflamme, Viola, PRL 84, 2525 (2000),
- Terhal and Burkard, PRA 71, 012336,
- Aliferis, Gottesman, Preskill, QIC 6, 97 (2006),
- Aharonov, Kitaev, Preskill, PRL 96, 050504 (2006),
- etc...

How correlated environments have been treated before?

define the evolution with at least one error

$$\mathcal{E}(t) = U(t) - 1 = -\frac{i}{\hbar}\int_0^t dt'\, V(t')\, U(t') \ .$$

find a bound for the norm of $\mathcal{E}$

$$||\mathcal{E}(t)|| \leq \frac{1}{\hbar}\int_0^t dt'\, ||V(t')|| \leq \frac{\Lambda t}{\hbar} \ ,$$

where Λ as the largest eigenvalue of V.

The problem

- it only makes sense when: $||\mathcal{E}|| \ll 1$, thus
- how to deal with the case $|\Lambda t| \gg 1$?

Different idea: study the stability of the perturbation theory

- free Hamiltonian: H_0;
- a "perturbation":

$$V(t) = \lambda \int_0^L d\mathbf{x} f(\mathbf{x}, t),$$

- "two point correlation function:

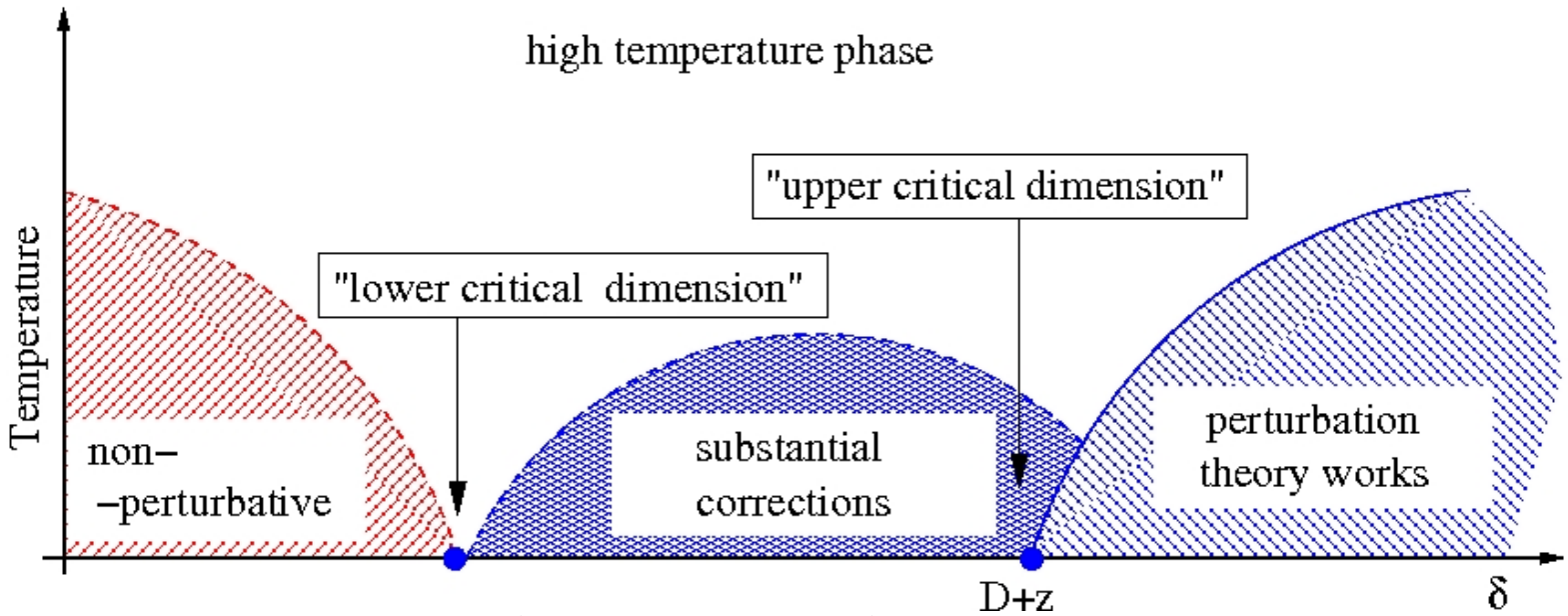
$$\langle \Psi | f(\mathbf{x}_1, t_1) f(\mathbf{x}_2, t_2) | \Psi \rangle \sim \mathcal{F}\left(\frac{1}{(\Delta x)^{2\delta}}, \frac{1}{(\Delta t)^{2\delta/z}} \right),$$

.

$$\frac{\langle \Psi | \mathcal{E}^\dagger(t) \mathcal{E}(t) | \Psi \rangle}{2} = 1 - \langle \Psi | T_t \cos\left[\frac{1}{\hbar} \int_0^t dt' V(t') \right] | \Psi \rangle .$$

- "RPA" series gives $\sim \lambda^{2m} (Lt)^{2m(D+z-\delta)}$.
- Perturbation theory around the "non-interacting" ground state is a bad start for $D + z - \delta > 0$.

A schematic phase-diagram

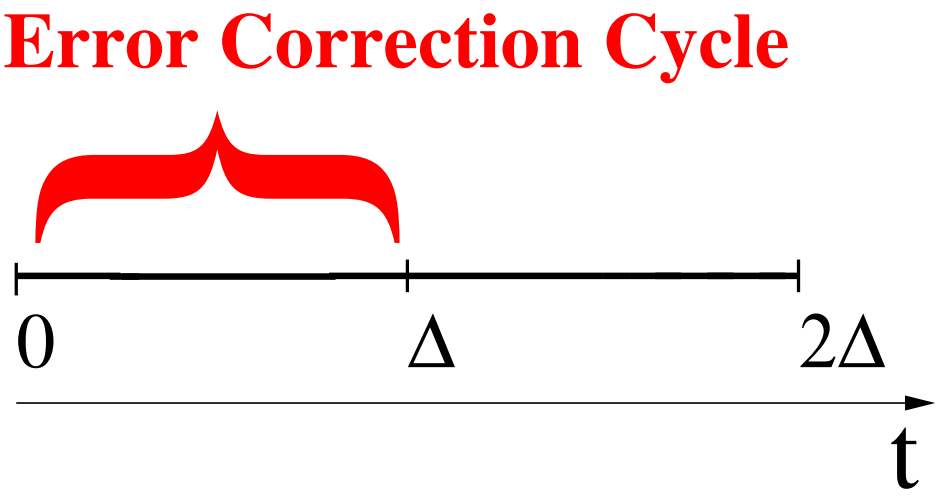


How QEC changes this picture?

main conclusions

main message

the dynamics imposed by QEC (syndrome extraction) naturally separates the environmental modes into a high and low frequency parts.



- QEC already provides some protection against correlated noise.
- if needed, additional protection can be built into the QEC code with a sort of "dynamical decoupling" of the logical qubit.
- the "threshold theorem" can be read as a sort of quantum phase transition.
- our discussion should be seen conjointly with Aharonov, Kitaev, Preskill, PRL 96, 050504 (2006).

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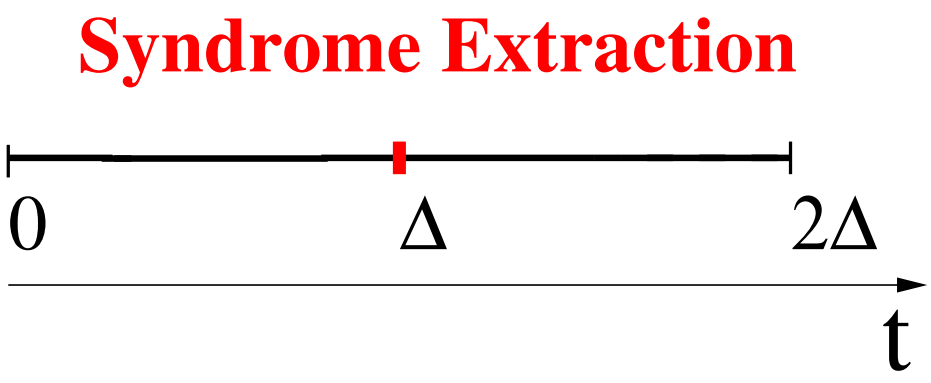

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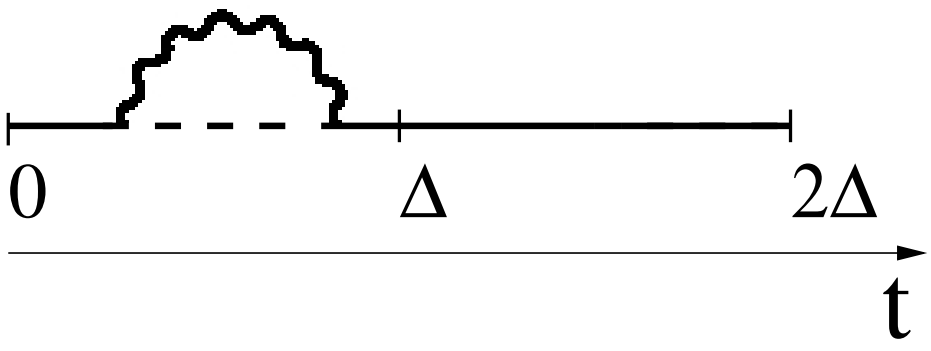

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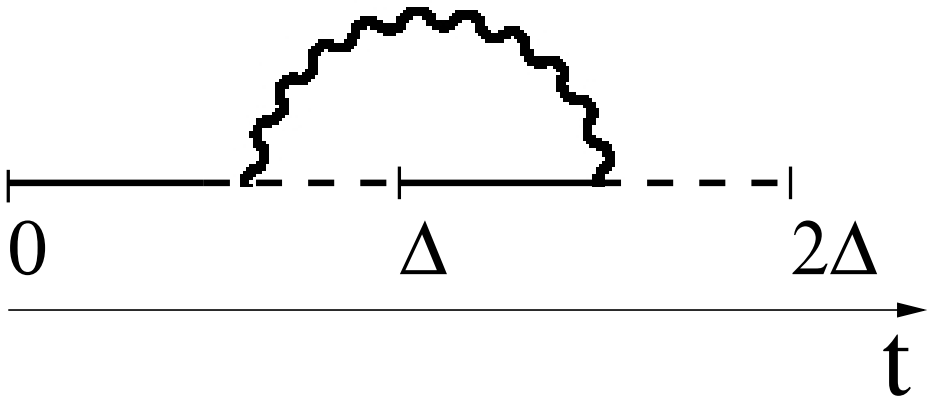

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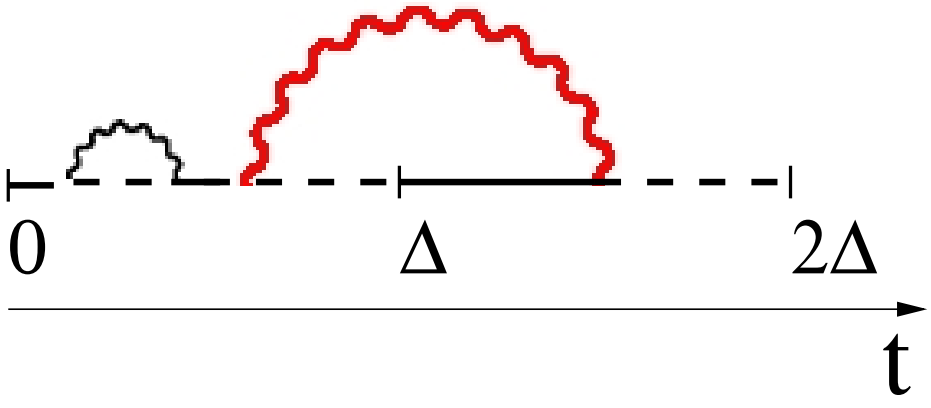

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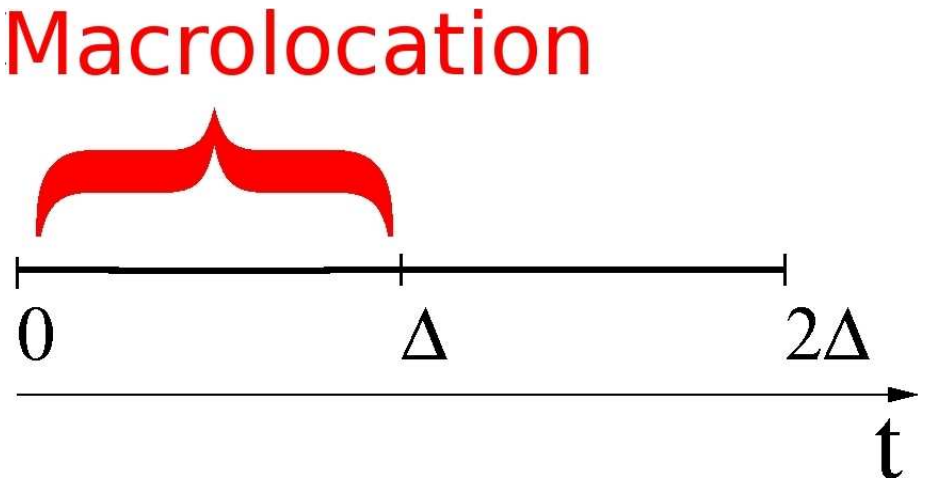
A small summary of the tradicional QEC

Steps in a QEC evolution

1. encode the information in a large Hilbert space;
2. let the system evolves;
3. extract the "syndrome";
4. correct the system;
5. start again;



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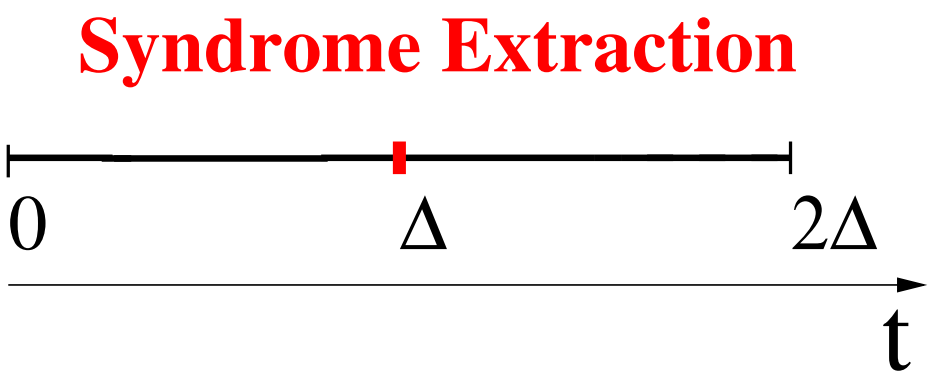

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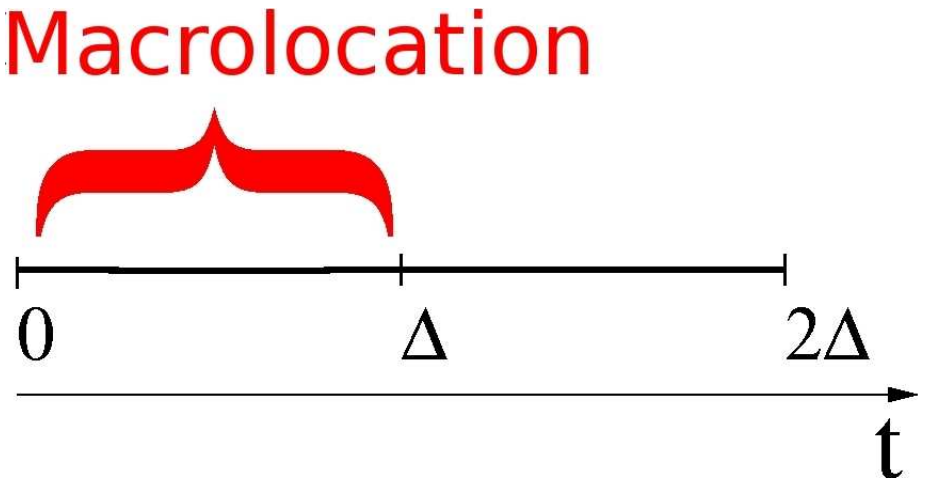
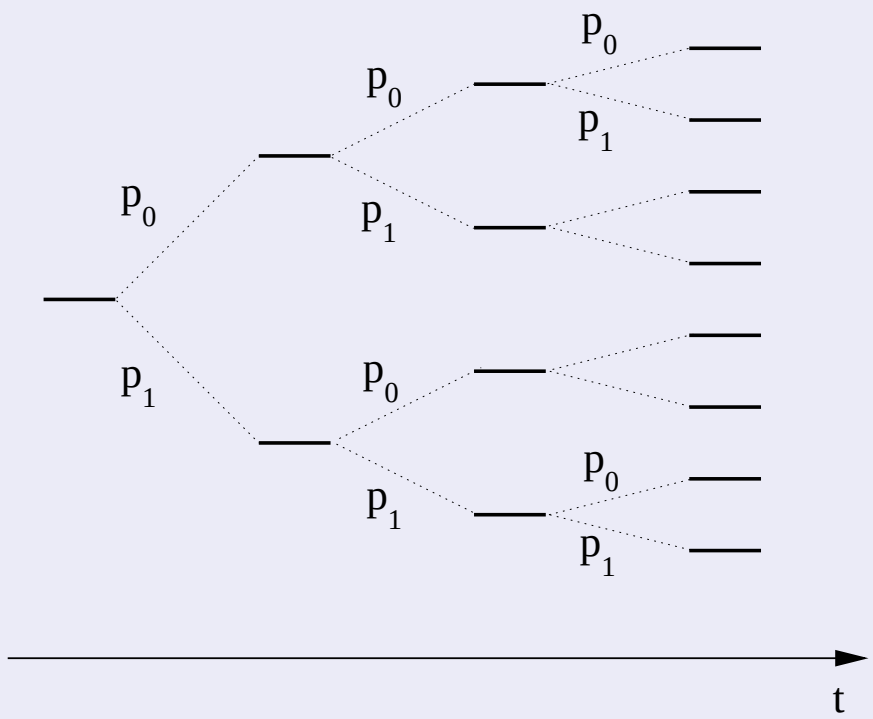


Quantities to calculate

When a probability can be defined:

probability of a history of syndromes

- p_0 probability of not having an error;
- p_1 probability of having an error.



residual decoherence

If error correction is not perfect (and usually it is not), there is always some residual decoherence to evaluate.

How this translates to the quantum evolution?

- consider and environment controlled by a "free" Hamiltonian:

$$H_0$$

- and an interaction:

$$V = \sum_{\mathbf{x},\alpha} \lambda_\alpha f_\alpha(\mathbf{x}) \sigma_\alpha(\mathbf{x}),$$

where $\vec{f}$ is a function of the environment degrees of freedom and $\vec{\sigma}$ are the Pauli matrices that parametrize the qubits.

How this translates to the quantum evolution?

$$V = \sum_{\mathbf{x},\alpha} \lambda_\alpha f_\alpha(\mathbf{x}) \sigma_\alpha(\mathbf{x}),$$

Slow or fast gates?

- fast gates - faster than the cut-off
  1. V is the Hamiltonian to consider;
  2. if we could do fast gates, better to use Bang-bang;
  3. nice for theory.
- slow gates - slower that the cut-off;
  1. need to define an *effective V* - upper bound;
  2. where QEC is the game in town;
  3. not so nice for theory (less structure).

Time evolution of encoded qubits

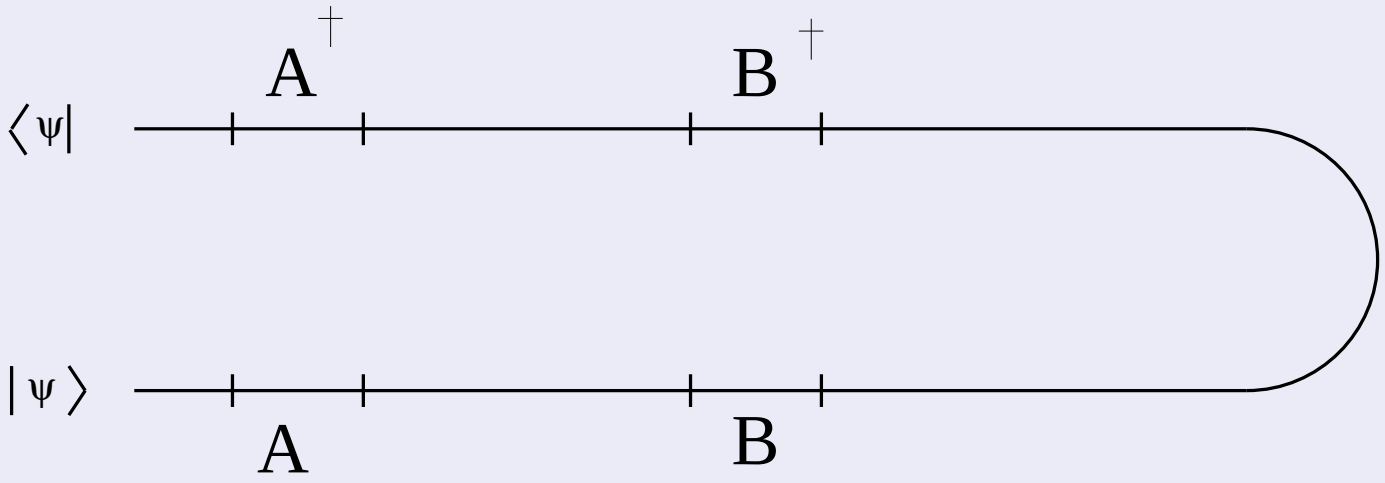
- usually the quantum evolution in the interaction picture is:

$$
\begin{aligned}
\hat{U}(\Delta,0) &= T_t e^{-i\frac{\lambda}{2}\sum_{\mathbf{x}}\int_0^{\Delta} dt\vec{f}(\mathbf{x},t)\vec{\sigma}(\mathbf{x})} \\
&= 1 - i\frac{\lambda}{2}\sum_{\mathbf{x}}\int_0^{\Delta} dt\vec{f}(\mathbf{x},t)\,\vec{\sigma}(\mathbf{x}) \\
&- \frac{\lambda^2}{4}\sum_{\substack{\mathbf{x},\mathbf{y} \\ \alpha,\beta=\{x,y,z\}}}\int_0^{\Delta} dt_1\int_0^{t_1} dt_2 f^{\alpha}(\mathbf{x},t_1)f^{\beta}(\mathbf{y},t_2)\,\sigma^{\alpha}(\mathbf{x})\,\sigma^{\beta}(\mathbf{y}) + \ldots
\end{aligned}
$$

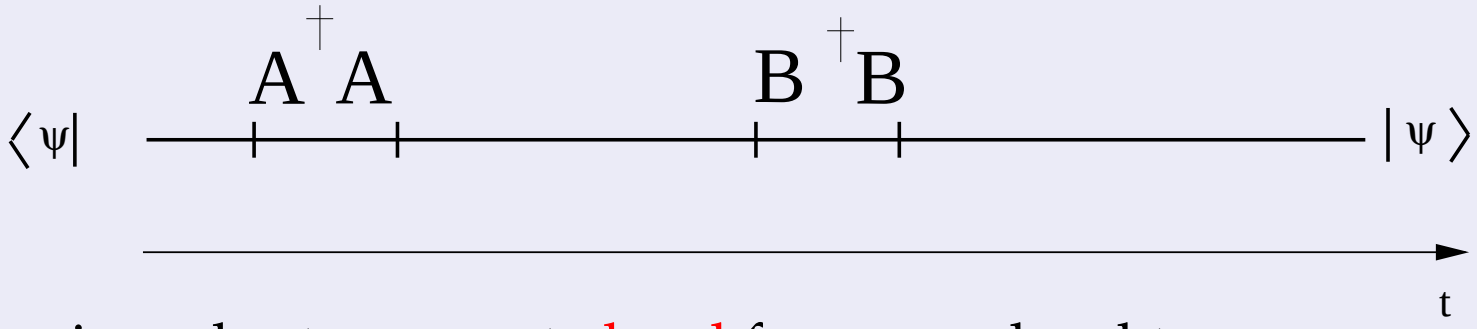
- when the syndrome is extracted only a set of terms is kept to the next QEC cycle, in other words,
- each syndrome defines a diferent "evolution" operator for a macrolocation.

Calculating the probability of an evolution

We start with:

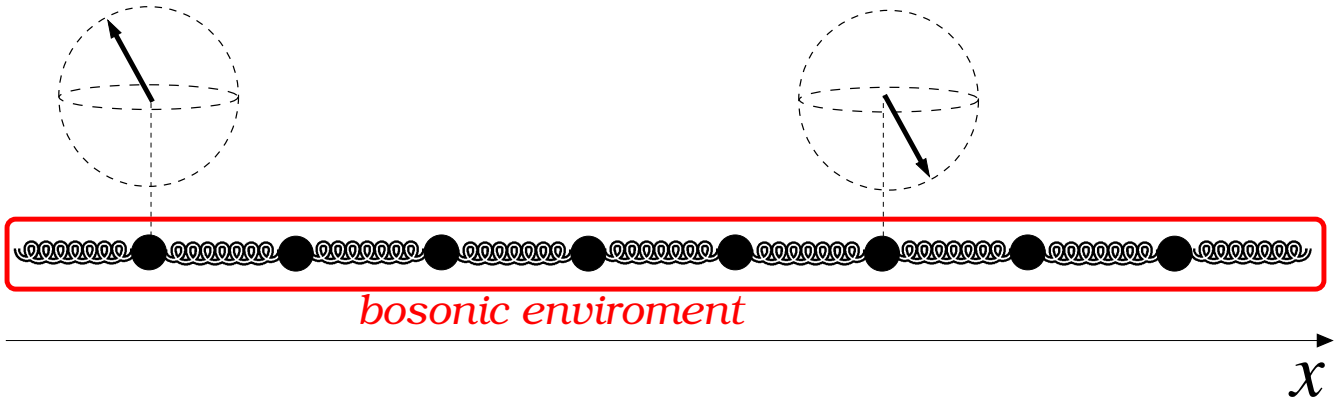


We would like to write...



... in order to separate local from non-local terms.

Example: the spin-boson model (ohmic dissipation)



$$H = \frac{v_b}{2}\int_{-\infty}^{\infty} dx\,[\partial_x\phi(x)]^2 + [\Pi(x)]^2 + \sqrt{\frac{\pi}{2}}\lambda\sum_n \partial_x\phi(n)\,\sigma_n^z$$

- ϕ and $\Pi = \partial_x\theta$ are canonical conjugate variables,
- σ_n^z act in the Hilbert space of the qubits,
- v_b is the velocity of the bosonic excitations,
- I define an ultraviolet cut-off, Λ.

bosonic field notation

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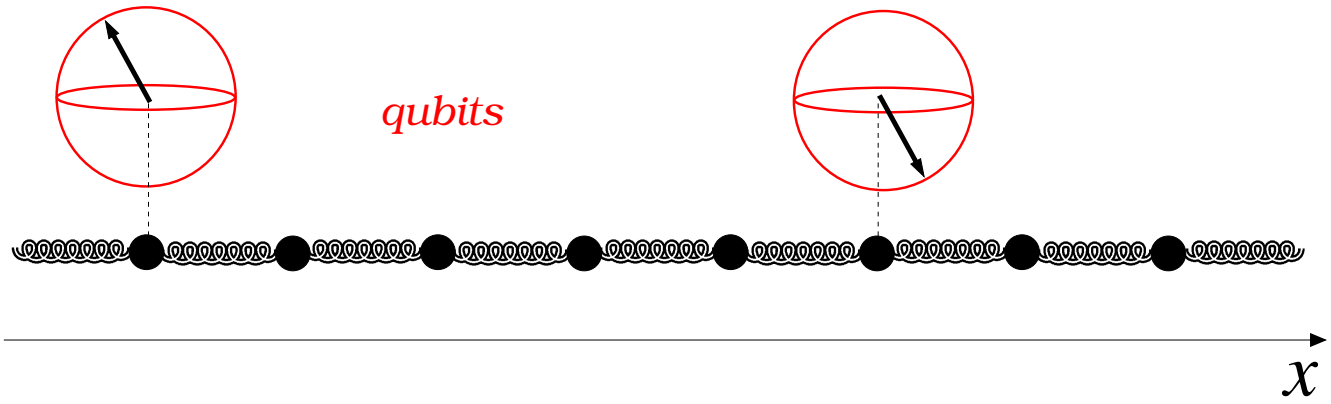

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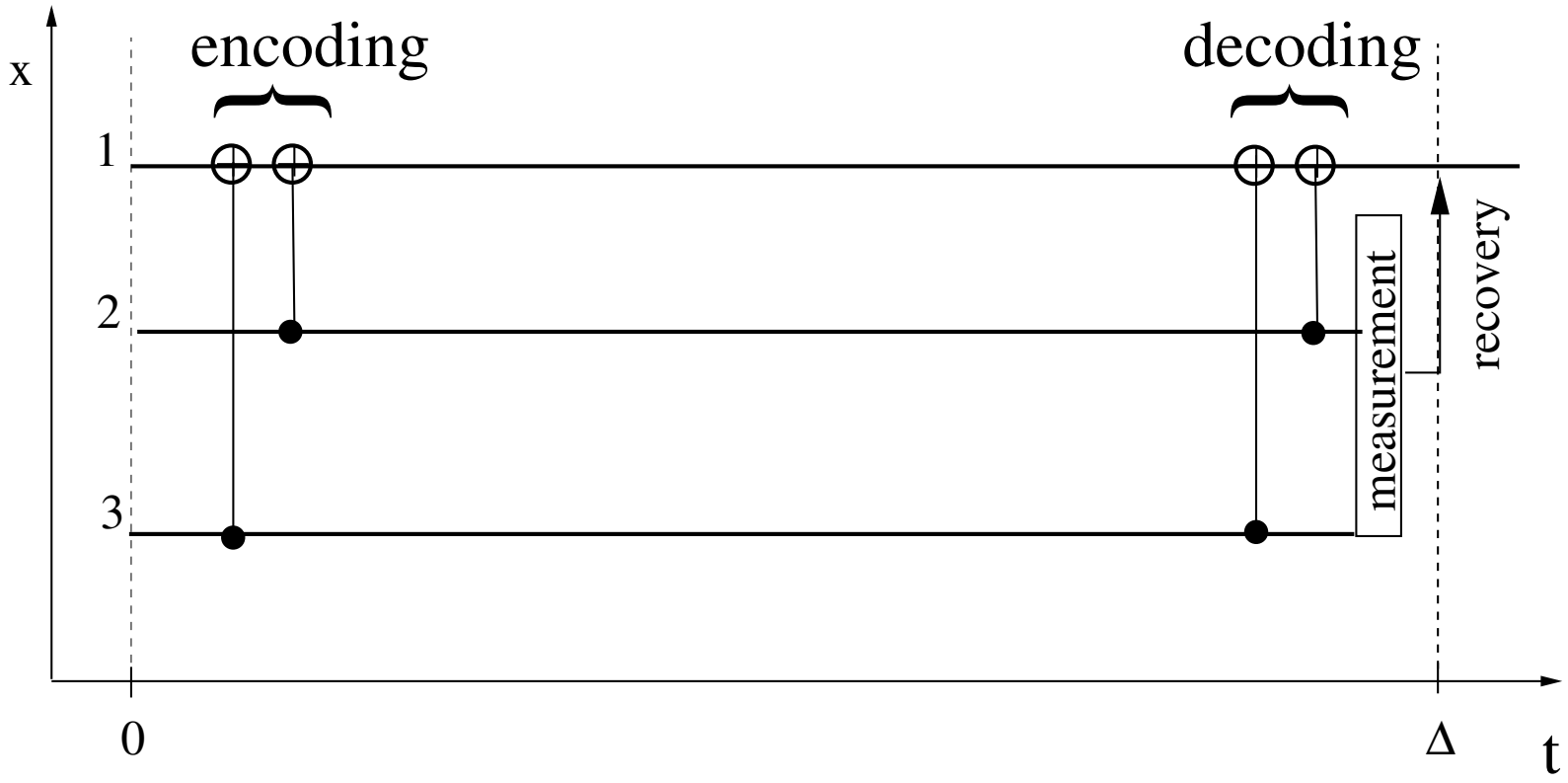
Example: a QEC that protects against phase flips

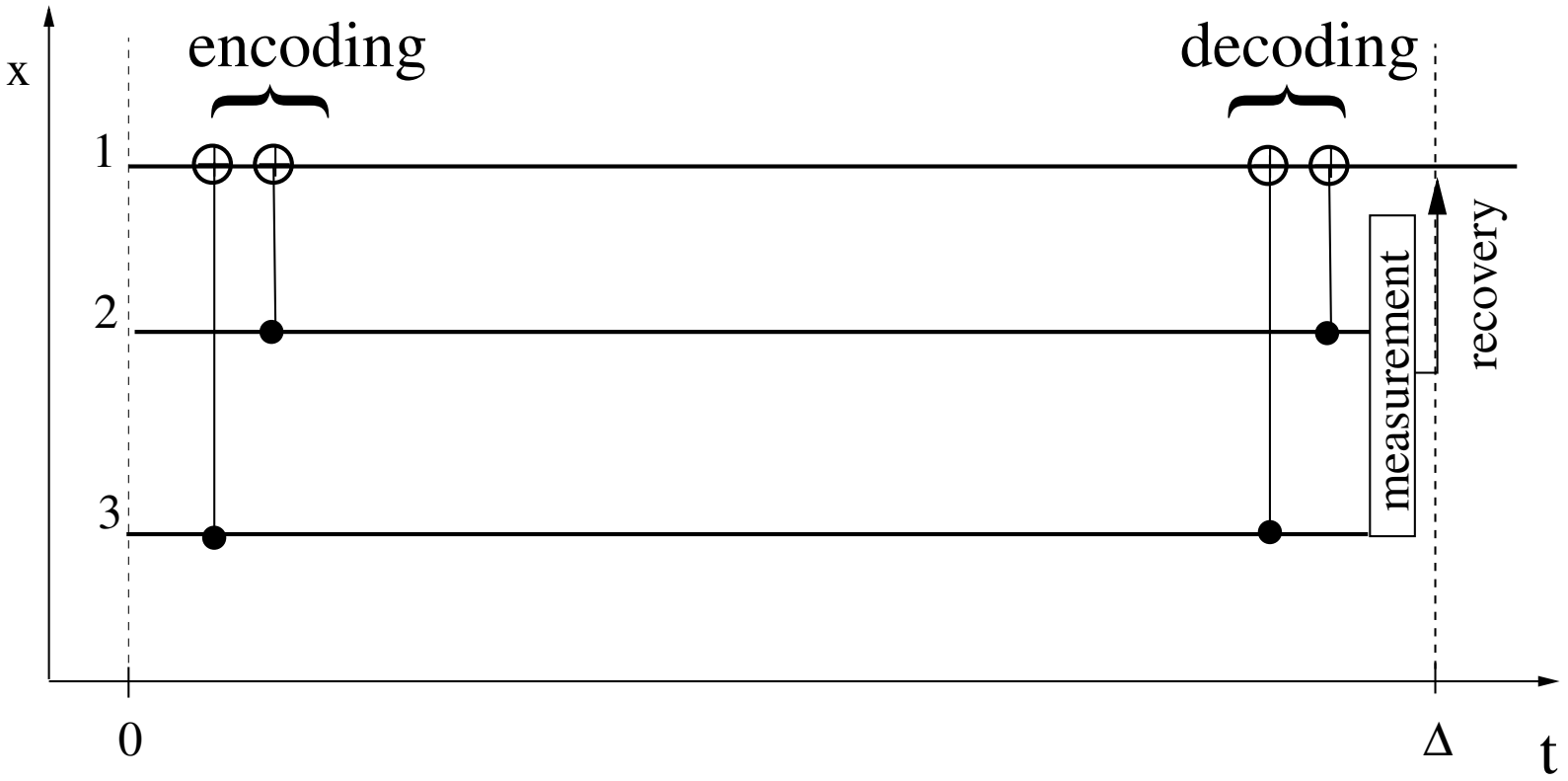


- encoded words:

$$
\begin{aligned}
|\overline{\uparrow}\rangle &= (|\uparrow\uparrow\uparrow\rangle + |\uparrow\downarrow\downarrow\rangle + |\downarrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\rangle)/2 \\
|\overline{\downarrow}\rangle &= (|\downarrow\downarrow\downarrow\rangle + |\downarrow\uparrow\uparrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\uparrow\uparrow\downarrow\rangle)/2
\end{aligned}
$$

time evolution of encoded qubits: spin-boson model

$$\hat{U}_{j=\{1,2,3\}}(\Delta,0) = \exp\left[i\sqrt{\frac{\pi}{2}}\lambda\left[\theta\left(x_j,\Delta\right)-\theta\left(x_j,0\right)\right]\sigma_j^z\right]$$



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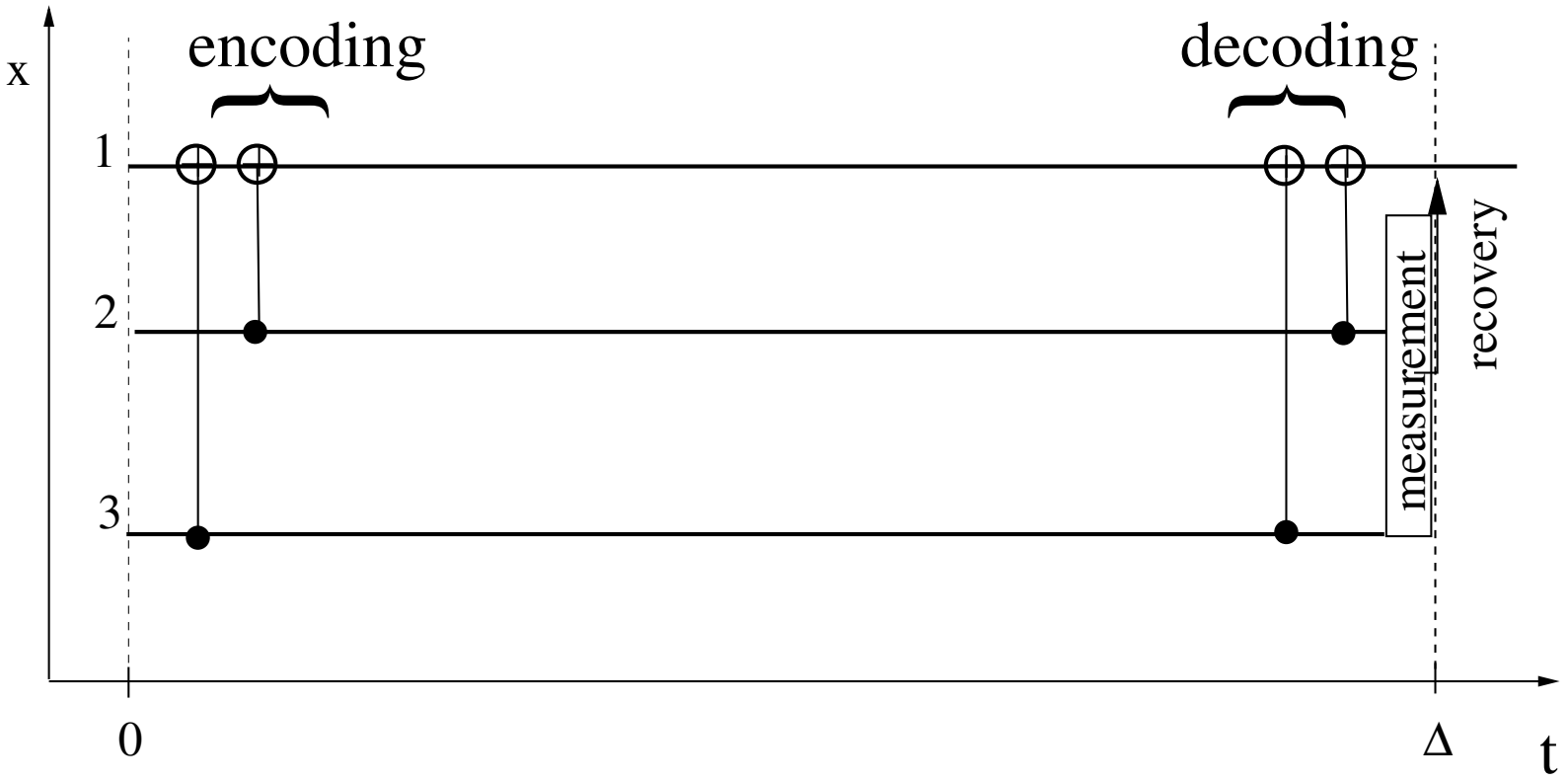

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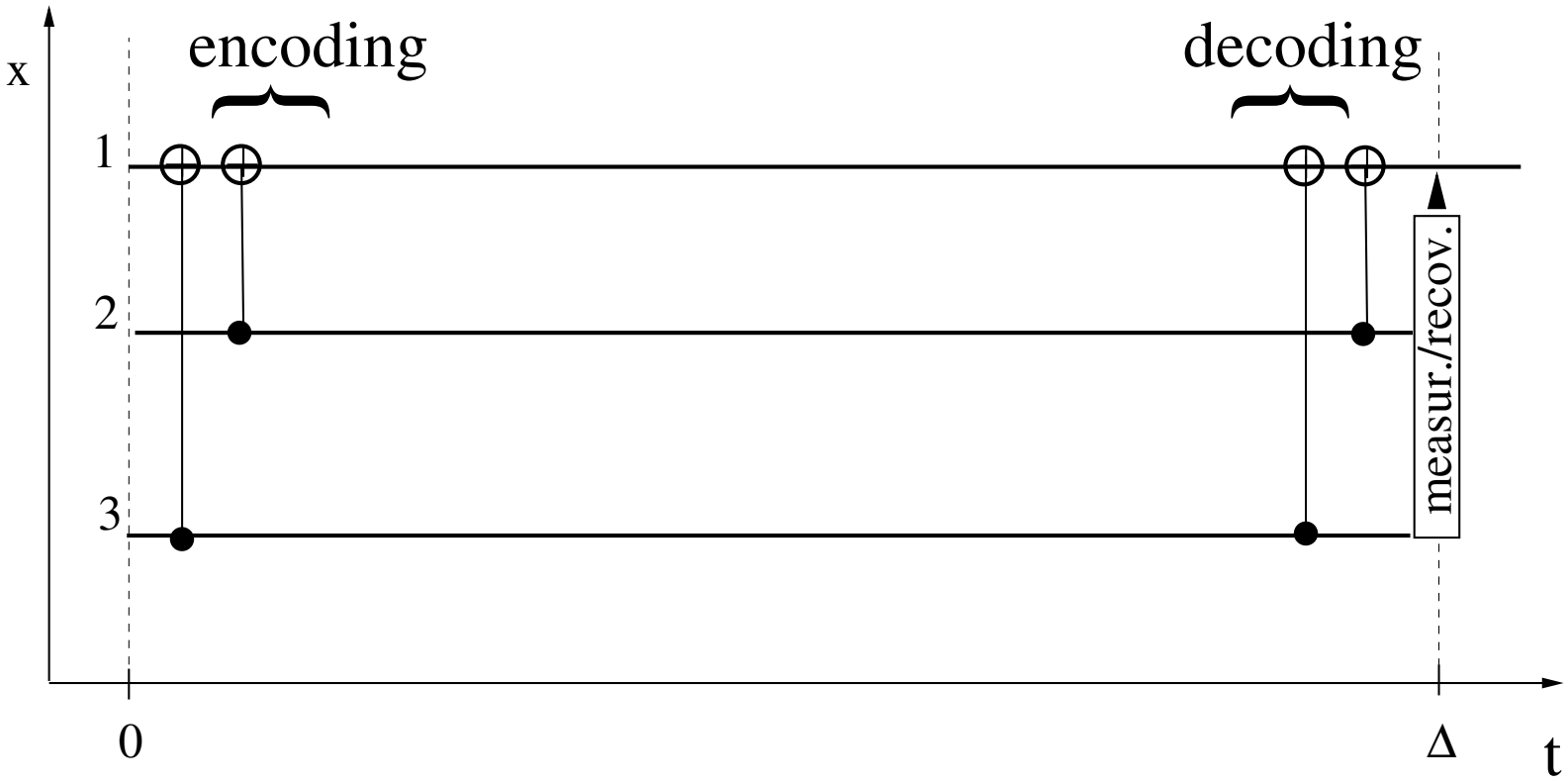

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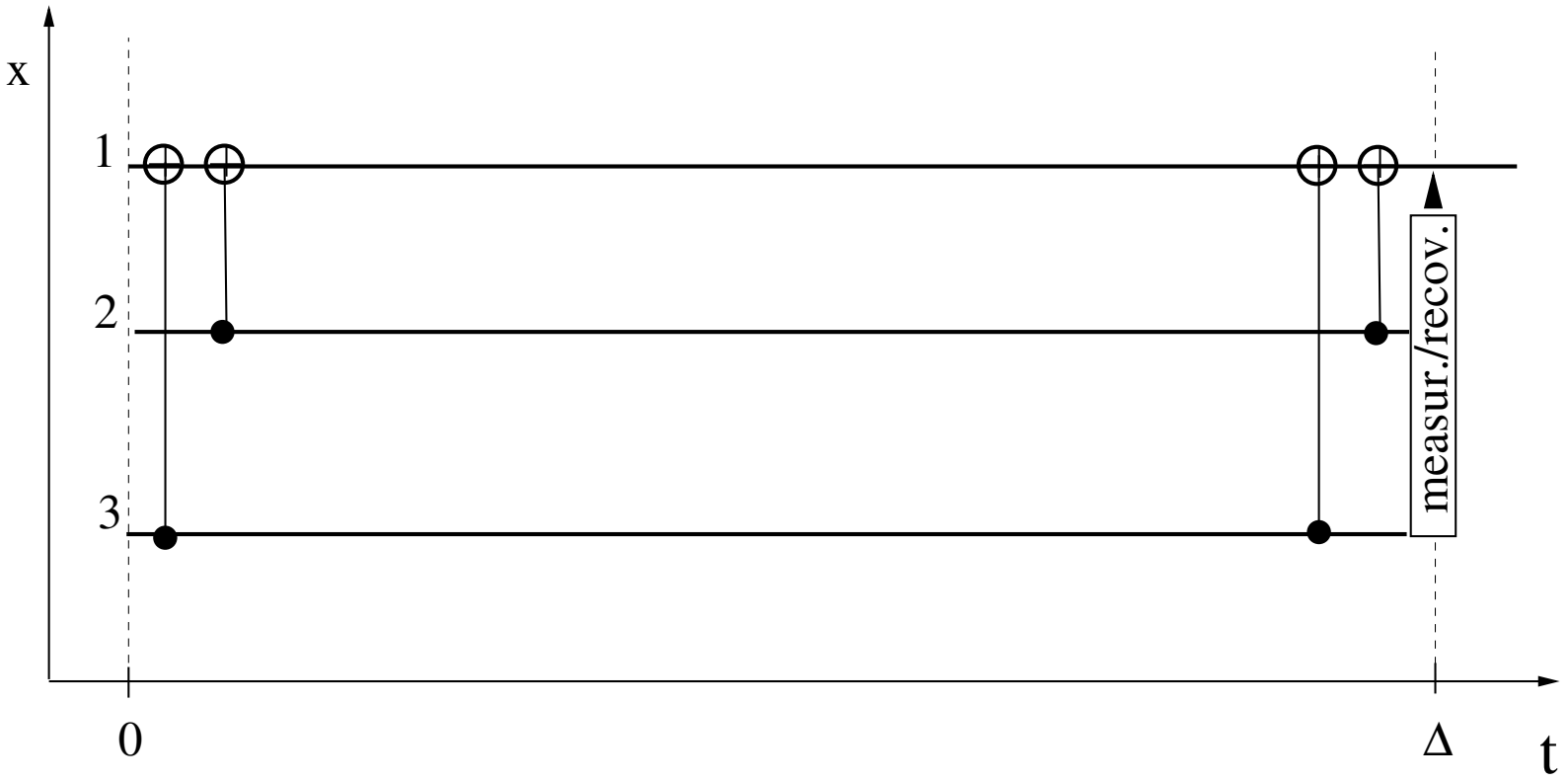

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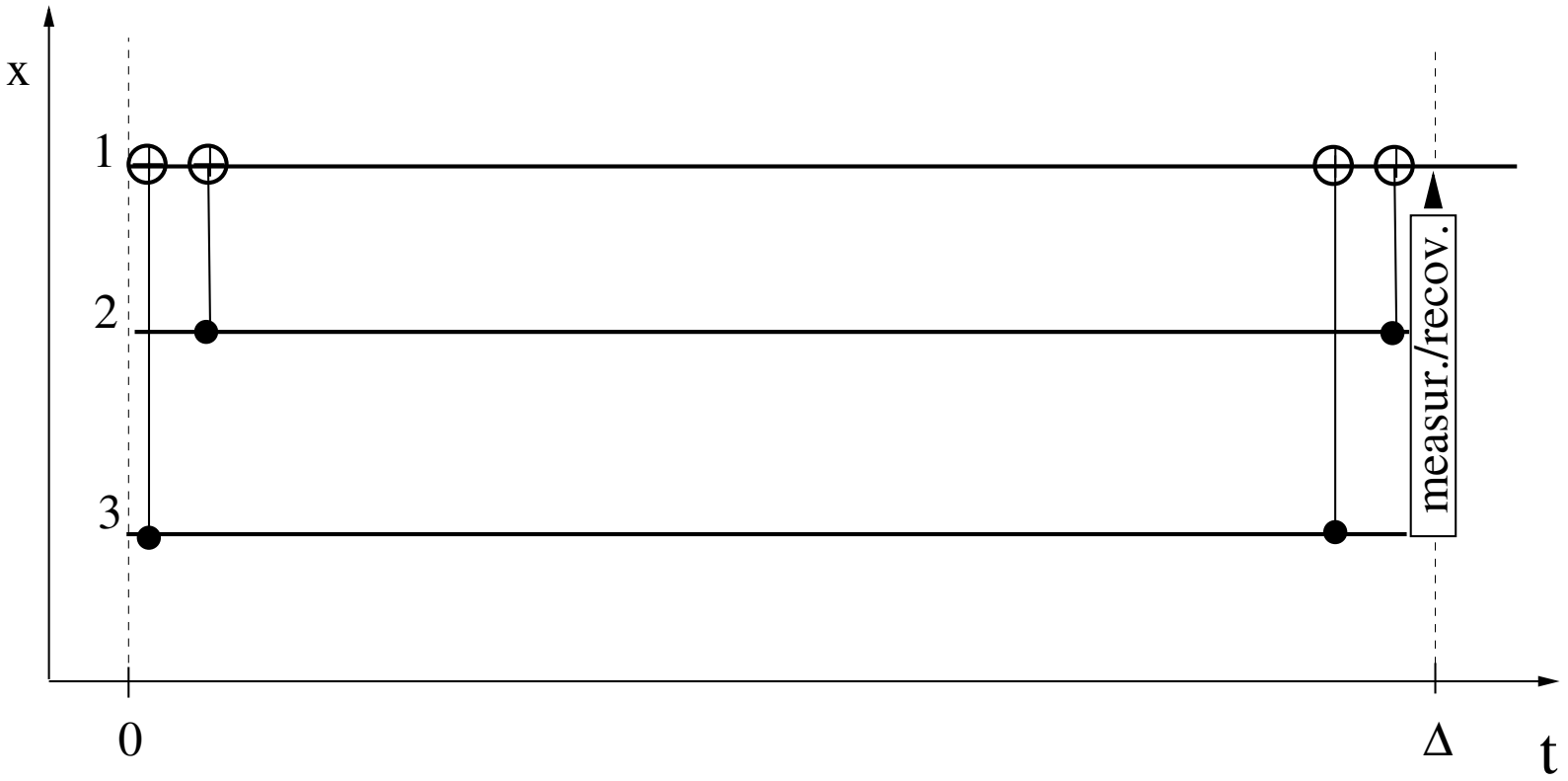

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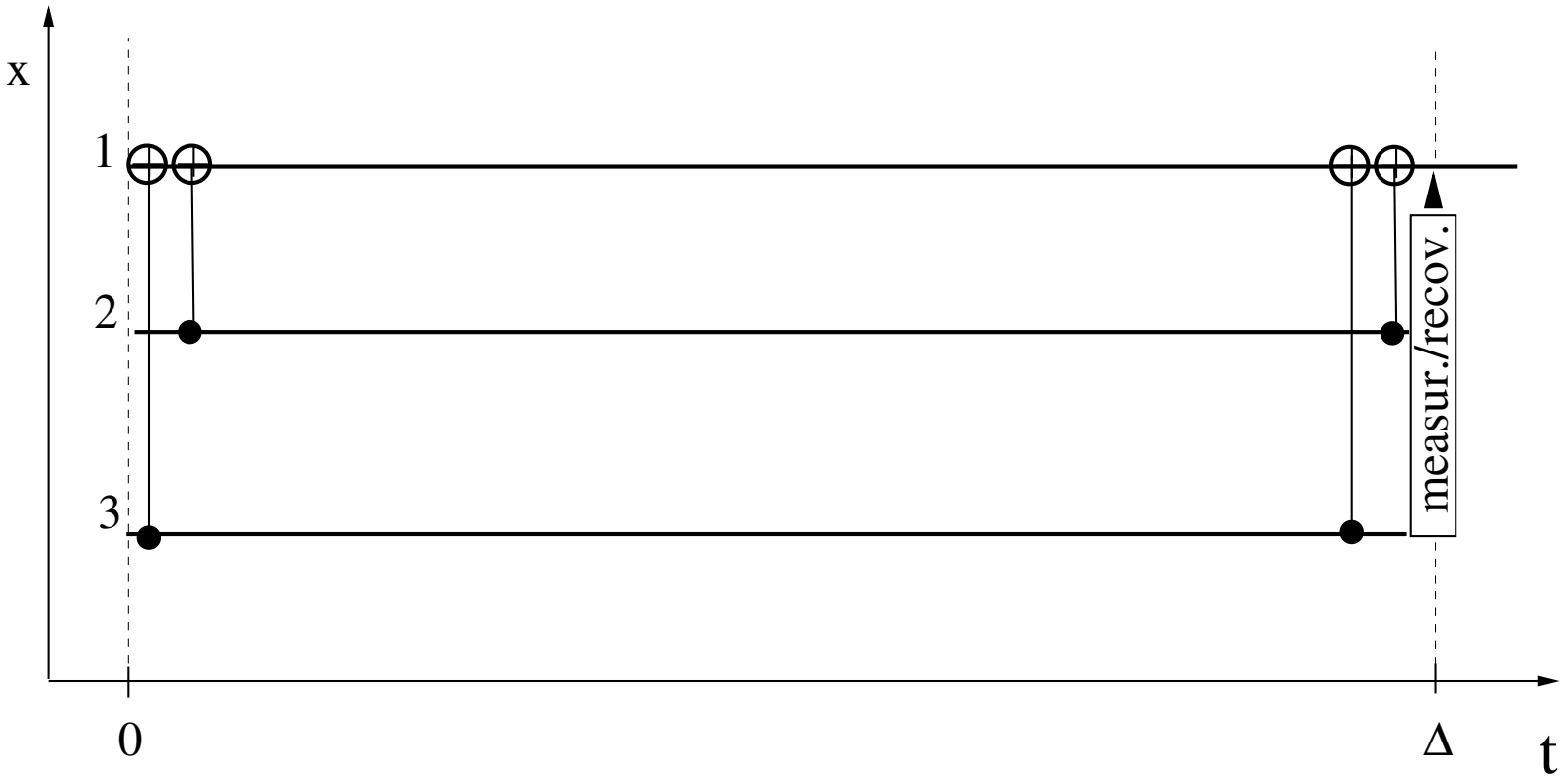

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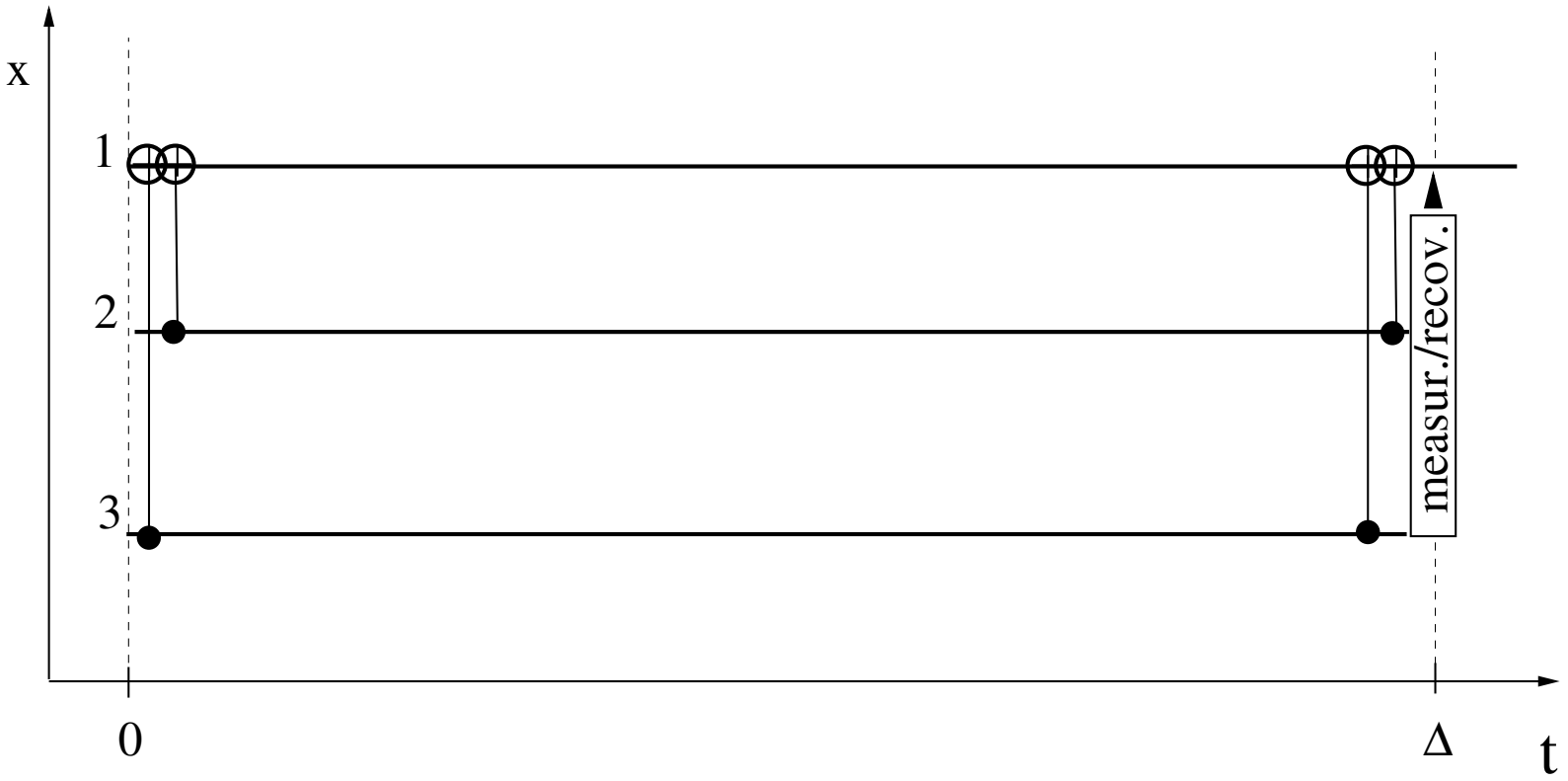

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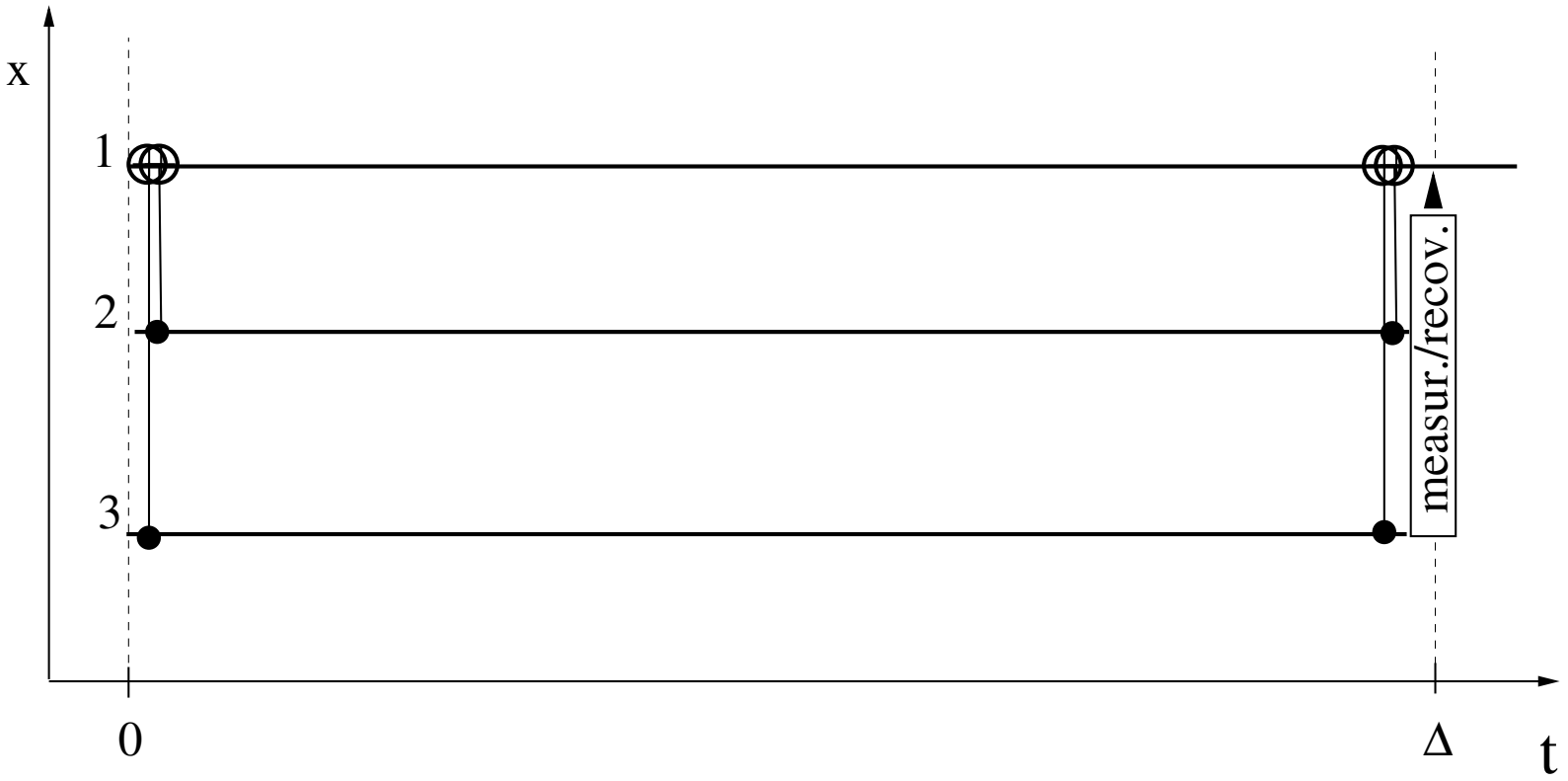

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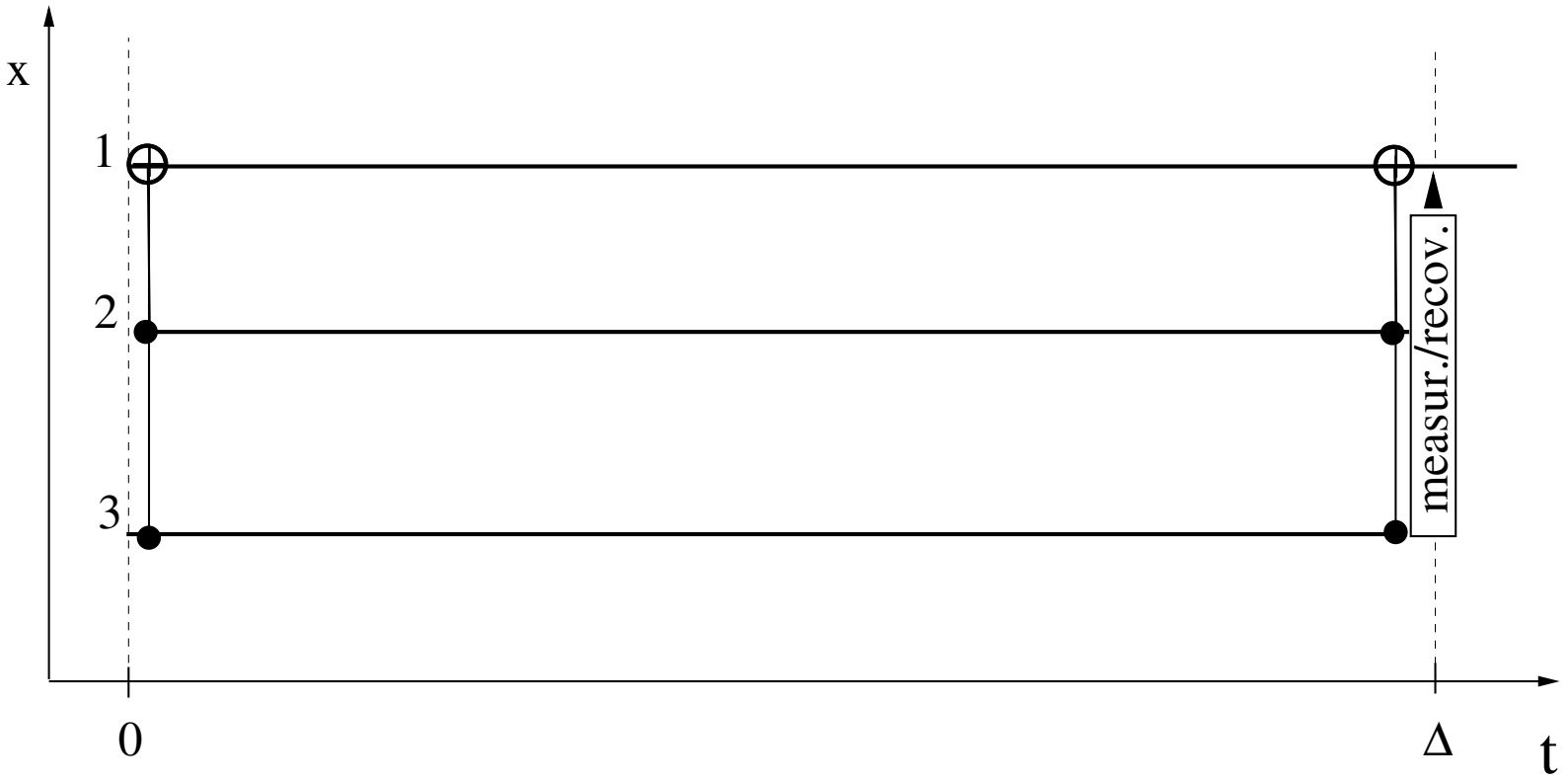

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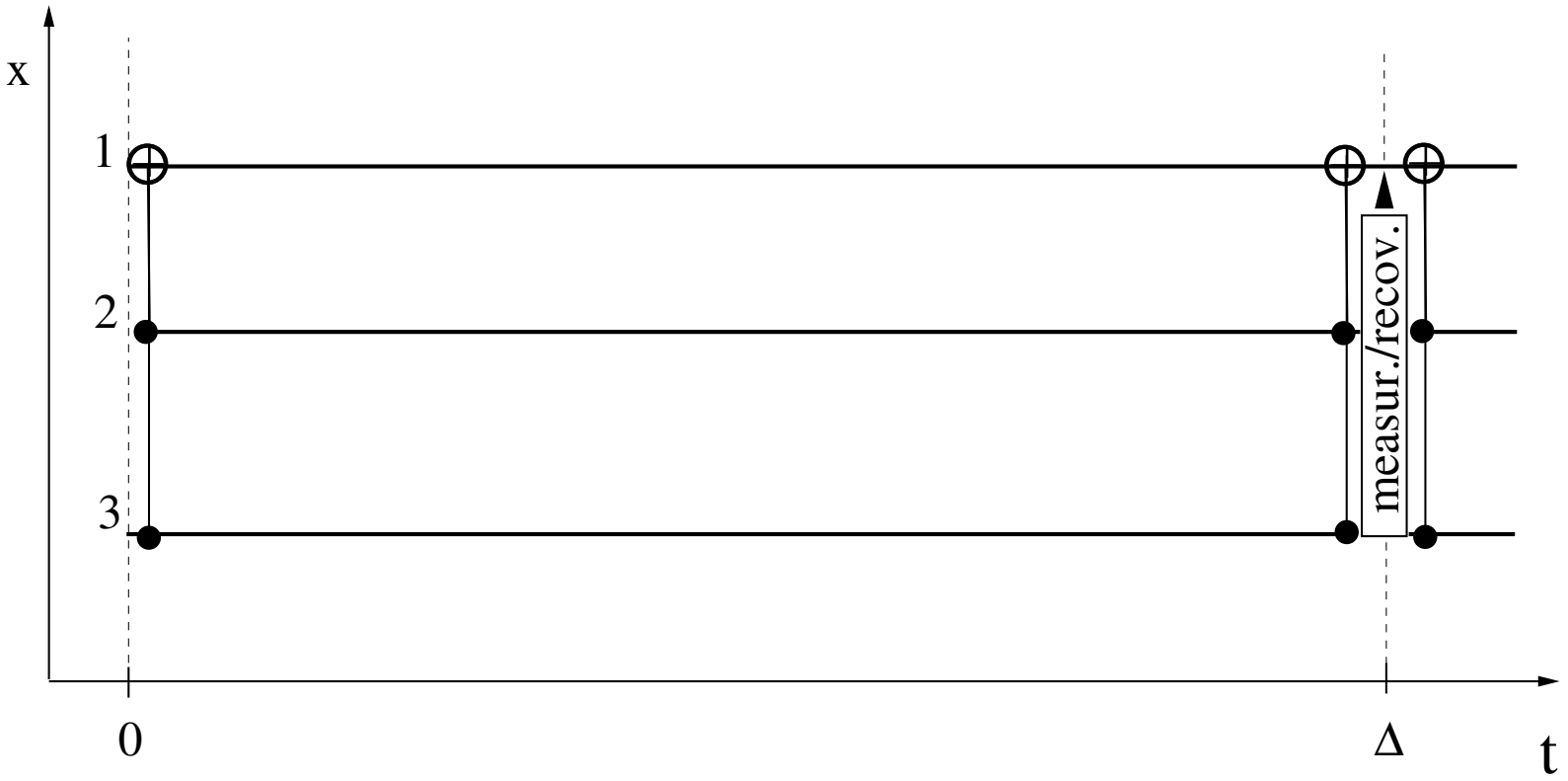

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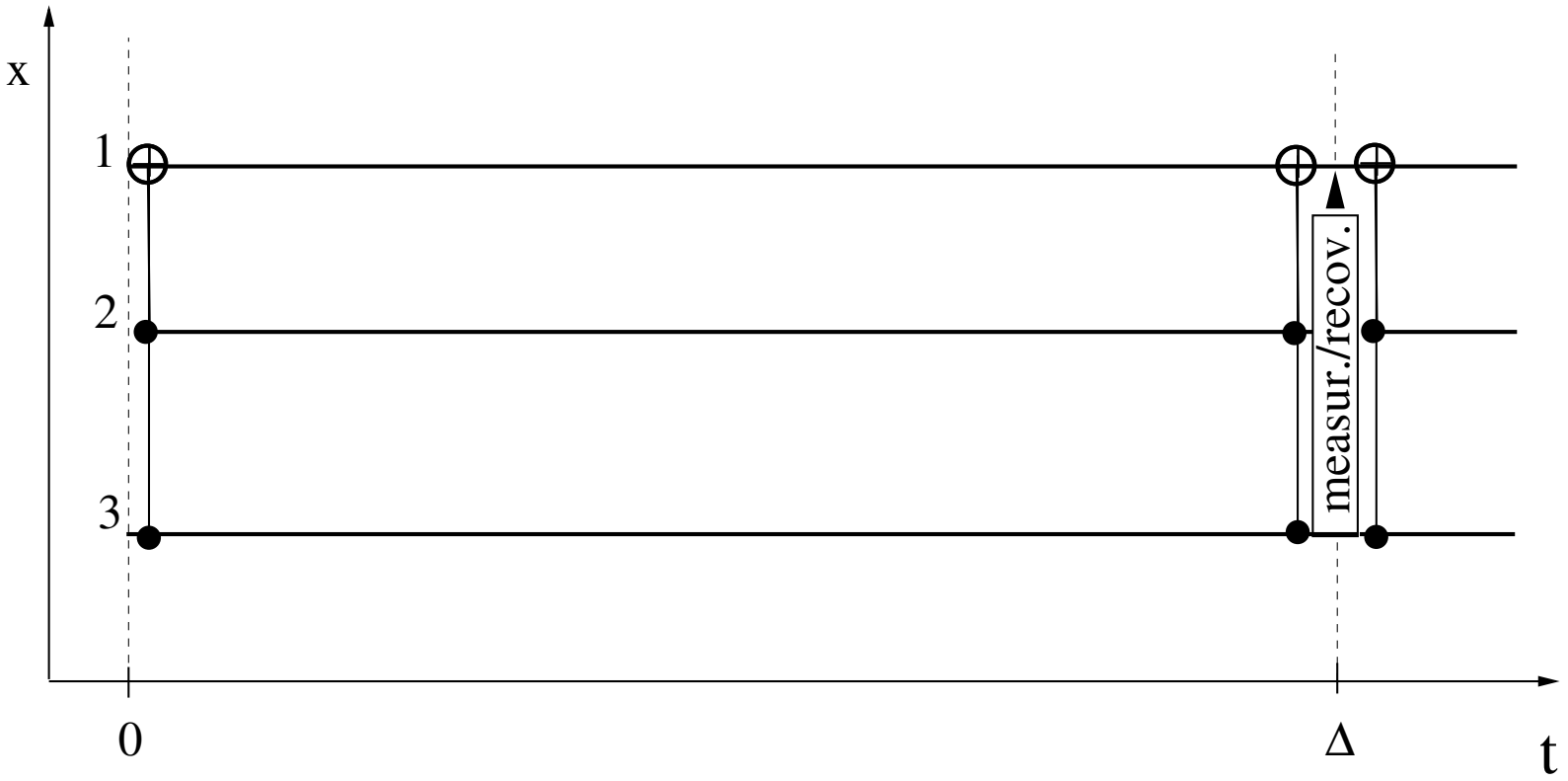

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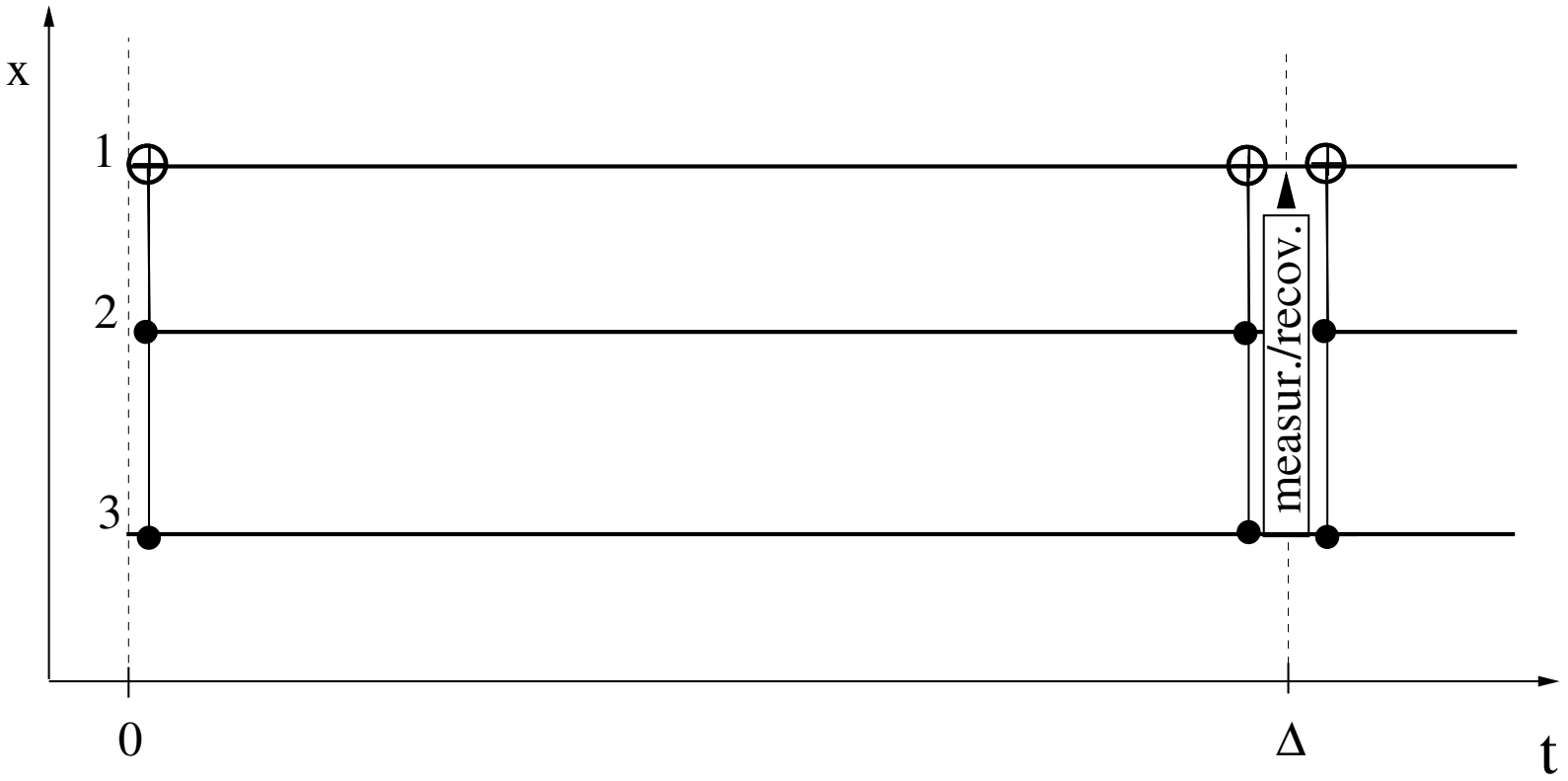

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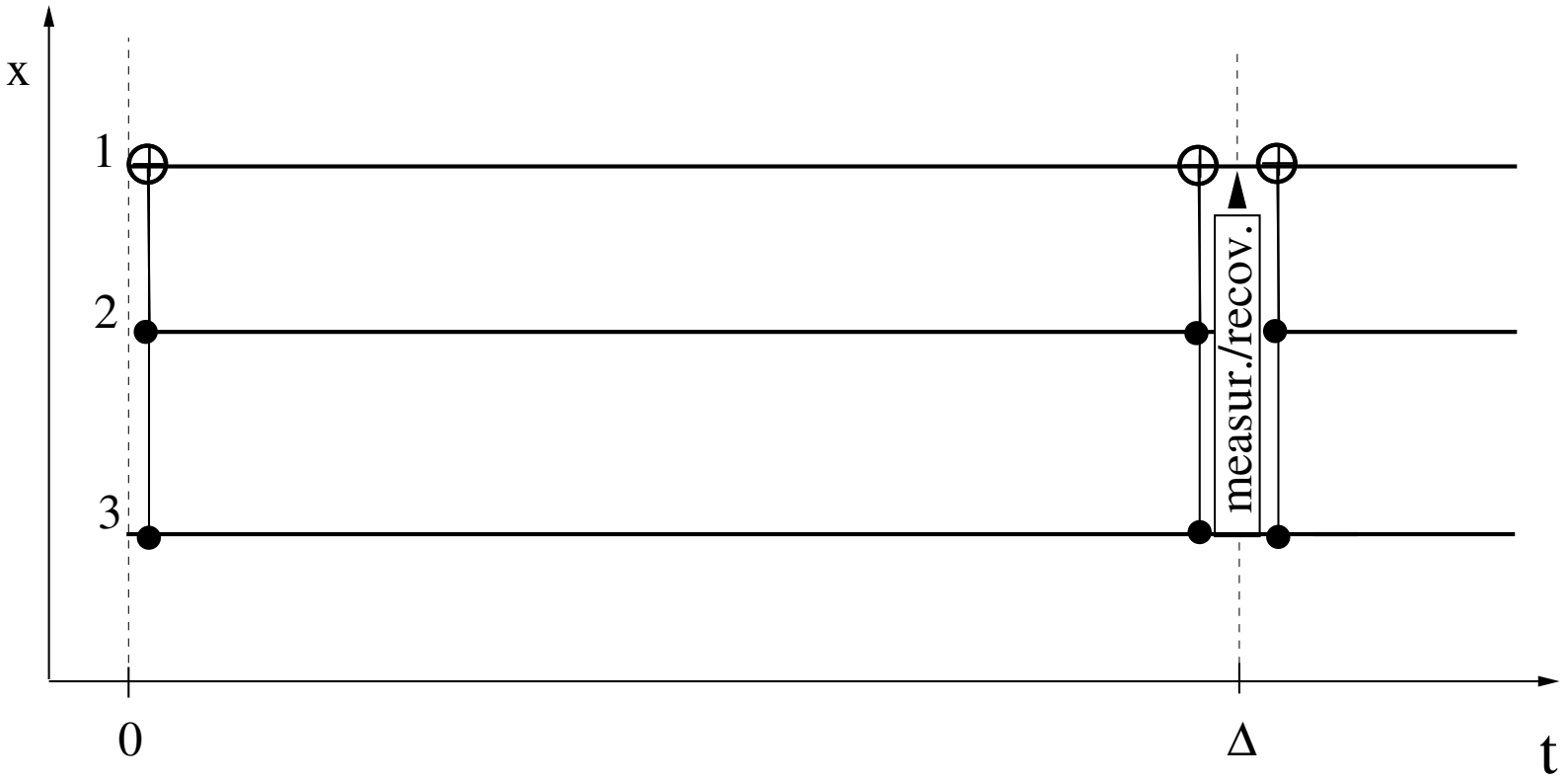

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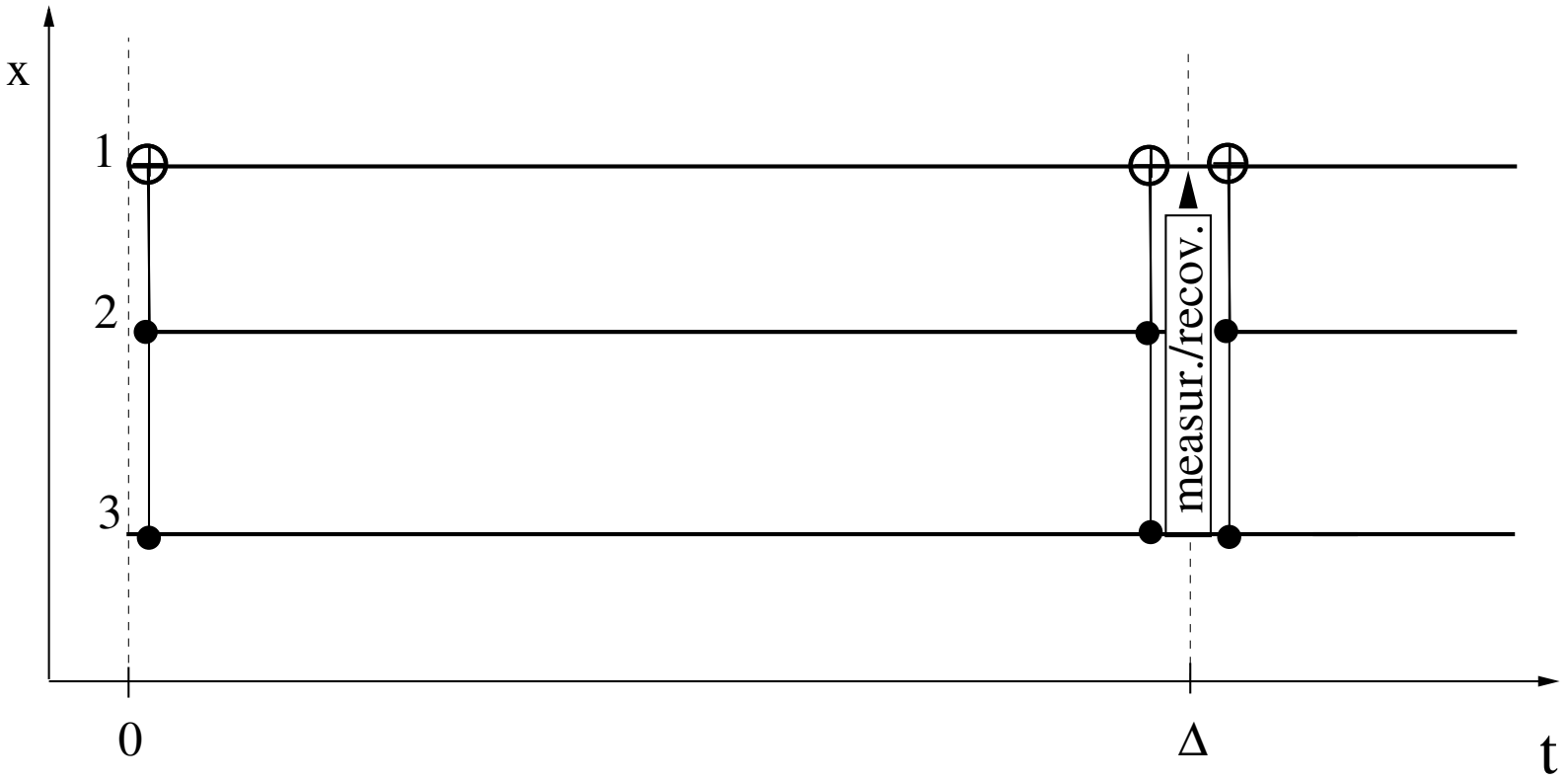

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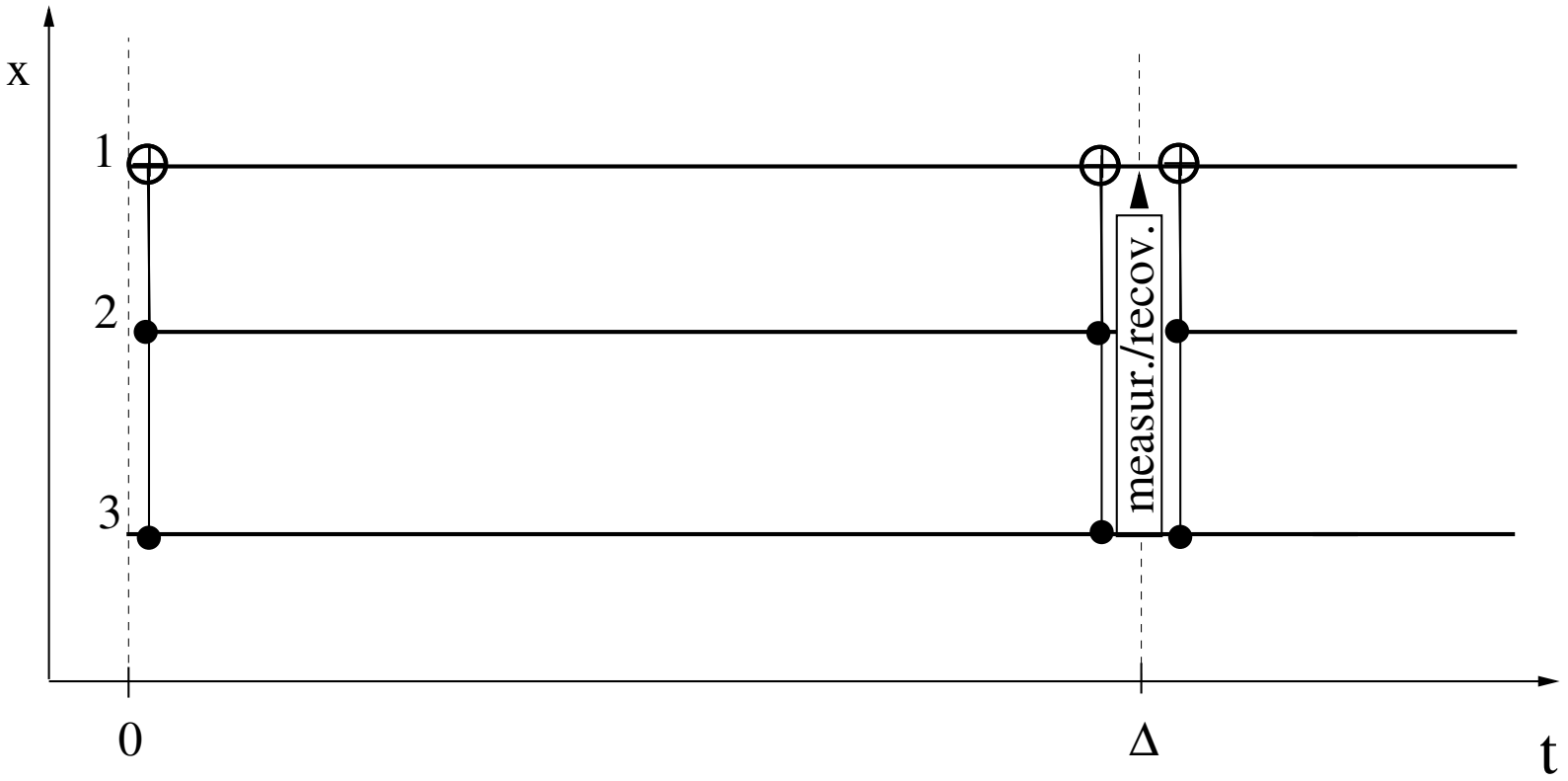

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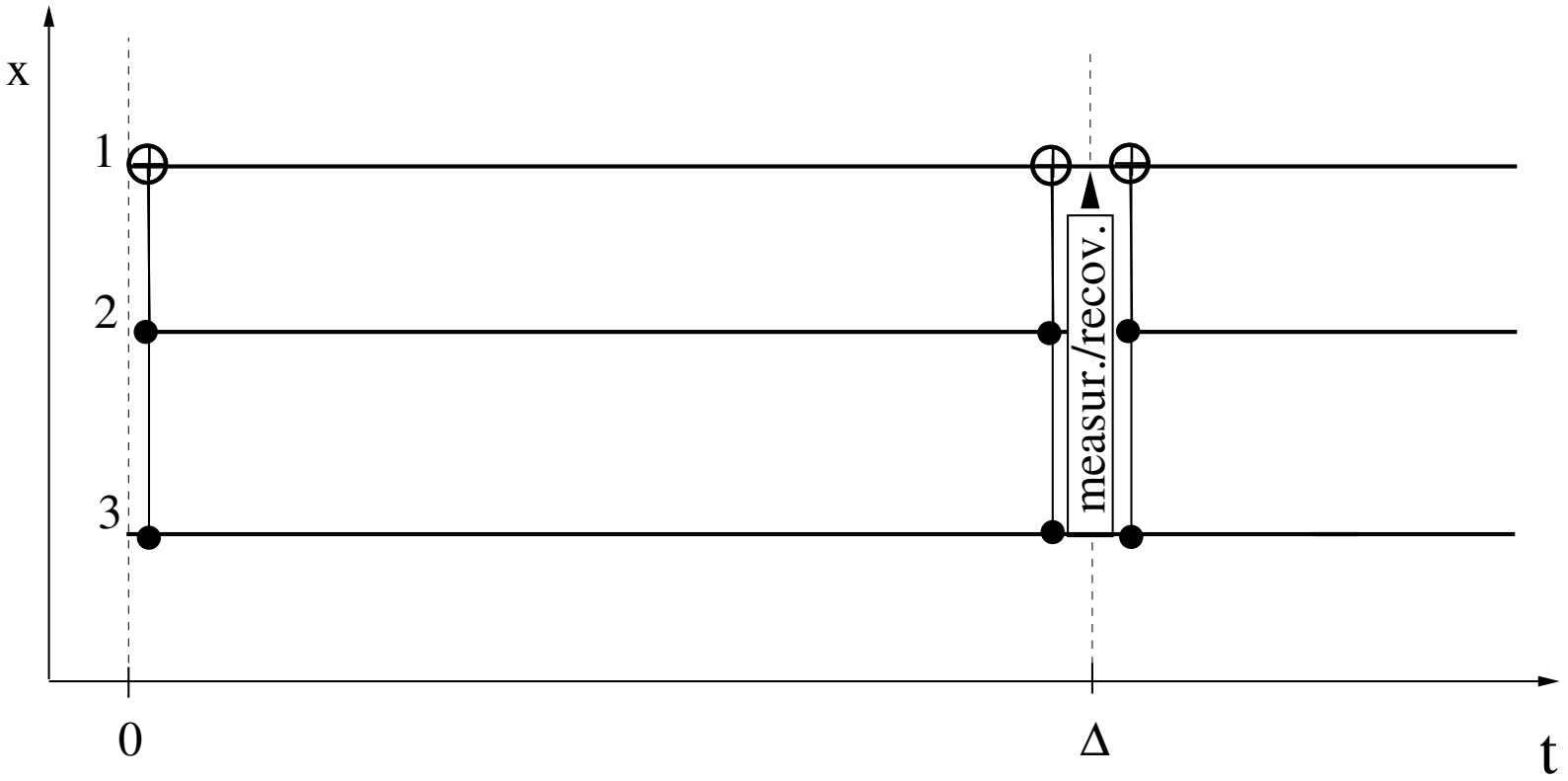

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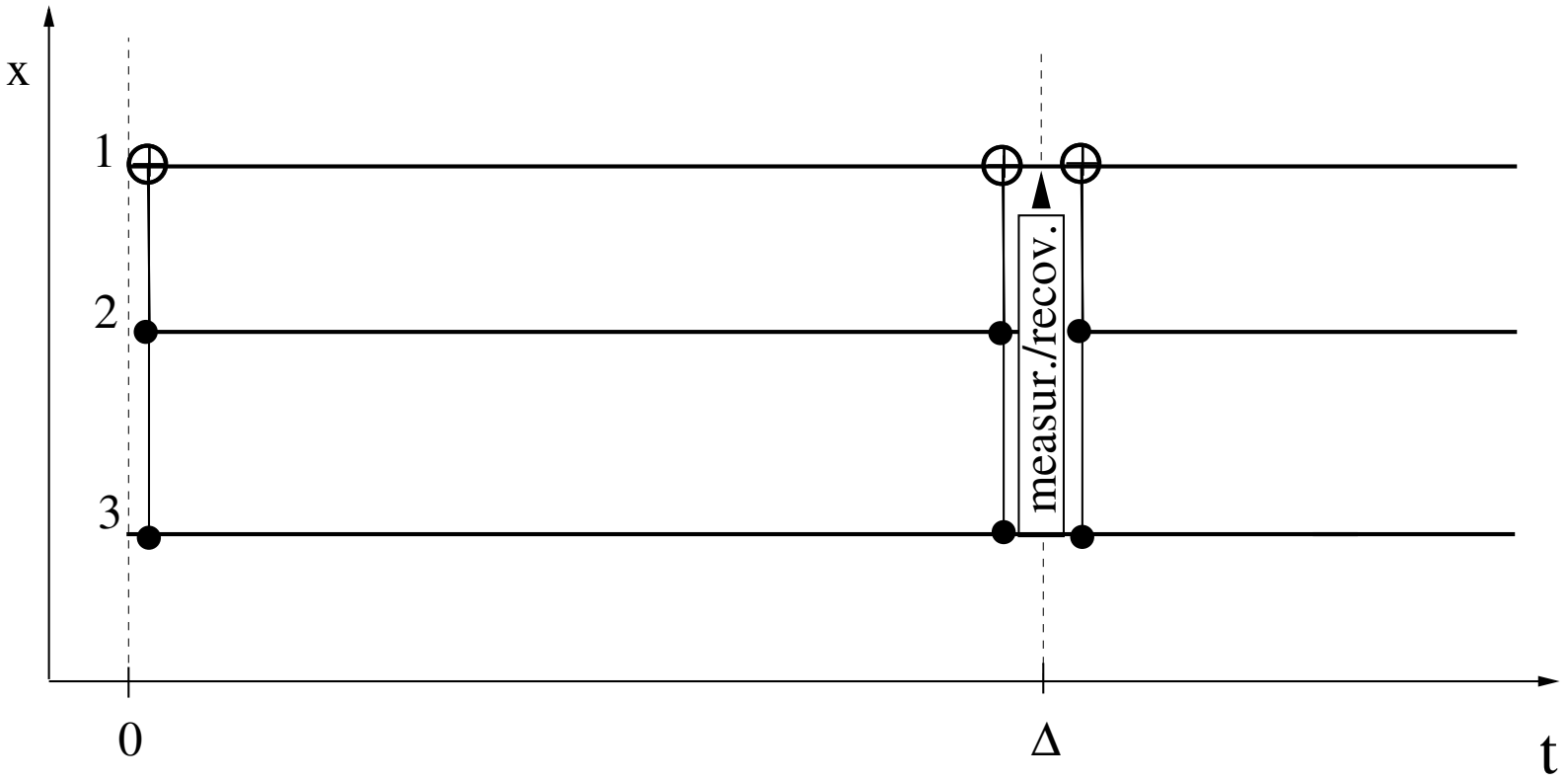

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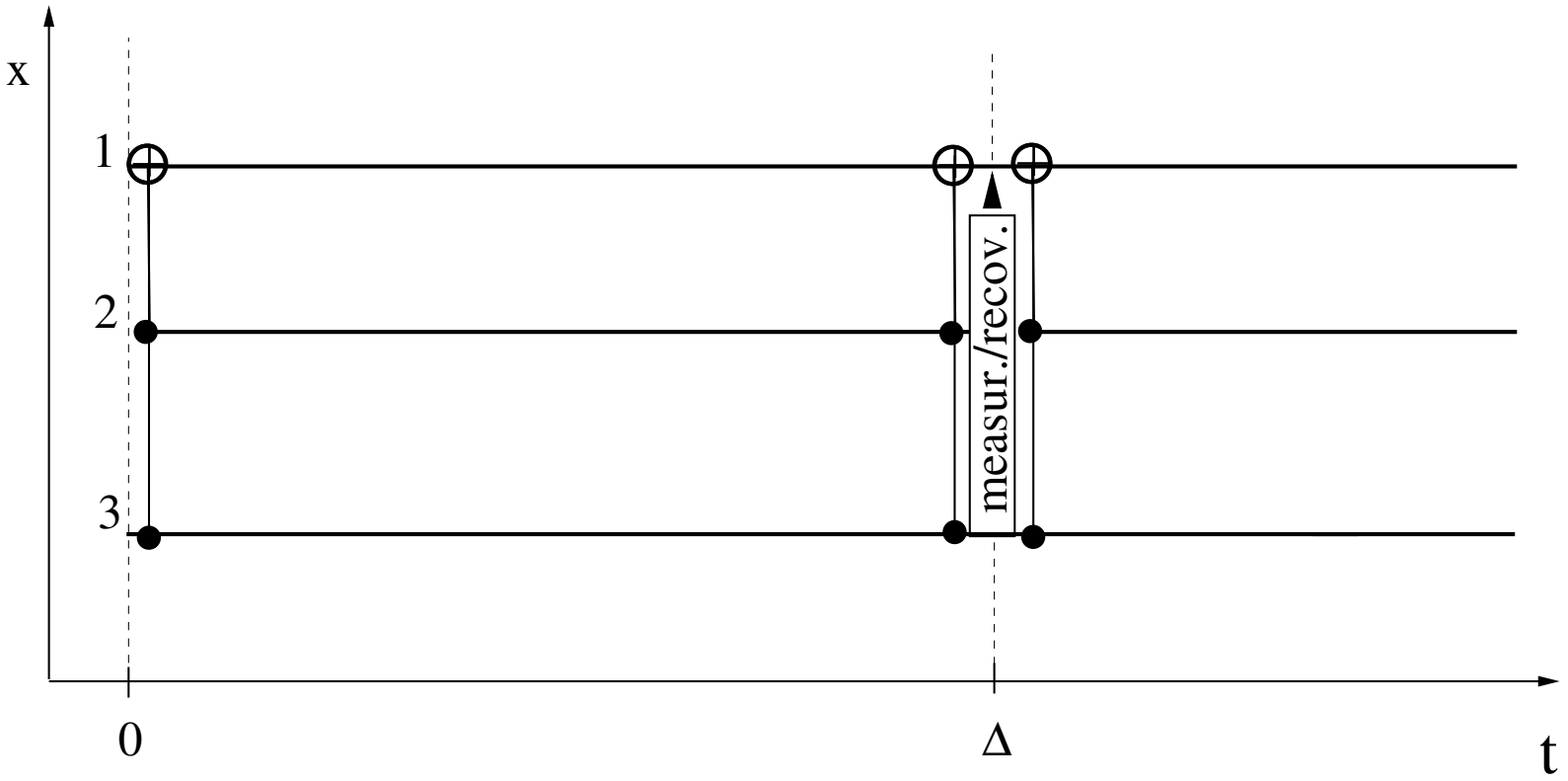

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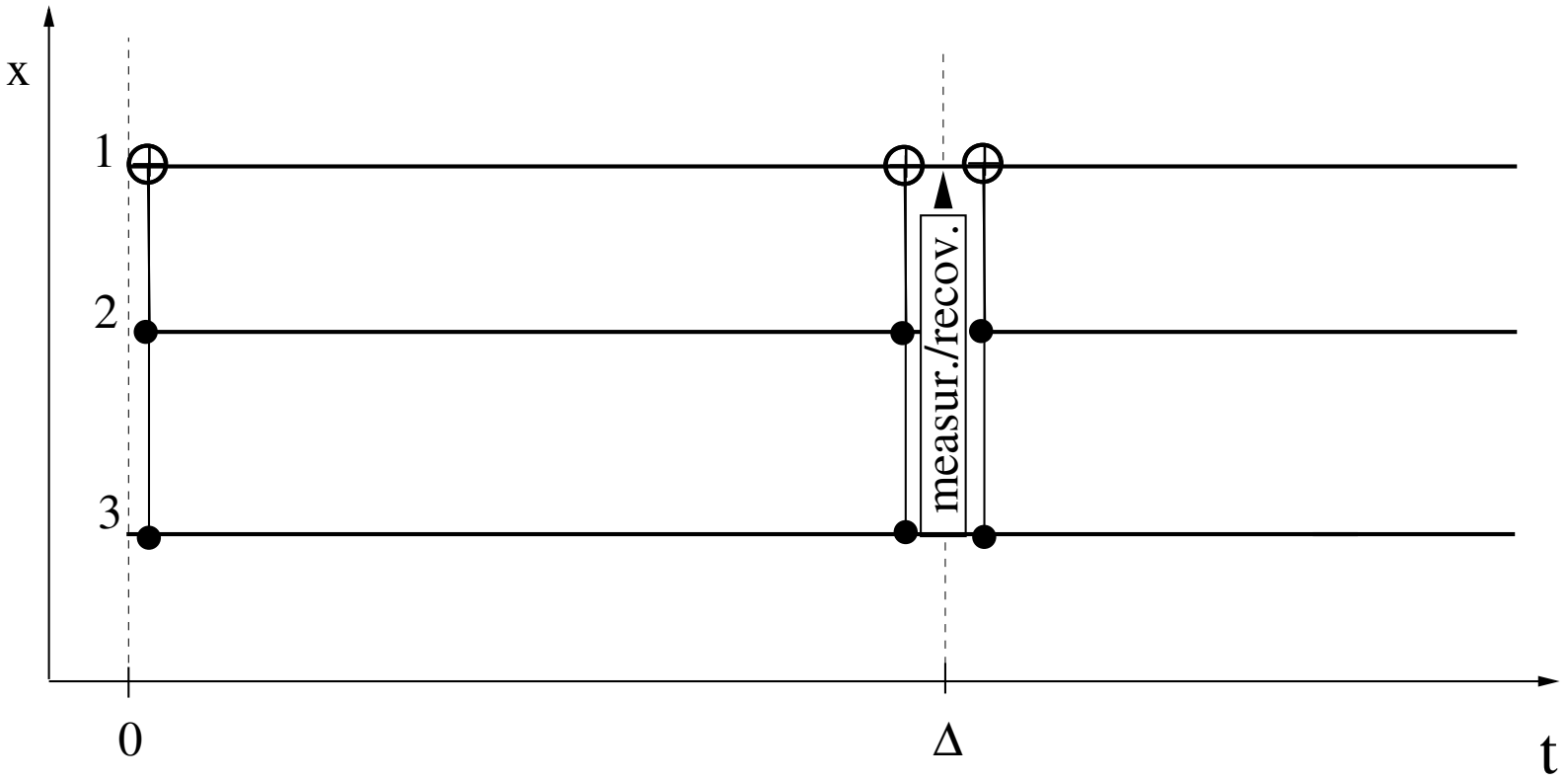

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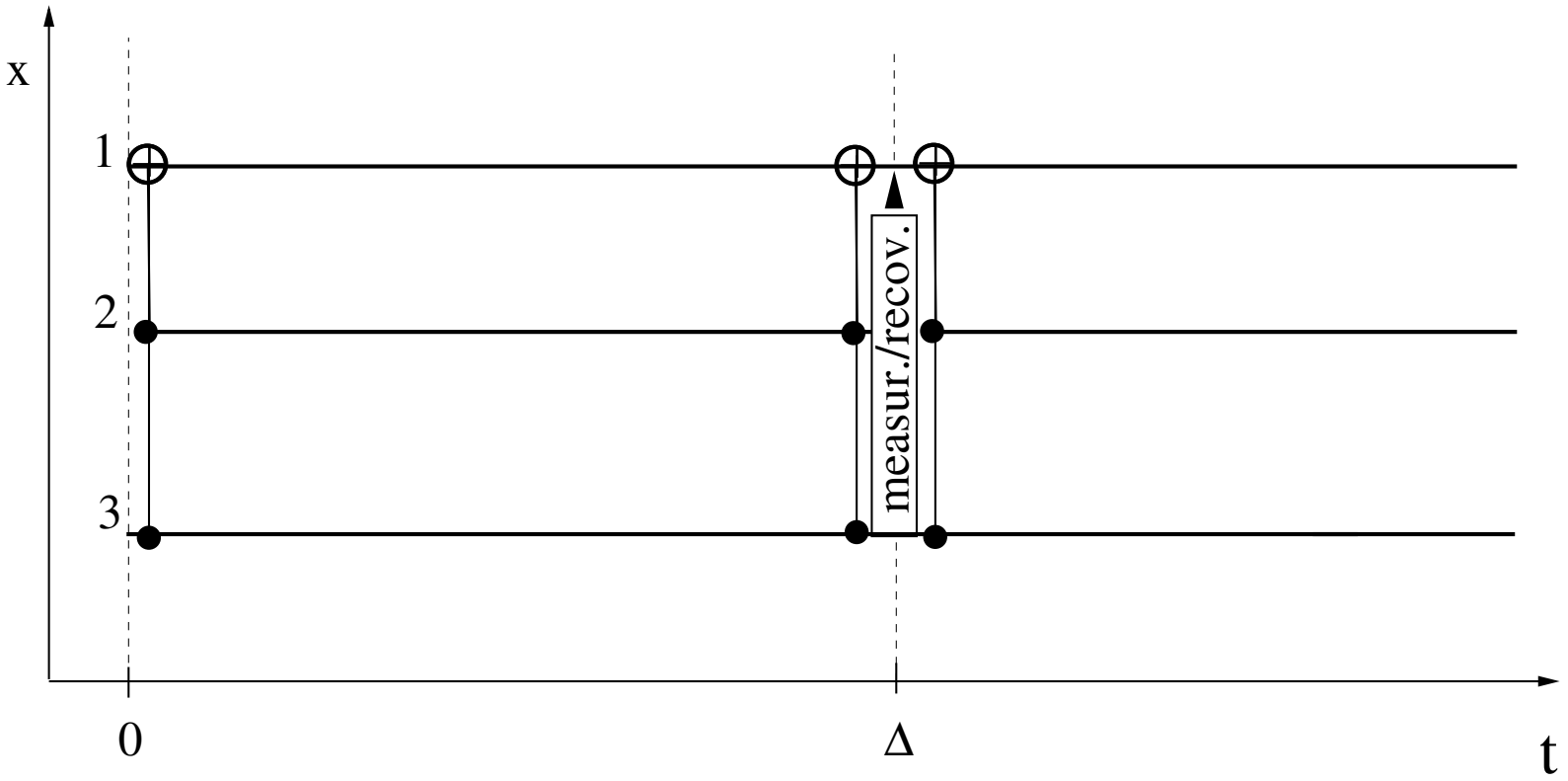

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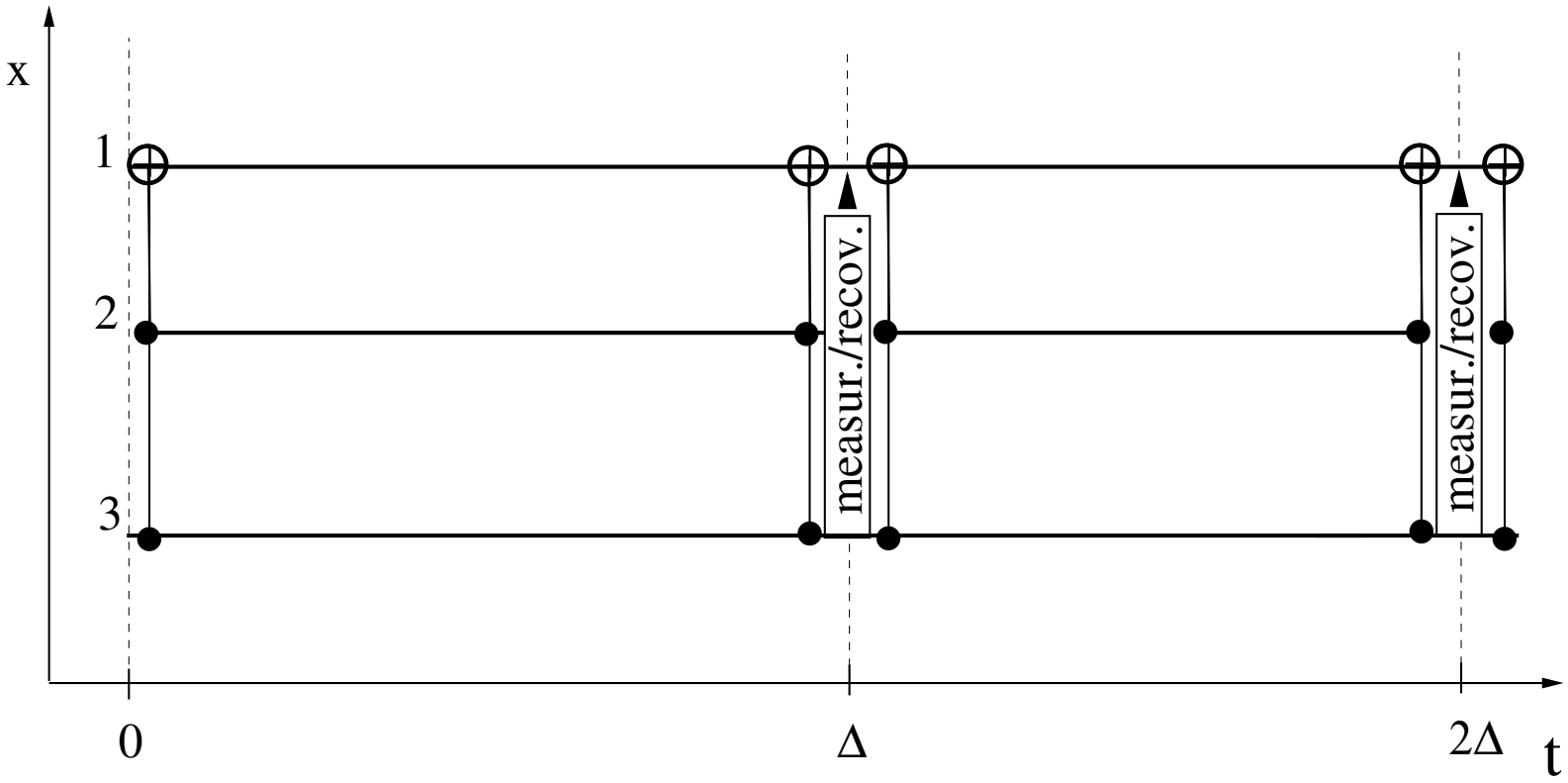

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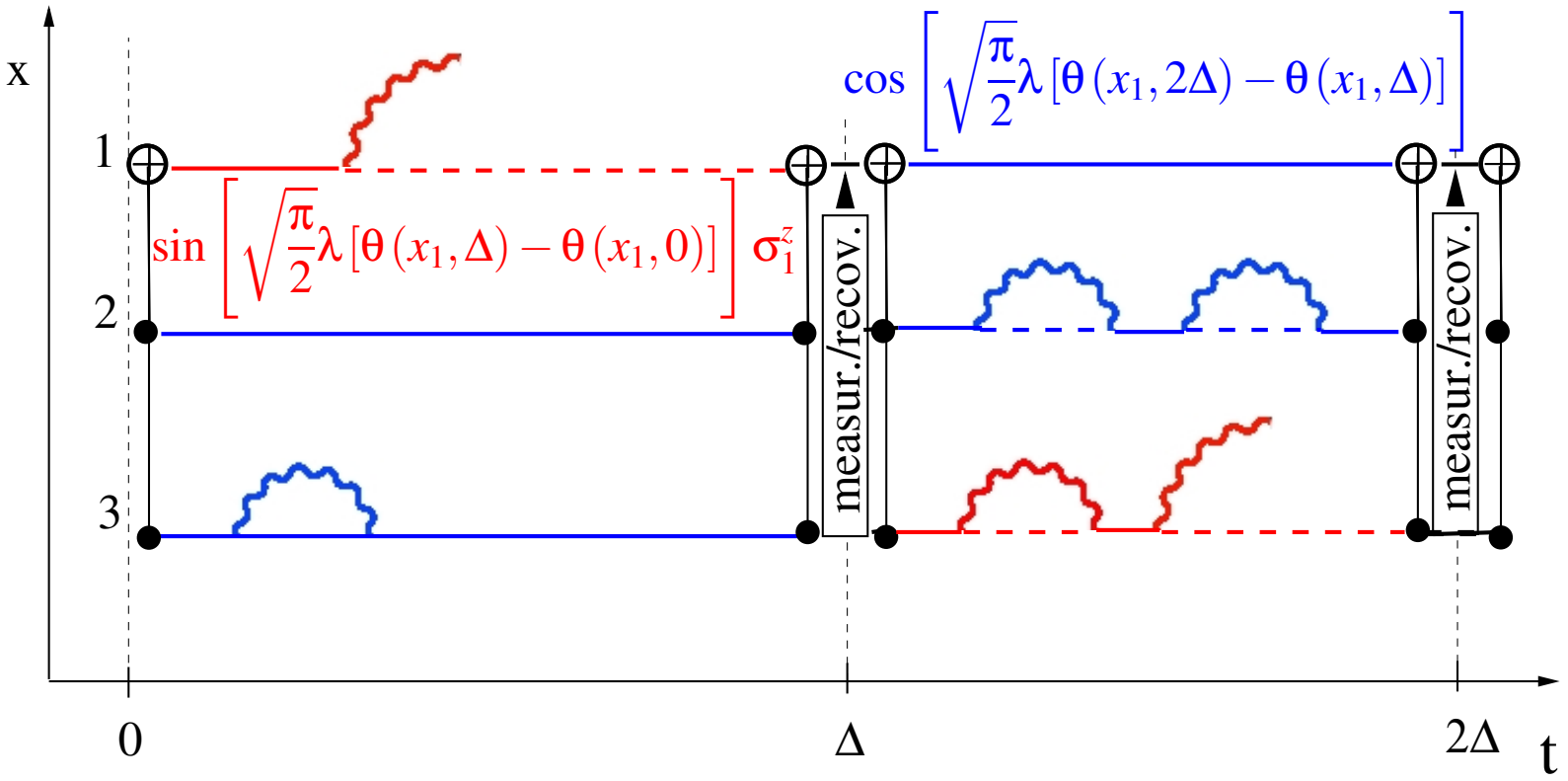

Example: time evolution of encoded qubits

- evolution with "no errors":

$$\upsilon_0(\Delta,0) = \eta_1\eta_2\eta_3 I - i\nu_1\nu_2\nu_3\bar{Z},$$

- evolution with "one error":

$$\begin{aligned}
\upsilon_1(\Delta,0) &= i\nu_1\eta_2\eta_3 I - \eta_1\nu_2\nu_3\bar{Z}, \\
\upsilon_2(\Delta,0) &= i\eta_1\nu_2\eta_3 I - \nu_1\eta_2\nu_3\bar{Z}, \quad \text{or} \\
\upsilon_3(\Delta,0) &= i\eta_1\eta_2\nu_3 I - \nu_1\nu_2\eta_3\bar{Z},
\end{aligned}$$

where:

$$\begin{aligned}
\eta_j &= \cos\left[\sqrt{\frac{\pi}{2}}\lambda\left[\theta(x_j,\Delta)-\theta(x_j,0)\right]\right], \\
\nu_j &= \sin\left[\sqrt{\frac{\pi}{2}}\lambda\left[\theta(x_j,\Delta)-\theta(x_j,0)\right]\right].
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Example: probability of a history of syndromes

- after N QEC cycles, the evolution of a logical qubit is

$$|\psi(N\Delta)\rangle = \ldots\upsilon_j((n+1)\Delta, n\Delta)\ldots\upsilon_k((m+1)\Delta, m\Delta)\ldots|\psi_0\rangle$$

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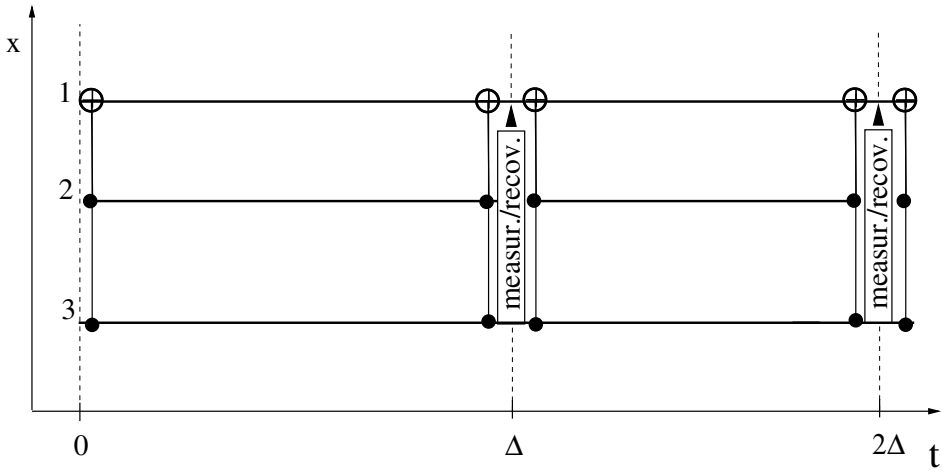

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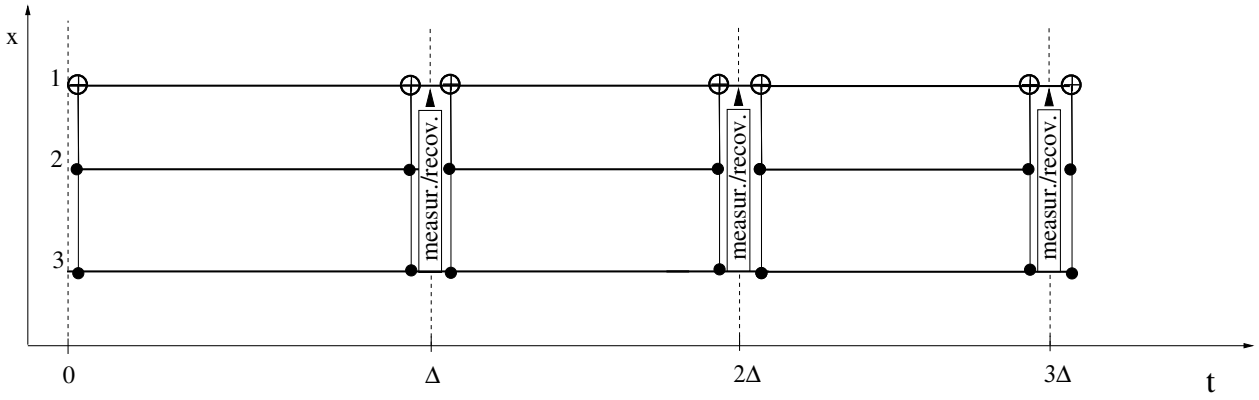

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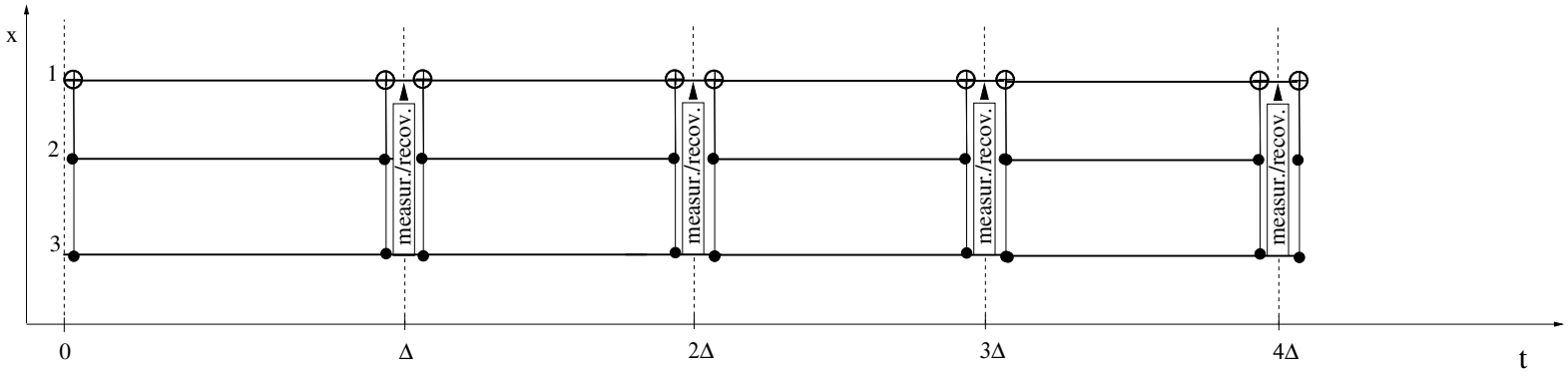

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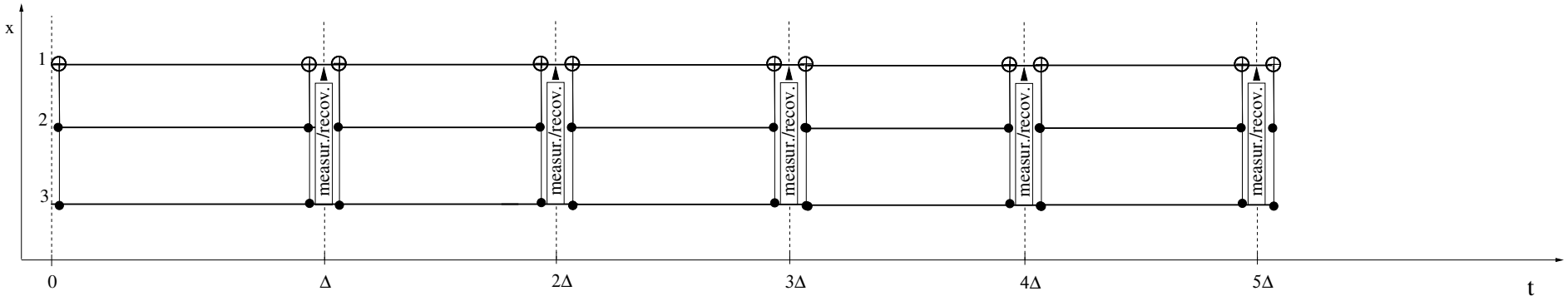

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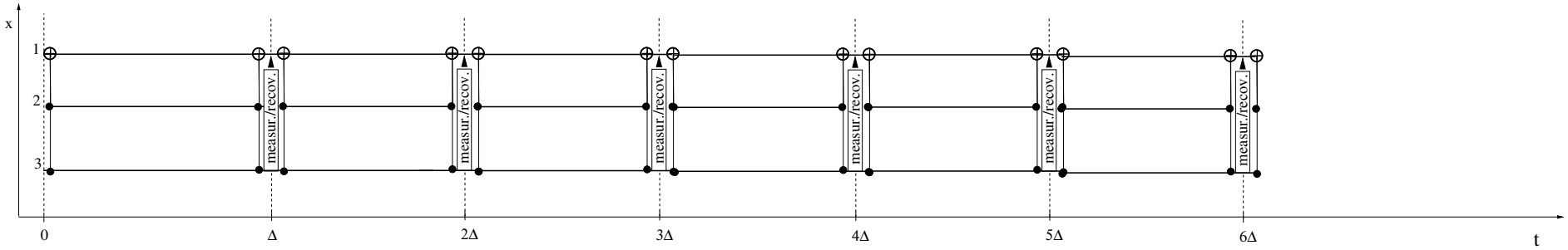

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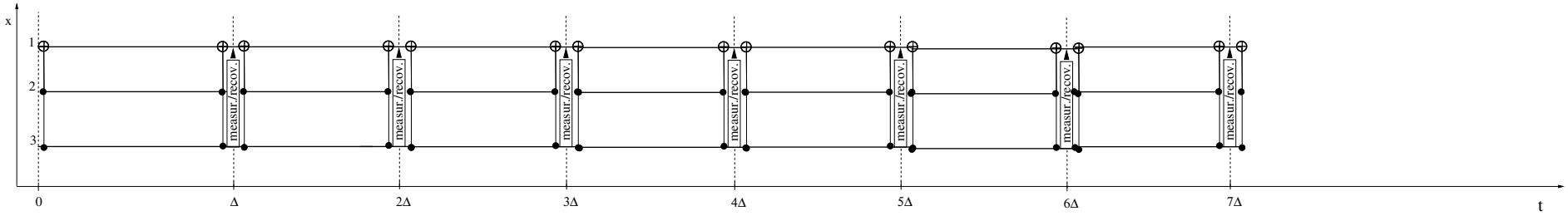

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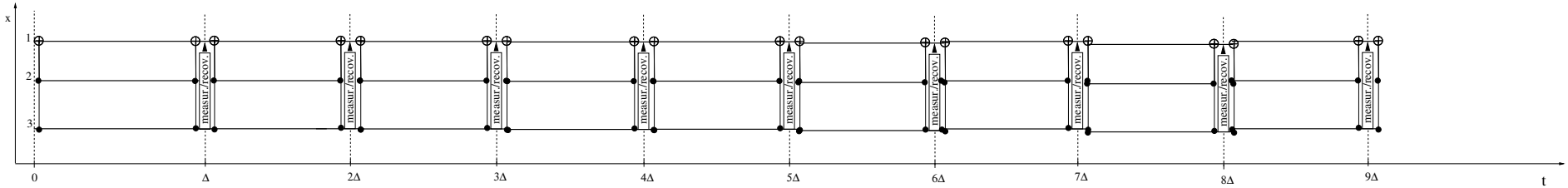

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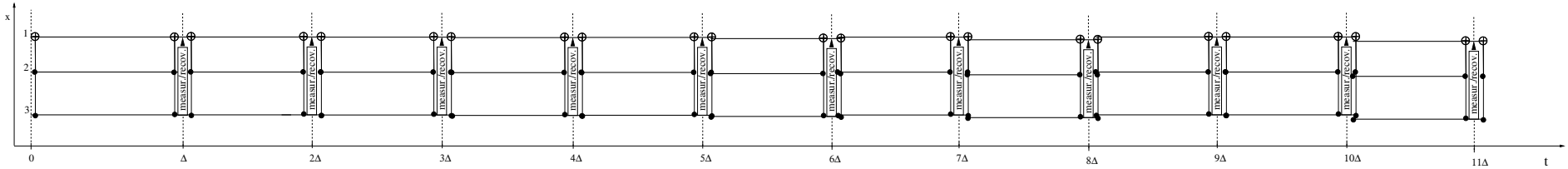

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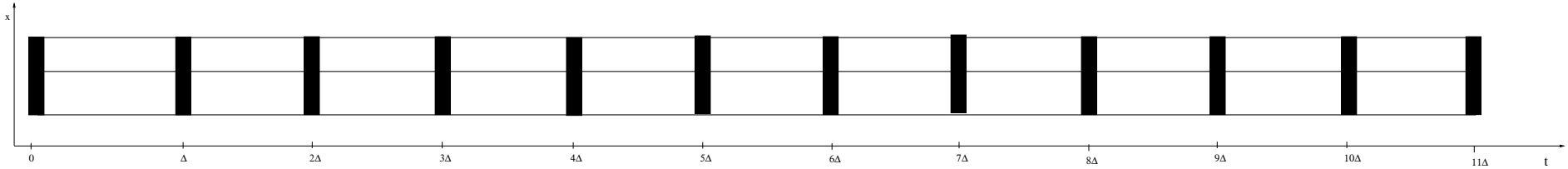

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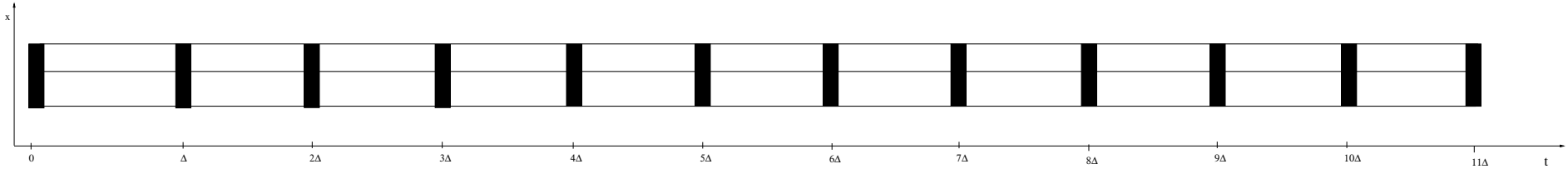

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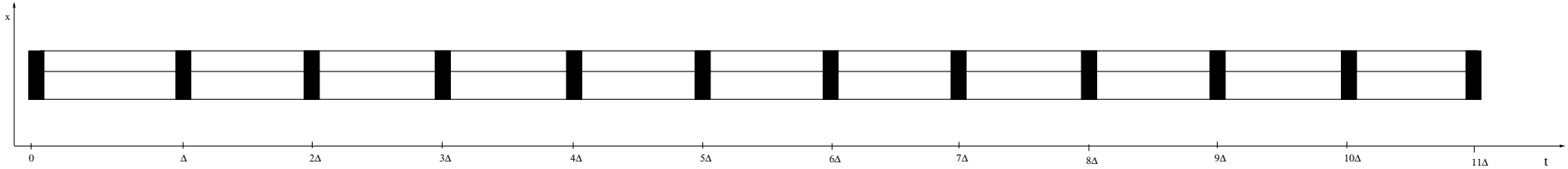

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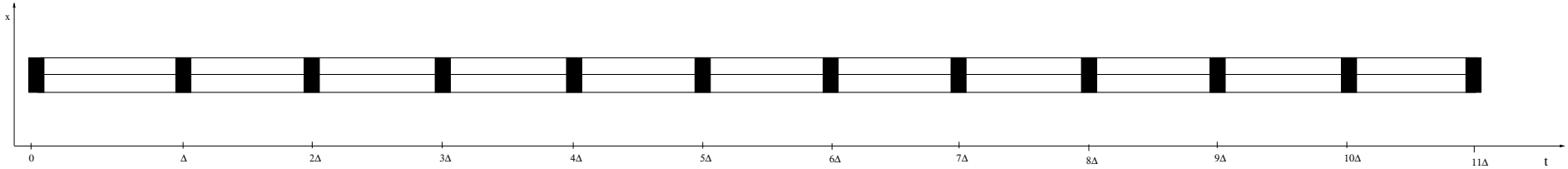

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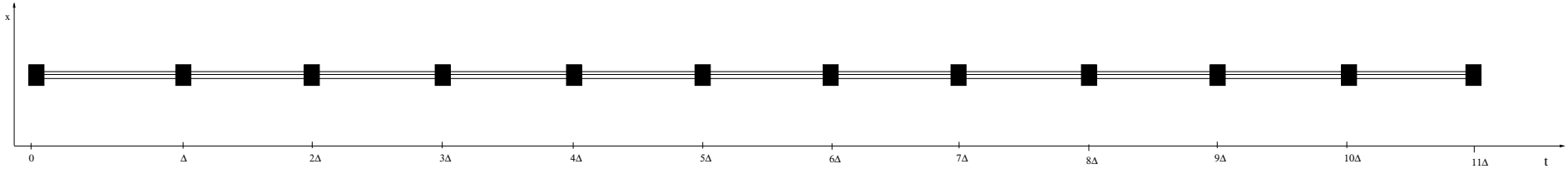

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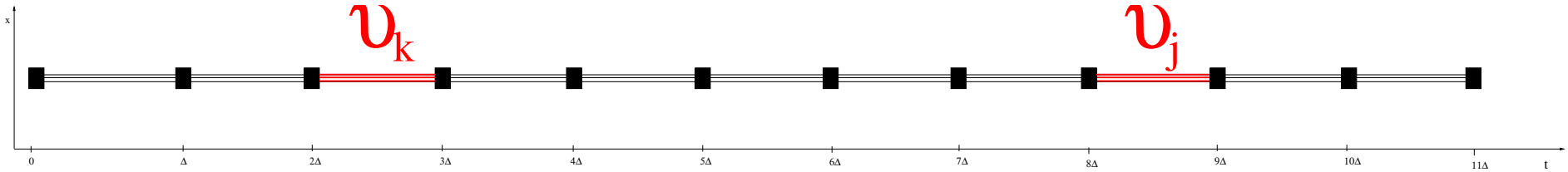

Example: consequences for the probability of events

- by the end of a QEC cycle

$$\upsilon_0^2 \sim 1 - \frac{3\varepsilon}{2} - \sum_{j=1}^{3} \frac{\pi\lambda^2\Delta^2}{2} :[\partial_t\theta(j,0)]^2: ,$$

uncorrelated probability

$$\upsilon_{j=\{1,2,3\}}^2 \sim \frac{\varepsilon}{2} + \frac{\pi\lambda^2\Delta^2}{2} :[\partial_t\theta(j,0)]^2: .$$

with $\varepsilon = \lambda^2 \ln\left[1 + (\Lambda v_b \Delta)^2\right]/2$.

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coarse-grained operators - correlated part

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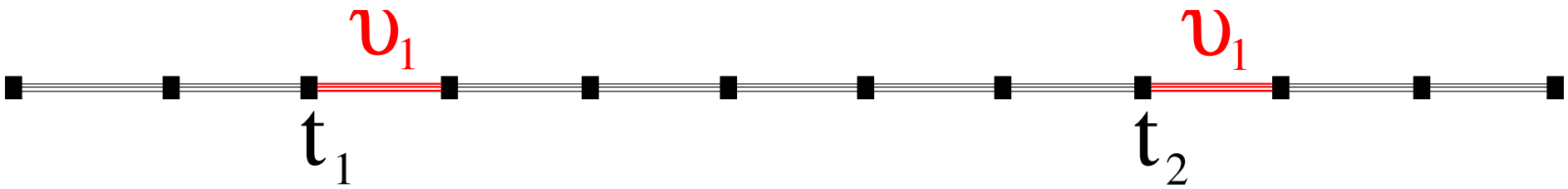
Example: consequences for the probability of events

example: 1- consider that at times $t_1 < t_2$
2- the syndromes gave errors at the first qubit,
3- for simplicity assume the qubits very far apart.

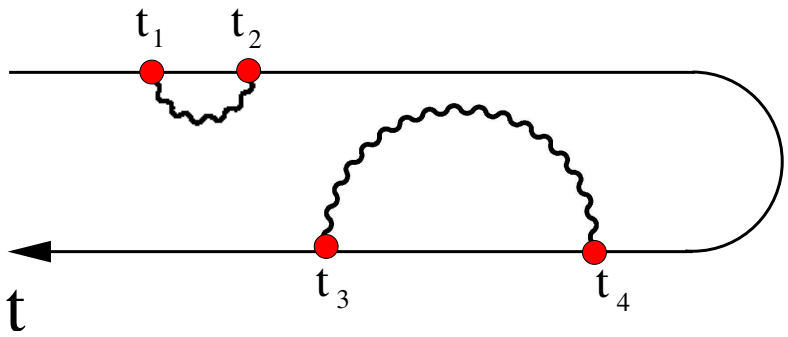
$$
\begin{aligned}
\mathcal{P} &= \langle \psi(N\Delta) | \psi(N\Delta) \rangle \\
&= \langle \psi_0 | \ldots \upsilon_1^2(t_2+\Delta, t_2) \ldots \upsilon_1^2(t_1+\Delta, t_1) \ldots | \psi_0 \rangle \\
&\approx \left(\frac{\varepsilon}{2}\right)^2 + \frac{\pi^2 \lambda^4 \Delta^4}{4} \left\langle : [\partial_t \theta(x_1, t_2)]^2 :: [\partial_t \theta(x_1, t_1)]^2 : \right\rangle + O(\lambda^6) \\
&\approx \left(\frac{\varepsilon}{2}\right)^2 + \frac{\lambda^4 \Delta^4}{8(t_1 - t_2)^4} + O(\lambda^6)
\end{aligned}
$$

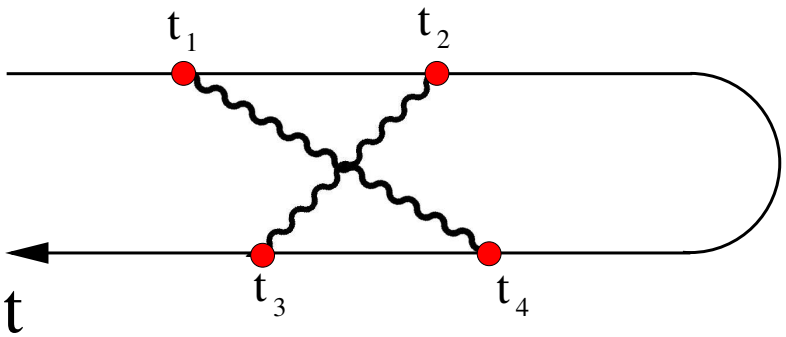


Why correlations are smaller for long times?

- for an ohmic environment the propagator is $\sim \frac{1}{(t_i - t_j)^2}$
- typically this gives terms like

$$\int dt_1 \int dt_2 \int dt_3 \int dt_4 \frac{1}{(t_1 - t_2)^2 (t_3 - t_4)^2}$$



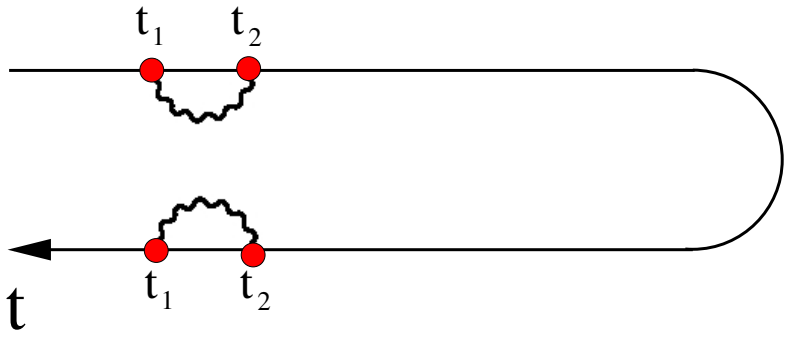


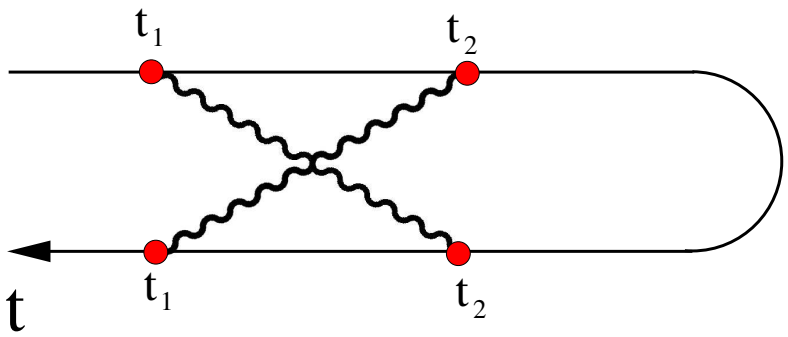
- that is why decoherence would grow as $\ln t$.
- that is what happens inside a QEC cycle.

Why correlations are smaller for long times?

- however in a QEC evolution, for long times we "know" where errors occurred

$$\int dt_1 \int dt_2 \frac{1}{(t_1 - t_2)^2 (t_1 - t_2)^2}$$





QEC steers a very peculiar evolution

decoherence for long times has very little to do with the microscopic Hamiltonian.

reducing the effects of correlations

dynamical decoupling;

L. Viola and S. Lloyd, Phys. Rev. A **58**, 2733 (1998).

L.Viola, E. Knill and S. Lloyd, Phys. Rev. Lett. **82**, 2417 (1999).

"However, a warning also emerges from Fig. 3: if flipping is not frequent enough, not only the correction effect disappears, but decoherence can actually be made worse compared to the absence of the pulse."

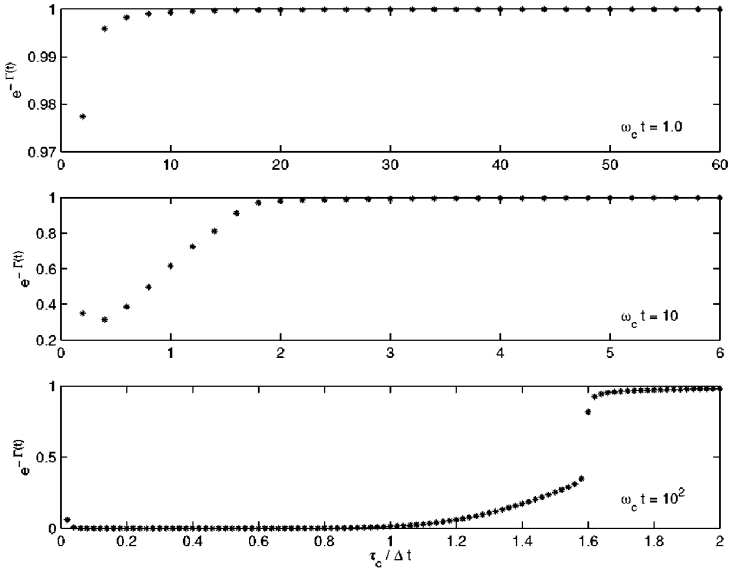


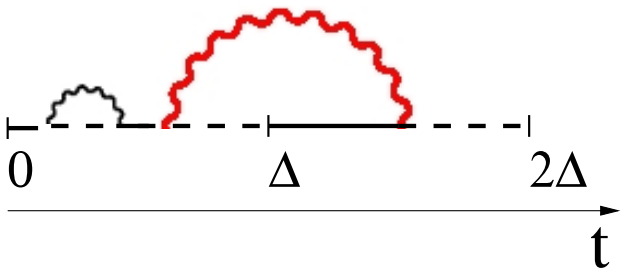
FIG. 3. Same as in Fig. 2 for the low-temperature configuration, $\omega_c/T = 10^2$. The maximum number of spin cycles is equal to $N_{\max} = 30$ in the simulations at $\omega_c t = 1.0, 10$, while $N_{\max} = 100$ at $\omega_c t = 10^2$. The unperturbed values of decoherence are read from Fig. 1(L).

$$v_b \Lambda \ll \omega_f,$$

where ω_f is the frequency of flips.

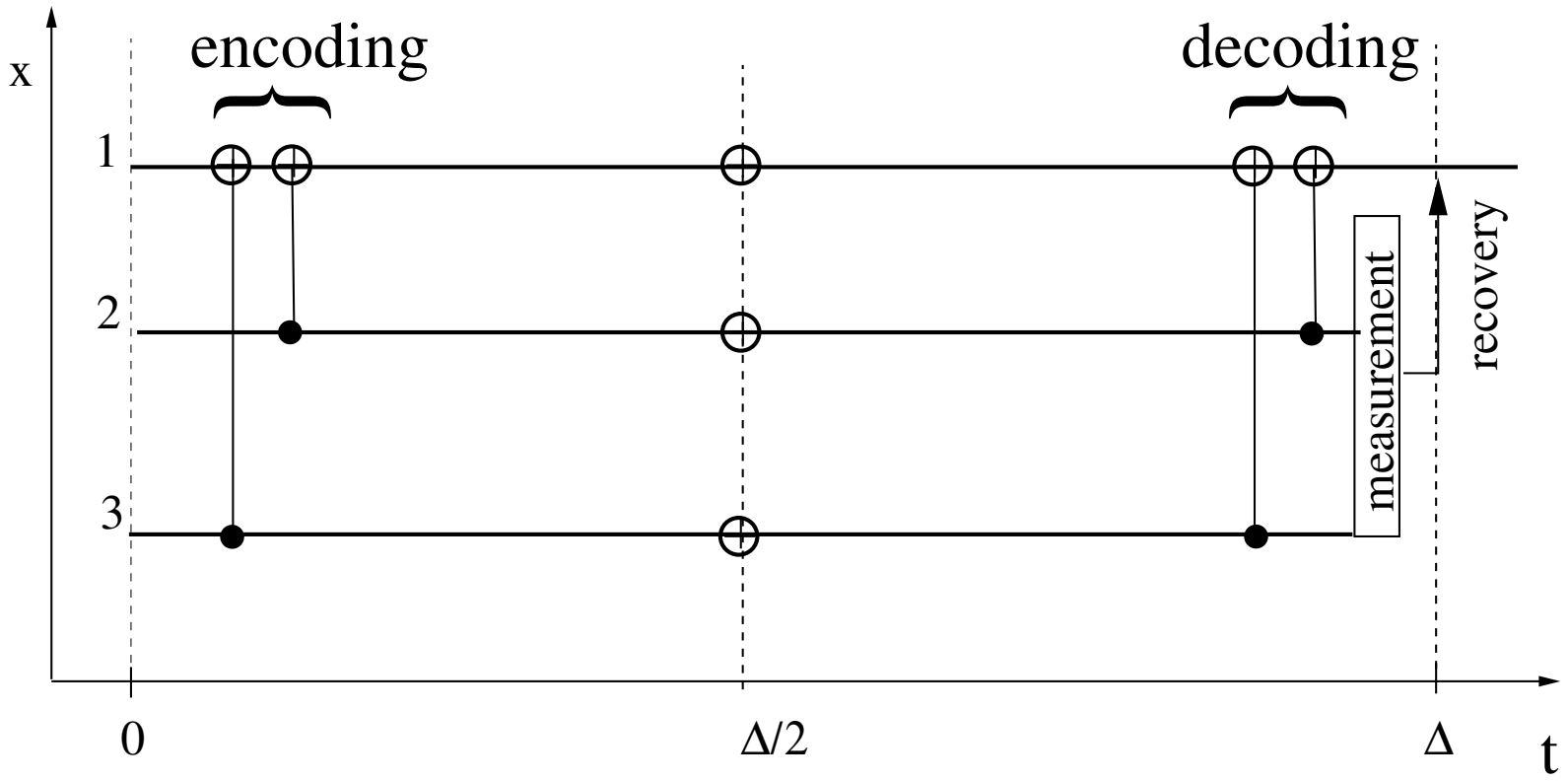
reducing the effects of correlations

- We saw that QEC defines a new scale: Δ^{-1}.



- We can use this new scale to reduce the effects of correlations.

including a logical NOT to the 3-qubits code



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consequences to decoherence

1. increases the local error probability,
2. correlations are more local.

$$\upsilon_0^2 \sim 1 - \frac{9\varepsilon}{2} - \sum_{j=1}^{3} \frac{\pi\lambda^2\Delta^2}{2} : \left[\partial_t^2 \theta(j,0)\right]^2 : ,$$

$$\upsilon_{j=\{1,2,3\}}^2 \sim \frac{3\varepsilon}{2} + \frac{\pi\lambda^2\Delta^2}{2} : \left[\partial_t^2 \theta(j,0)\right]^2 : .$$

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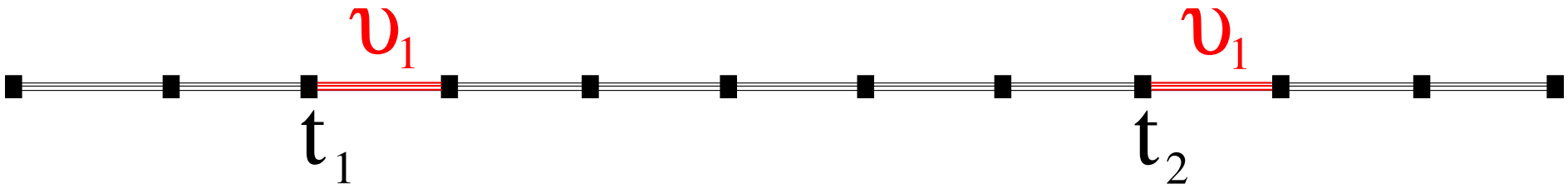
including a logical NOT to the 3-qubits code

example: 1- consider that at times $t_1 < t_2$
2- the syndromes gave errors at the first qubit,
3- for simplicity assume the qubits very far apart.

$$
\begin{aligned}
\mathcal{P} &= \langle \psi(N\Delta) | \psi(N\Delta) \rangle \\
&= \langle \psi_0 | \ldots \upsilon_1^2(t_2+\Delta, t_2) \ldots \upsilon_1^2(t_1+\Delta, t_1) \ldots | \psi_0 \rangle \\
&\approx \left(\frac{3\varepsilon}{2}\right)^2 + \frac{\pi^2 \lambda^4 \Delta^8}{32^2} \left\langle : \left[\partial_t^2 \theta(x_1, t_2)\right]^2 :: \left[\partial_t^2 \theta(x_1, t_1)\right]^2 : \right\rangle + O(\lambda^6) \\
&\approx \left(\frac{3\varepsilon}{2}\right)^2 + \frac{\lambda^4 \Delta^8}{228 (t_1 - t_2)^8} + O(\lambda^6)
\end{aligned}
$$



How to deal with the general problem?

The old trick of ...

... reducing the problem to a known one.
We want to use the usual threshold theorem for Markovian noise.

Main steps to be done

- start with a Hamiltonian,
- define a local error probability,
- identify the long range operator,
- study how long range component alters the local part.

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Returning to the general problem

- For an environment controlled by a "free" Hamiltonian:

$$H_0$$

- there are three important parameters:
  1. the number of spatial dimensions, D;
  2. the wave velocity, v and;
  3. the strength, η.
- a general form for the interaction is:

$$V = \sum_{\mathbf{x},\alpha} \lambda_\alpha f_\alpha(\mathbf{x})\, \sigma_\alpha(\mathbf{x}), \tag{1}$$

where $\vec{f}$ is a function of the environment degrees of freedom and $\vec{\sigma}$ are the Pauli matrices that parametrize the qubits.

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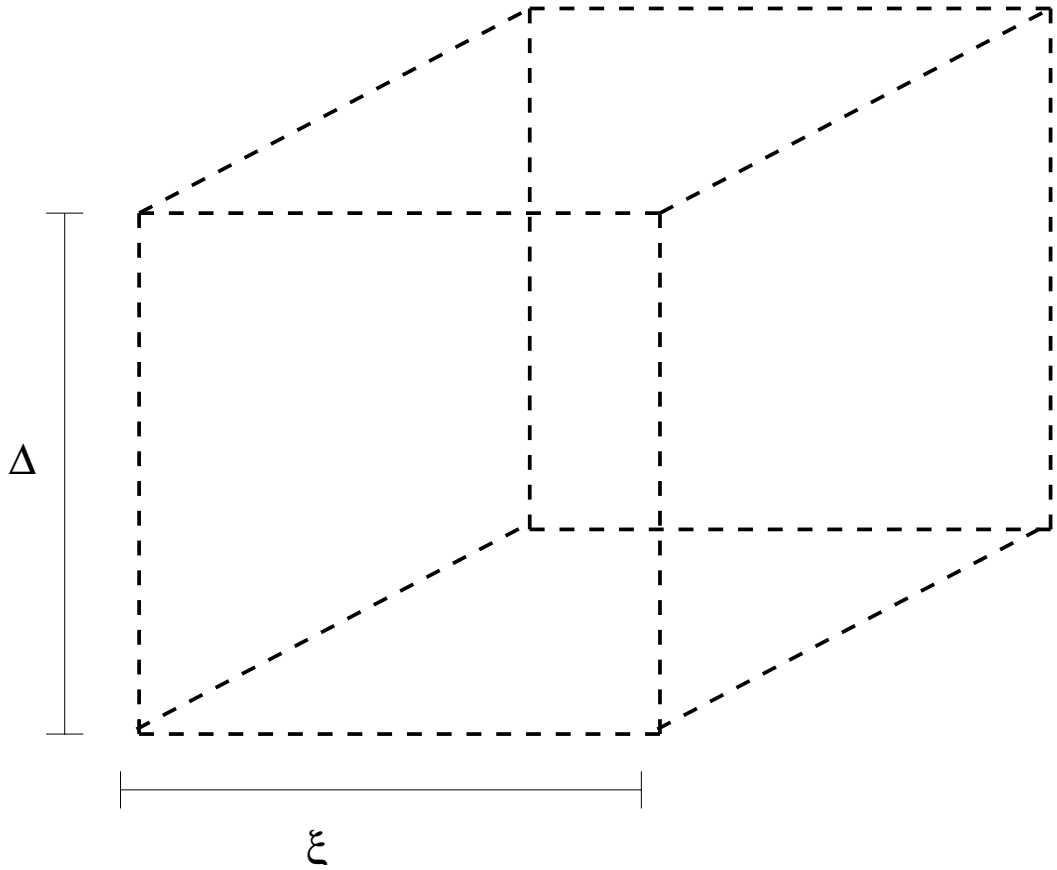
Defining a coarse-grain grid

Hypercube - MOST IMPORTANT SIMPLIFICATION

of size Δ in time and $\xi = (v\Delta)^{\frac{1}{z}}$ in space.



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There is only one qubit in each hypercube.

- allows to define the probability of an error in a qubit.
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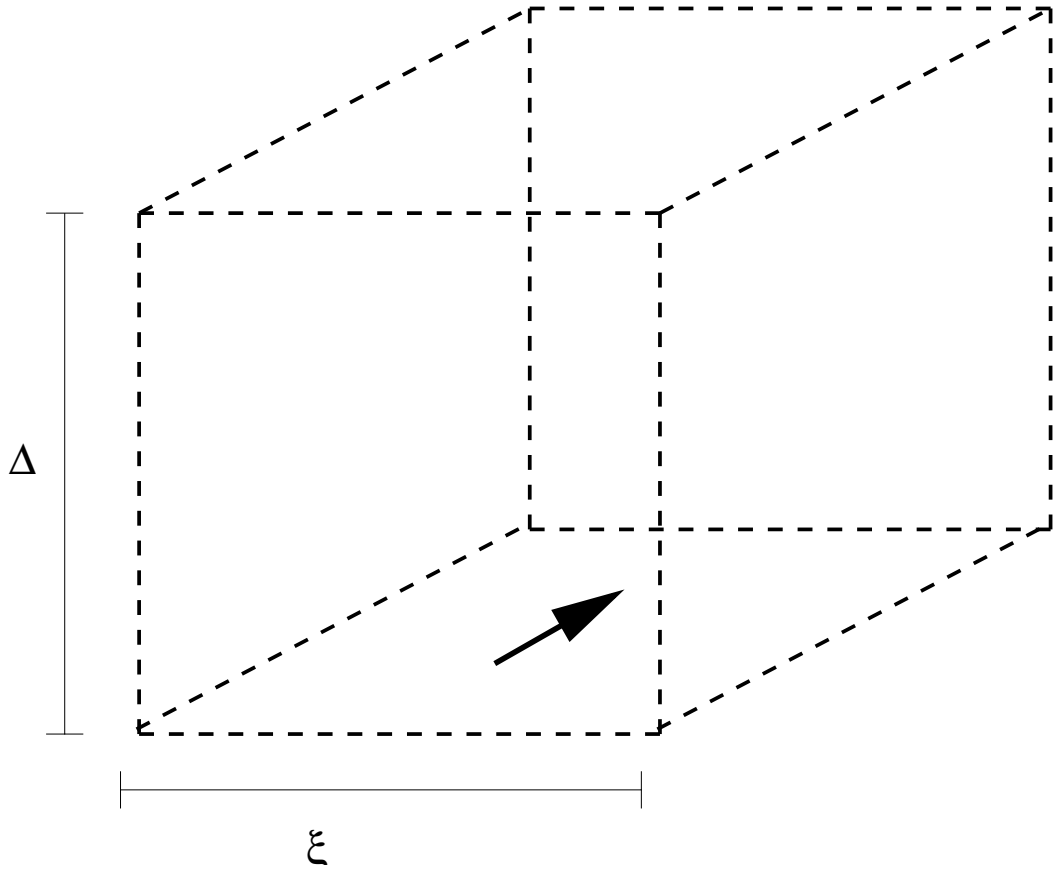

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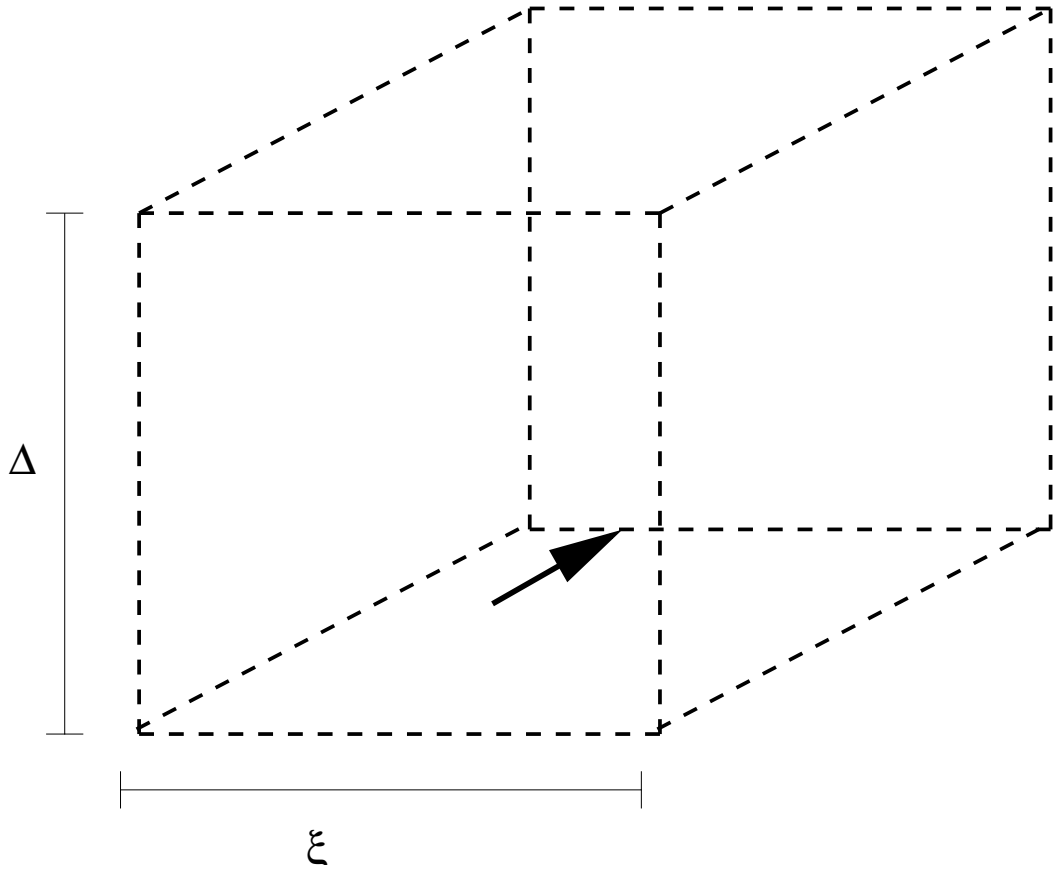

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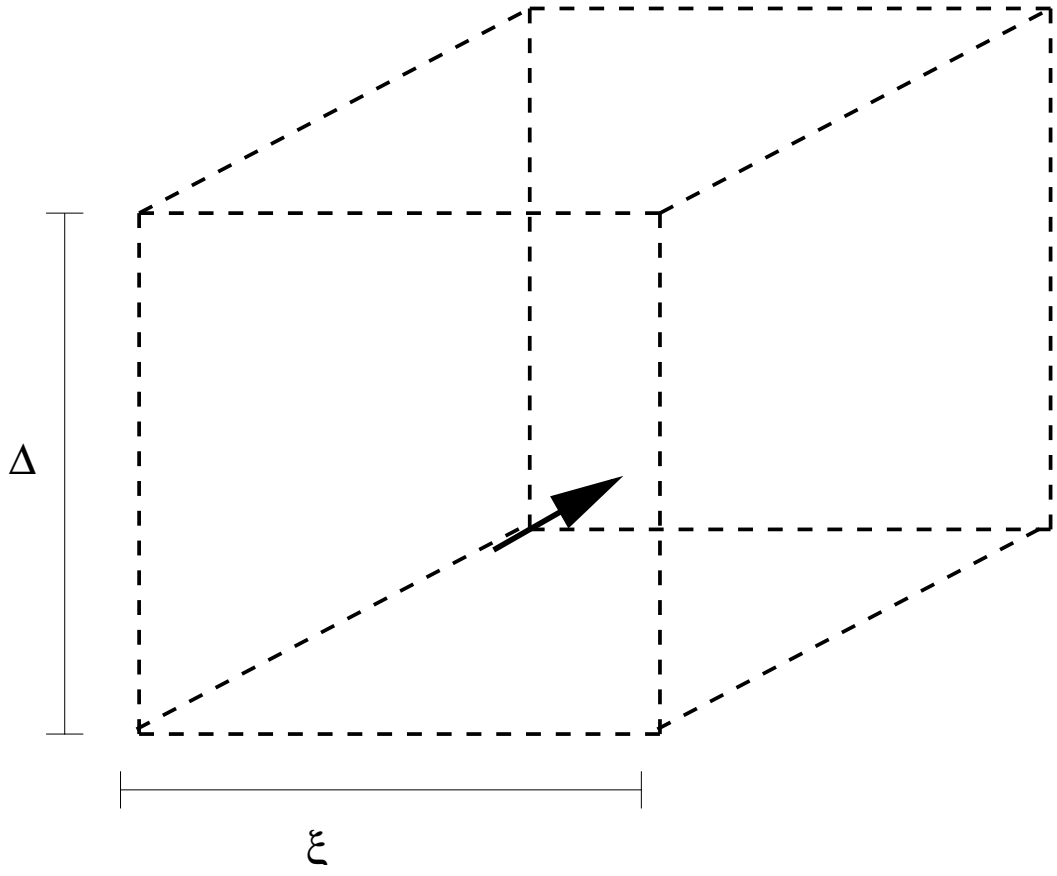

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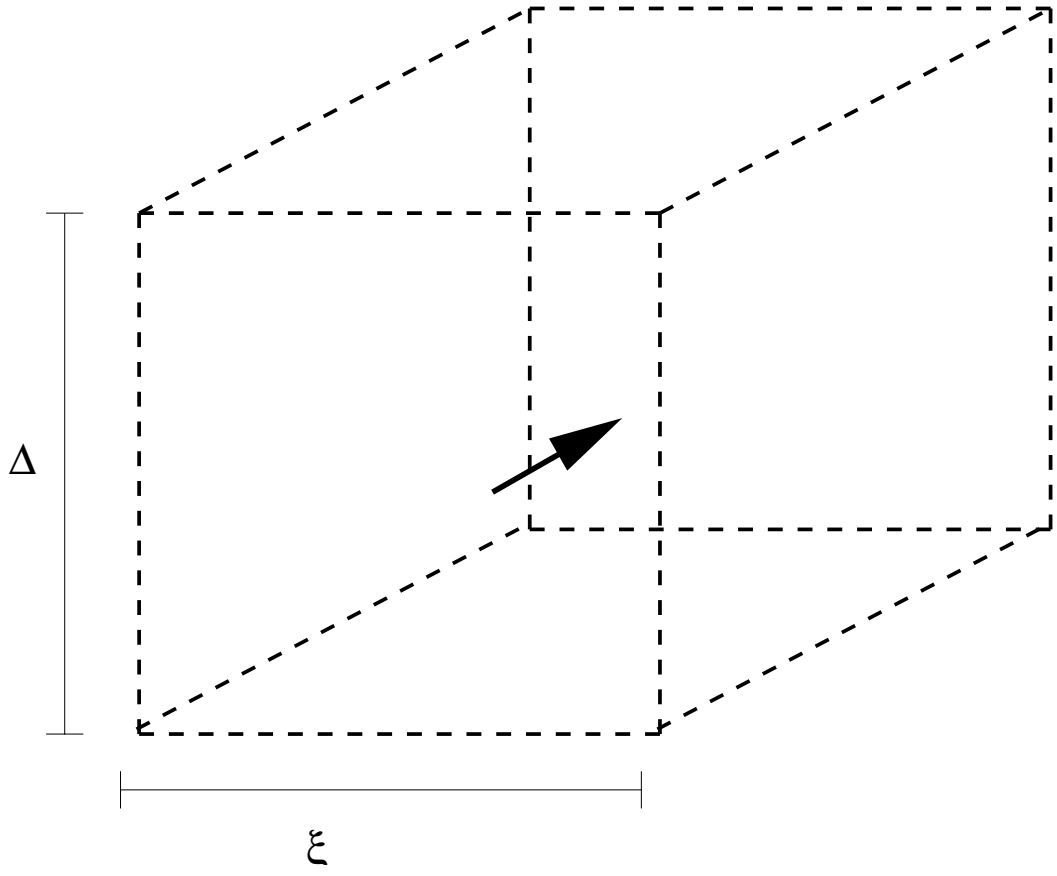

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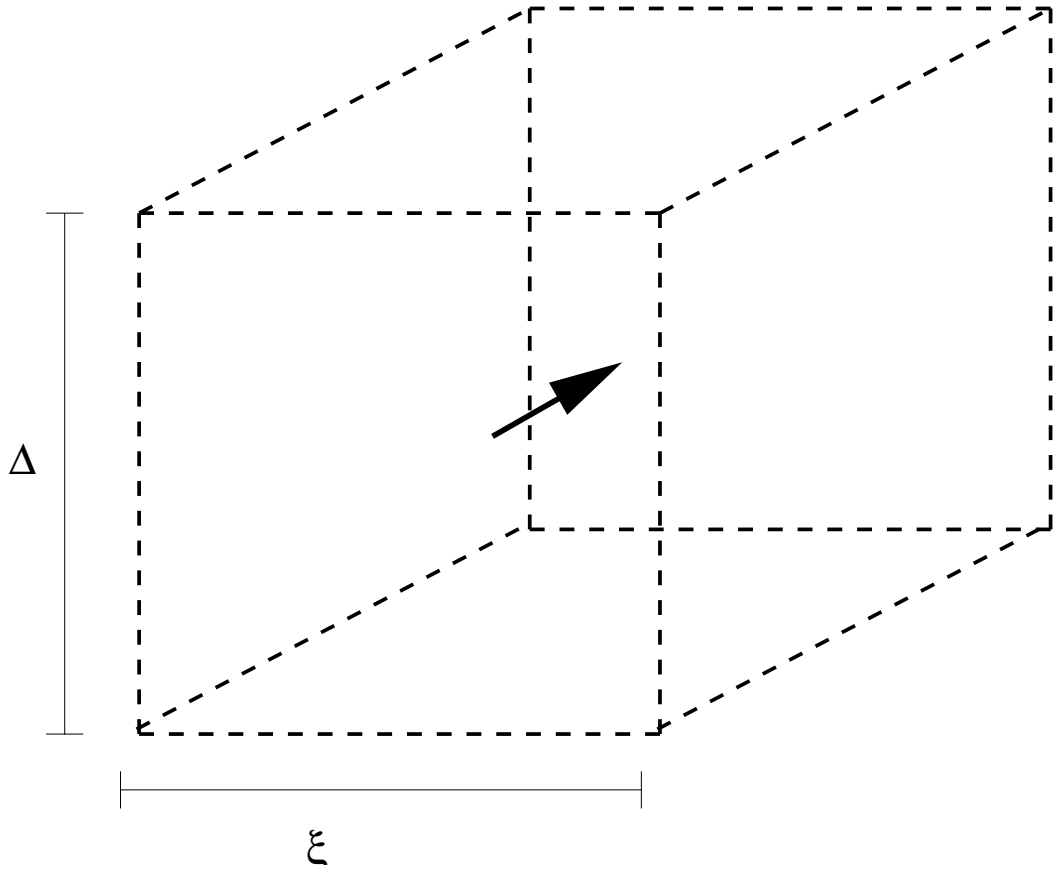

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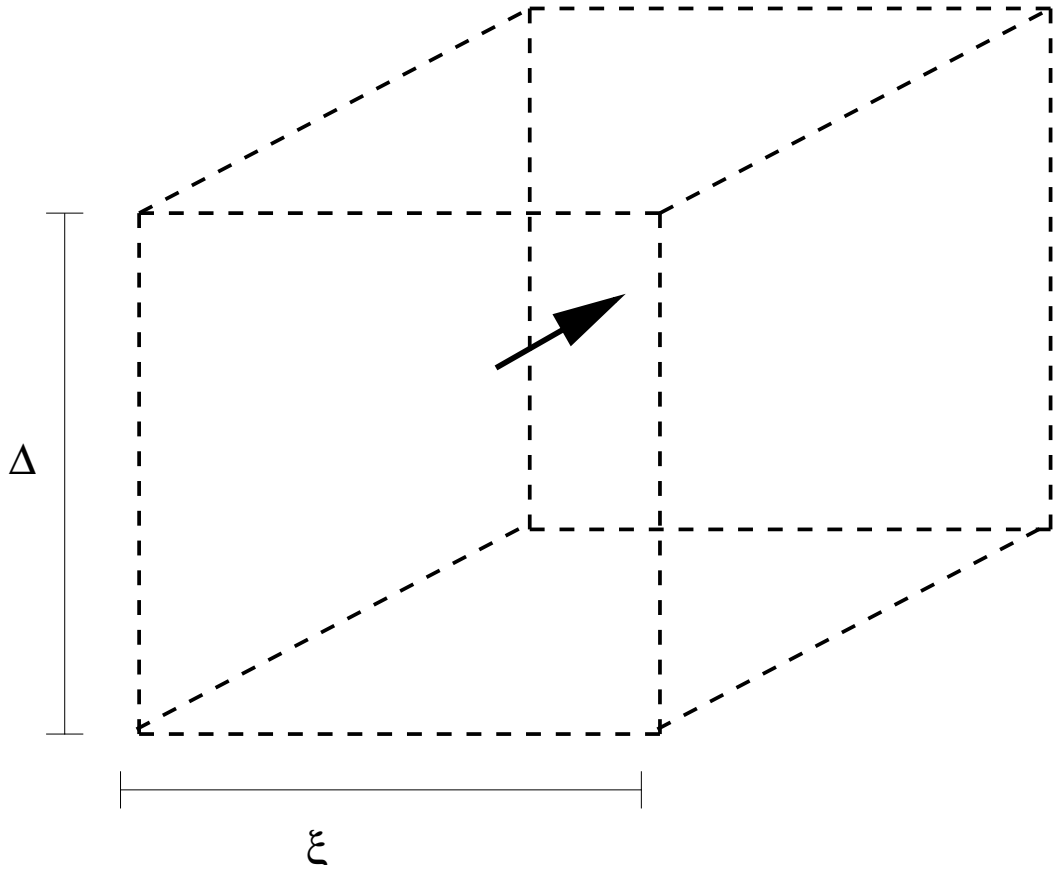

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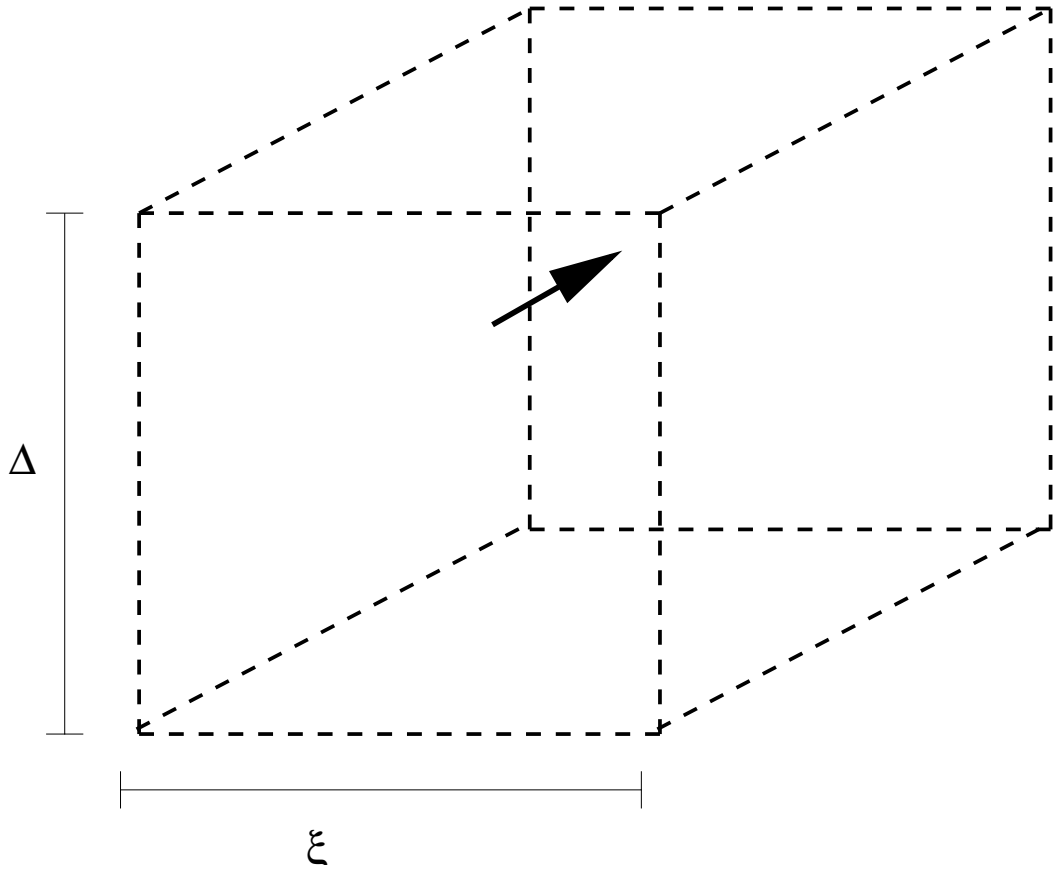

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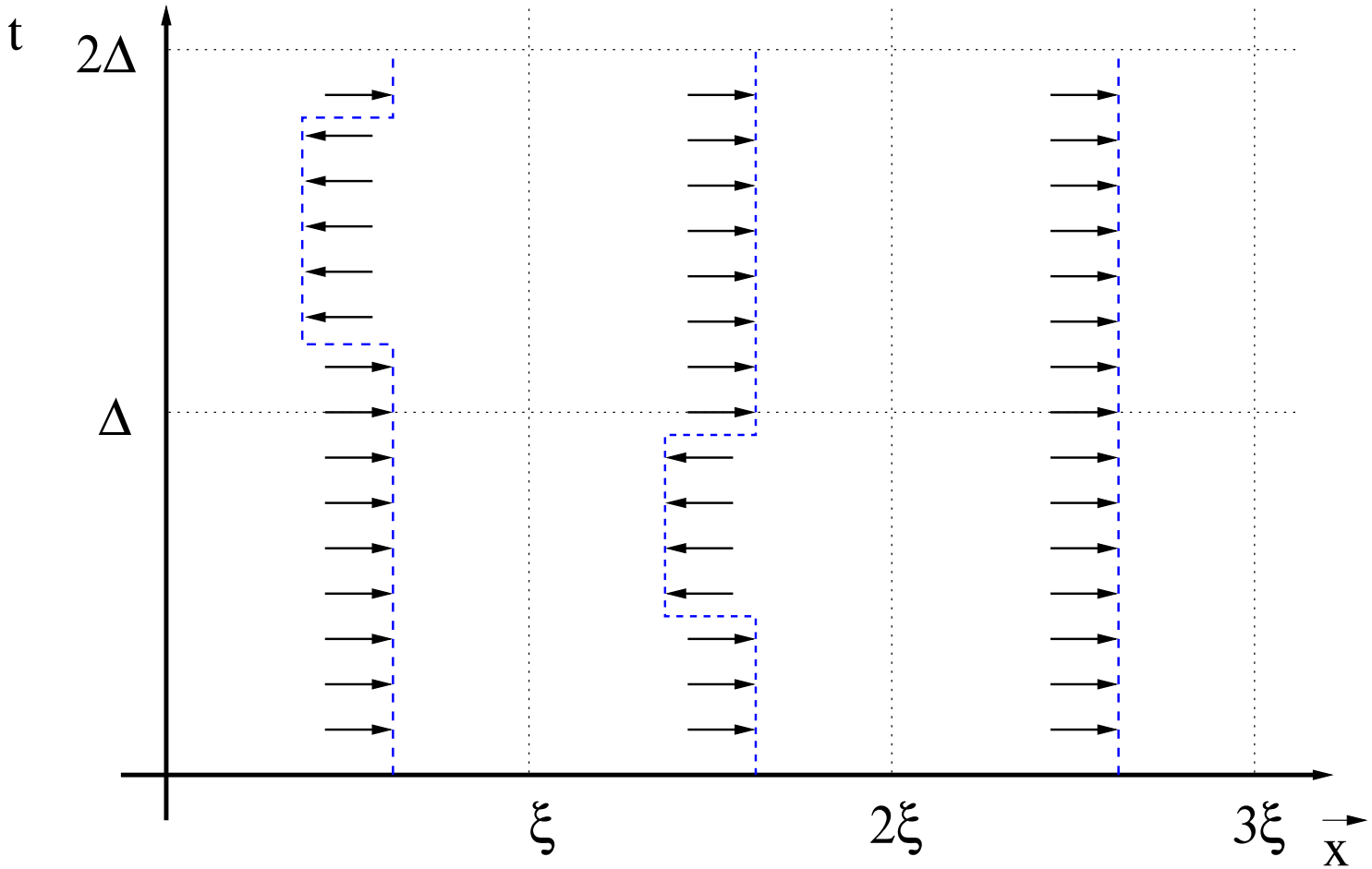

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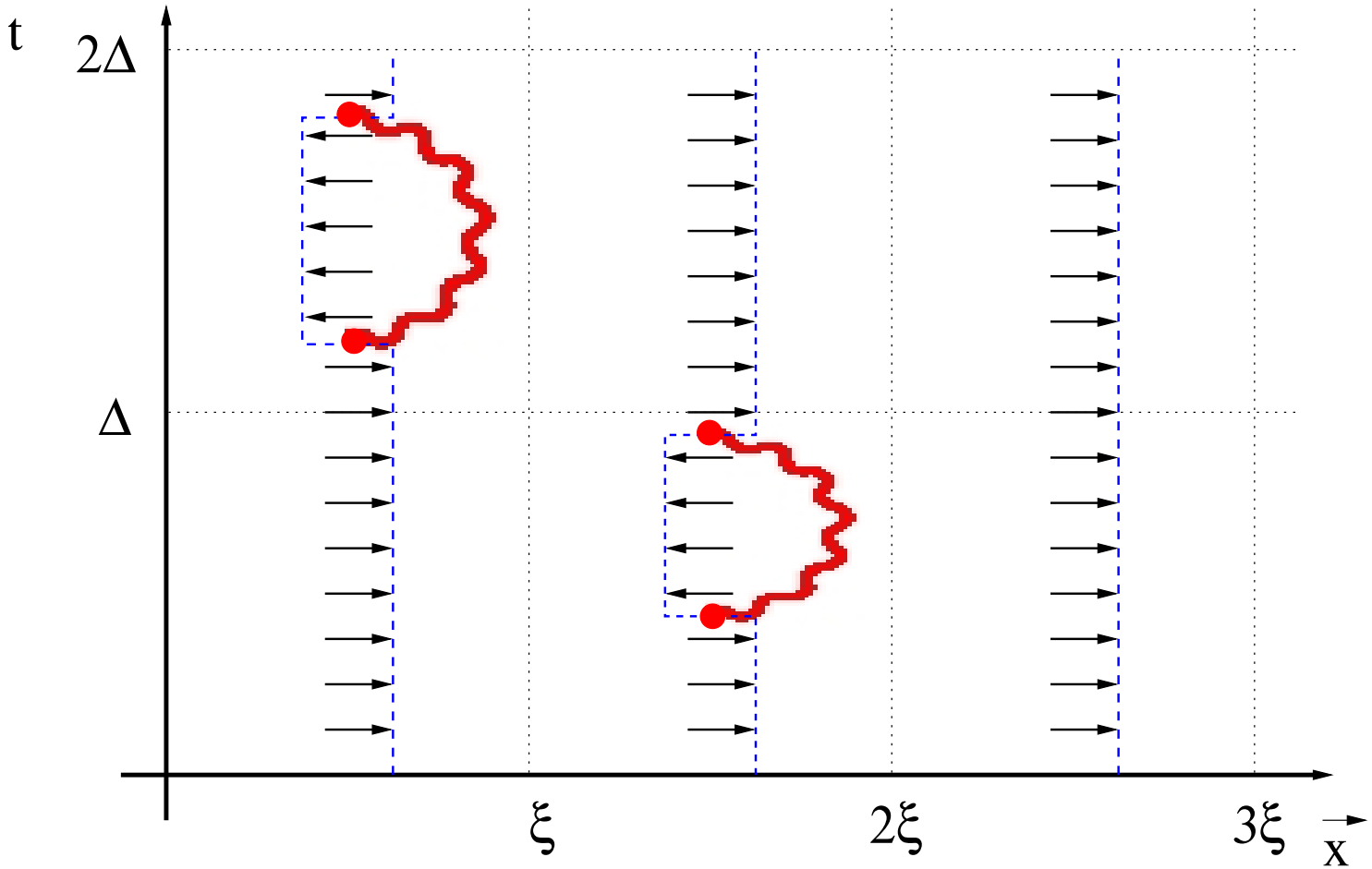

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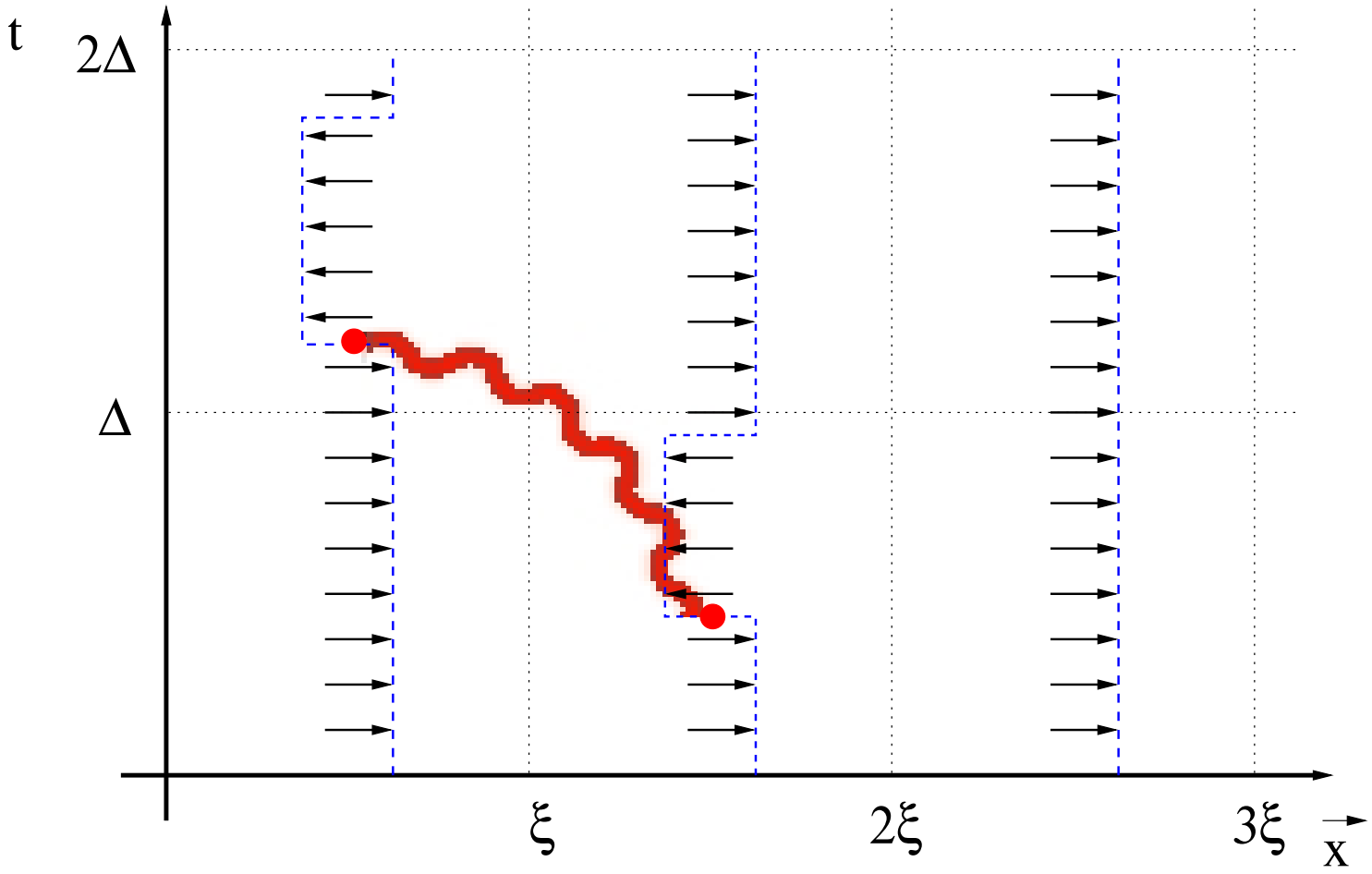


Qubit evolution in a QEC cycle

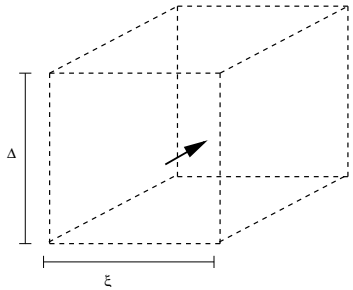
Lowest order perturbation theory for an error of type α.

$$\hat{\upsilon}_\alpha(\mathbf{x}_1, \lambda_\alpha) \approx -i\lambda_\alpha \int_0^\Delta dt f_\alpha(\mathbf{x_1}, t),$$

Note: QEC removed the qubit operator!!! I only need to use the field's commutation relations.

Perturbation theory improved by RG or any other method...

$$\begin{aligned}
\hat{\upsilon}_\alpha(\mathbf{x}_1, \lambda_\alpha) \quad &\approx \quad -i\lambda_\alpha \int_0^\Delta dt f_\alpha(\mathbf{x_1}, t) \\
&- \quad \frac{1}{2}\left|\varepsilon_{\alpha\beta\gamma}\right| \lambda_\beta \lambda_\gamma \sigma_\alpha(\Delta) T_t \int_0^\Delta dt_1\, dt_2 f_\beta(\mathbf{x_1}, t_1) f_\gamma(\mathbf{x_1}, t_2) \sigma_\beta(t_1) \sigma_\gamma(t_2) \\
&+ \quad \frac{i}{6} \sum_\beta \lambda_\alpha \lambda_\beta^2 \sigma_\alpha(\Delta) \\
&\times \quad T_t \int_0^\Delta dt_1\, dt_2\, dt_3 f_\alpha(\mathbf{x_1}, t_1) f_\beta(\mathbf{x_1}, t_2) f_\beta(\mathbf{x_1}, t_3) \sigma_\alpha(t_1) \sigma_\beta(t_2) \sigma_\beta(t_3),
\end{aligned}$$



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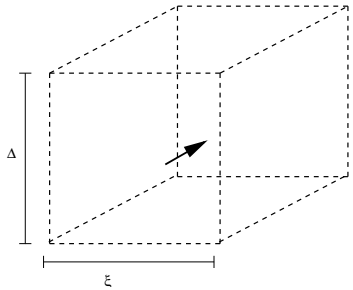
$$\frac{d\lambda_\alpha}{d\ell} = g_{\beta\gamma}(\ell)\lambda_\beta\lambda_\gamma + \sum_\beta h_{\alpha\beta}(\ell)\lambda_\alpha\lambda_\beta^2,$$

integrate from the ultraviolet cut-off to Δ^{-1}, defines λ^*.



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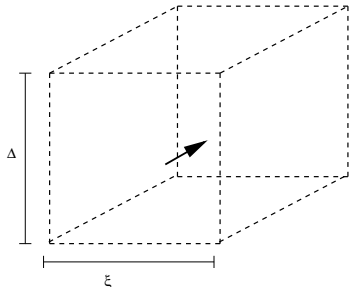

Correlations can:

- increase λ^* - Kondo problem for instance,
- decrease λ^* - quantum frustrated system.

Separating intra- and inter-hypercube components

Calculating a probability

$$P\left(\ldots;\alpha,\mathbf{x}_1;\ldots\right)\approx\left\langle\ldots\hat{\upsilon}_\alpha^\dagger\left(\mathbf{x}_1,\lambda_\alpha^*\right)\ldots\hat{\upsilon}_\alpha\left(\mathbf{x}_1,\lambda_\alpha^*\right)\ldots\right\rangle.$$

$$\upsilon_\alpha^2\left(\mathbf{x}_1,\lambda_\alpha^*\right)\approx\varepsilon_\alpha+\left(\lambda_\alpha^*\Delta\right)^2:\left|f_\alpha\left(\mathbf{x_1},0\right)\right|^2:,$$

$$\varepsilon_\alpha=\left(\lambda_\alpha^*\right)^2\int_0^\Delta dt_1\int_0^\Delta dt_2\left\langle f_\alpha^\dagger\left(\mathbf{x_1},t_1\right)f_\alpha\left(\mathbf{x_1},t_2\right)\right\rangle$$

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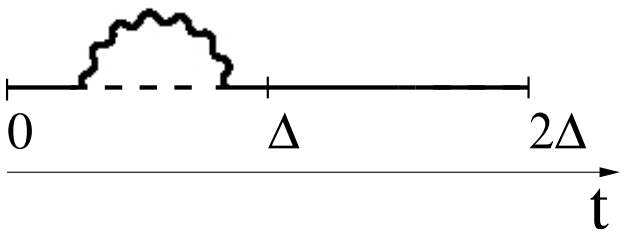

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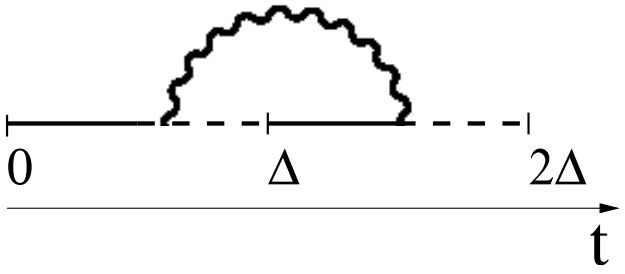


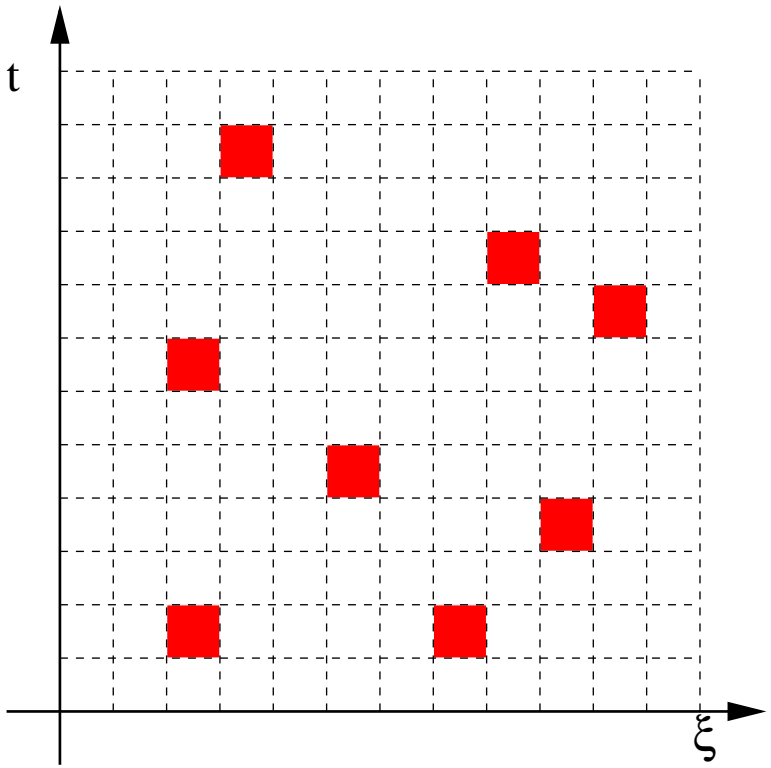
Probability of having m errors after N steps in R qubits

$$P_m^\alpha = p_m \int \frac{d\mathbf{x}_1}{(v\Delta)^{D/z}} \cdots \frac{d\mathbf{x}_m}{(v\Delta)^{D/z}} \int_0^{N\Delta} \frac{dt_1}{\Delta} \cdots \int_0^{t_{m-1}} \frac{dt_m}{\Delta}$$
$$\times \left\langle \left[\prod_\zeta F_0(\mathbf{x}_\zeta, t_\zeta) \right] \left[1 + F_\alpha(\mathbf{x_1}, t_1) \right] \ldots \left[1 + F_\alpha(\mathbf{x_m}, t_m) \right] \right\rangle$$

$$F_0(\mathbf{x_1}, 0) = 1 - \frac{\sum_\beta \left(\lambda_\beta^* \Delta \right)^2 : |f_\beta(\mathbf{x_1}, 0)|^2 :}{1 - \sum_{\beta = x,y,z} \varepsilon_\beta},$$

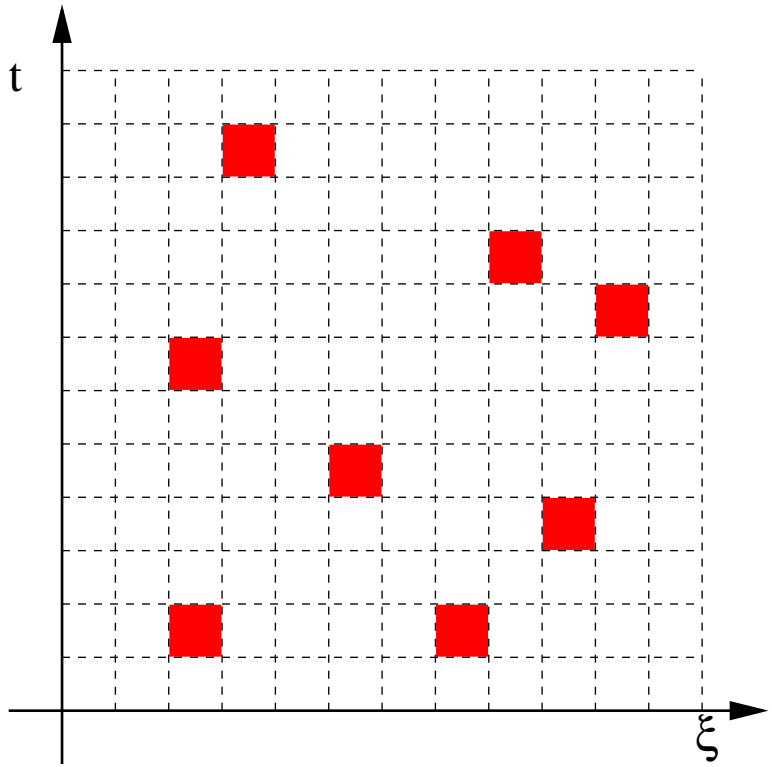
$$F_\alpha(\mathbf{x_1}, 0) = \frac{(\lambda_\alpha^* \Delta)^2}{\varepsilon_\alpha} : |f_\alpha(\mathbf{x_1}, 0)|^2 : .$$

"New" perturbation theory
in the coarse grain grid.



Probability of having m errors after N steps in R qubits

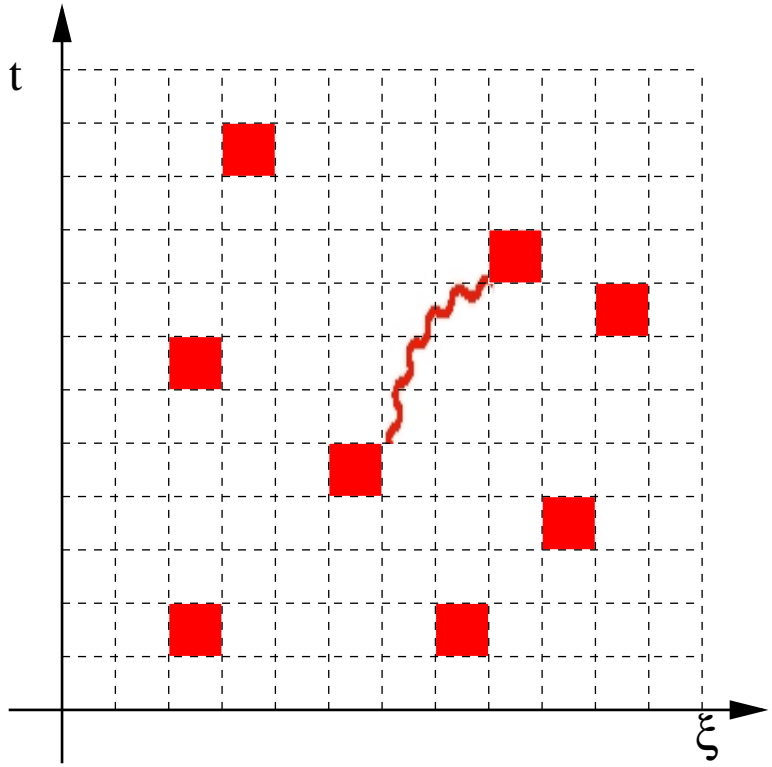
- zeroth order terms is just the stochastic probability:



$$p_m \int \prod_{k=1}^{m} \frac{d\mathbf{x}_k}{(v\Delta)^{D/z}} \frac{dt_k}{\Delta} = p_m \binom{NR}{m}.$$

Probability of having m errors after N steps in R qubits

- second order term is the first correction due to long range correlations



$$p_m \int \prod_{k=1}^{m} \frac{d\mathbf{x}_k}{(v\Delta)^{D/z}} \frac{dt_k}{\Delta} \left\langle F_\alpha\left(\mathbf{x_i}, t_i\right) F_\alpha\left(\mathbf{x_j}, t_j\right) \right\rangle .$$

Probability of having m errors after N steps in R qubits

correlation function

$$\left\langle F_\alpha\left(\mathbf{x_i},t_i\right)F_\alpha\left(\mathbf{x_j},t_j\right)\right\rangle \sim \mathcal{F}\left(\left|\mathbf{x}_i-\mathbf{x}_j\right|^{-4\delta_\alpha},\left|t_i-t_j\right|^{-4\delta_\alpha/z}\right)$$

where δ_α is the scaling dimension of f_α

using scaling again

$$p_m\int\prod_{k=1}^{m}\frac{d\mathbf{x}_k}{(v\Delta)^{D/z}}\frac{dt_k}{\Delta}\left\langle F_\alpha\left(\mathbf{x_i},t_i\right)F_\alpha\left(\mathbf{x_j},t_j\right)\right\rangle \to \frac{d\lambda_\alpha^*}{d\ell}=\left(D+z-2\delta_\alpha\right)\lambda_\alpha^*.$$

defines the stability of the perturbation theory:

1. $D+z-2\delta_\alpha<0$ corrections are small,
2. $D+z-2\delta_\alpha>0$ new derivation needed.

Probability of having m errors after N steps in R qubits

correlation function

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Threshold theorem as a quantum phase transition

low temperature - disentangled phase

$$|\psi_{total}\rangle = |\psi_{computer}\rangle \otimes |\psi_{environment}\rangle$$

1. qubits and environment are weakly entangled,
2. the qubits are strongly entangle among themselves,
3. low decoherence.

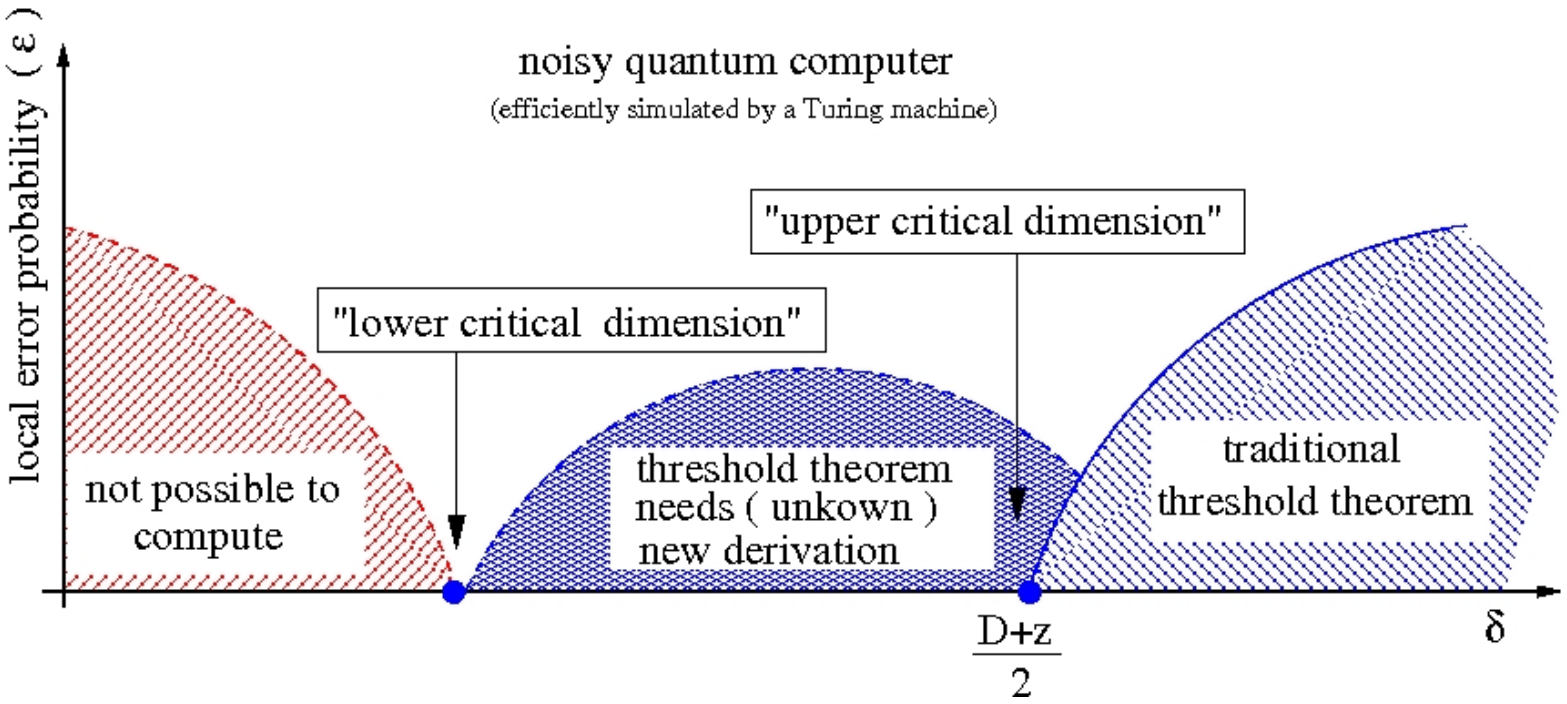
high temperature - entangled phase

1. qubits and environment are strongly entangled,
2. the qubits are weakly entangle among themselves,
3. strong decoherence.

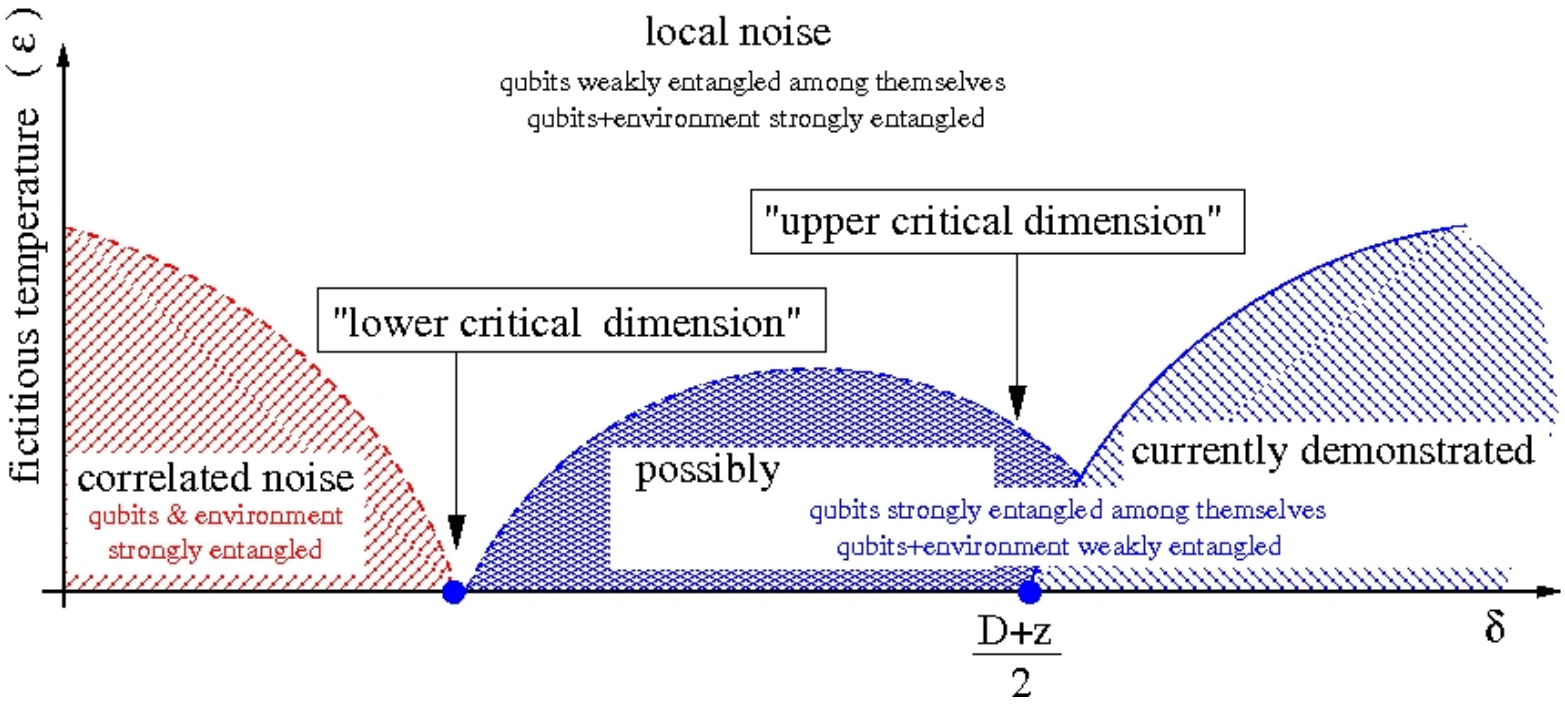
low temperature - entangled phase

1. computer and qubits are strongly entangle,
2. strong decoherence.

Schematic phase diagram



Schematic phase diagram



Conclusions

We...

1. ... studied decoherence in quantum computers in a correlated environment,
2. ... identify "software" methods to reduce the effects of correlations,
3. ... derive when is possible to use the "threshold theorem",
4. ... produce a parallel with the theorem of quantum phase transitions.

references

1. Phys. Rev. Lett. 97, 040501 (2006).
2. Phys. Rev. Lett. 98, 040501 (2007).
3. arxiv.org: quant-ph/0710.1624.

the bosonic field - definitions

Let us consider a bosonic environment

$$H_0 = \sum_{k>0} k a_k^\dagger a_k + k b_k^\dagger b_k$$

where a and b are bosonic operators, $\left[a_k, a_q^\dagger\right] = \delta_{k,q}$ and $\left[b_k, b_q^\dagger\right] = \delta_{k,q}$.
With them we can define the bosonic chiral fields

$$\phi_R(x) = \frac{1}{\sqrt{2L}} \sum_{k>0} \frac{e^{ikx}}{\sqrt{k}} a_k^\dagger + \frac{e^{-ikx}}{\sqrt{k}} a_k$$

$$\phi_L(x) = \frac{1}{\sqrt{2L}} \sum_{k>0} \frac{e^{-ikx}}{\sqrt{k}} b_k^\dagger + \frac{e^{ikx}}{\sqrt{k}} b_k$$

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the bosonic field - definitions

From the chiral fields we define the bosonic fields

$$\begin{aligned}\phi(x) &= \phi_R(x) + \phi_L(x) \\ \theta(x) &= \phi_R(x) - \phi_L(x)\end{aligned}$$

$$H_0 = \int_{-\infty}^{\infty} dx \left[\partial_x \phi_L(x)\right]^2 + \left[\partial_x \phi_R(x)\right]^2 = \frac{1}{2}\int_{-\infty}^{\infty} dx \left[\partial_x \phi(x)\right]^2 + \left[\partial_x \theta(x)\right]^2$$

We can define the fields in the interaction picture

$$\phi(x,t) = \frac{1}{\sqrt{2L}} \sum_{k>0} \frac{e^{ikx+ikt}}{\sqrt{k}} a_k^\dagger + \frac{e^{-ikx-ikt}}{\sqrt{k}} a_k + \frac{e^{-ikx+ikt}}{\sqrt{k}} b_k^\dagger + \frac{e^{ikx-ikt}}{\sqrt{k}} b_k$$

$$\theta(x,t) = \frac{1}{\sqrt{2L}} \sum_{k>0} \frac{e^{ikx+ikt}}{\sqrt{k}} a_k^\dagger + \frac{e^{-ikx-ikt}}{\sqrt{k}} a_k - \frac{e^{-ikx+ikt}}{\sqrt{k}} b_k^\dagger - \frac{e^{ikx-ikt}}{\sqrt{k}} b_k$$

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commutation relations

$$[\phi(x,t),\phi(y,\tau)] = \frac{-i}{4}\{\mathrm{sgn}(t-\tau+x-y)+\mathrm{sgn}(t-\tau+y-x)\}$$

where

$$\mathrm{sgn}(\varepsilon) = \begin{cases} +1, \varepsilon > 0 \\ 0, \varepsilon = 0 \\ -1, \varepsilon < 0 \end{cases}$$

$$[\phi(x,t),\theta(y,\tau)] = \frac{-i}{4}\{\mathrm{sgn}(t-\tau+x-y)-\mathrm{sgn}(t-\tau+y-x)\}$$

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the interaction

$$
\begin{aligned}
V &= \frac{i\lambda}{\sqrt{2L}} \sum_x \sum_{k>0} \sqrt{k} \left(e^{ikx} a_k^\dagger - e^{-ikx} a_k \right) \sigma_x^z - \sqrt{k} \left(e^{-ikx} b_k^\dagger - e^{ikx} b_k \right) \sigma_x^z \\
&= \lambda \sum_j \partial_x \phi \left(x_j \right) \sigma_j^z
\end{aligned}
$$

in the interaction picture

$$
V(t) = \lambda \sum_j \partial_t \theta \left(x_j, t \right) \sigma_j^z
$$

where the bosonic velocity was set to one (thus space and time have the same dimension).

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Evolution operator

$$
\begin{aligned}
U(t,0) &= T_t e^{i\lambda \sum_j \int dt' \partial_t \theta\left(x_j,t'\right)\sigma_j^z} \\
&= e^{i\varphi t} e^{i\lambda \sum_j \theta\left(x_j,t\right)\sigma_j^z} e^{-i\lambda \sum_j \theta\left(x_j,0\right)\sigma_j^z}
\end{aligned}
$$

where φ is a constant phase.

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correlation functions

$$\left\langle e^{i\alpha\theta(x,t)+i\beta\theta(0,0)} \right\rangle = \delta_{\alpha,-\beta} \exp\left[-\frac{\alpha^2}{4\pi} \left\{ \frac{1}{2} \ln\left[1+\Lambda^2 (x+t)^2\right] + \frac{1}{2} \ln\left[1+\Lambda^2 (x-t)^2\right] \right\} \right]$$

$$\langle \partial_t \theta(x,t)\, \partial_t \theta(0,0) \rangle = \frac{1}{4\pi} \frac{\Lambda^{-2}-(x+t)^2}{\left[\Lambda^{-2}+(x+t)^2\right]^2} + \frac{1}{4\pi} \frac{\Lambda^{-2}-(x-t)^2}{\left[\Lambda^{-2}+(x-t)^2\right]^2}$$

where Λ is the bosonic cut-off.

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