Dynamical Decoupling and Quantum Error Correction Codes

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GAPS and DAL paper in preparation
Dynamical Decoupling and Quantum Error Correction Codes (SXDD)

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Motivation

$q\text{Mac}$

$H_S$
Motivation

$H_B$

$qMac$

$H_S$
Motivation
Motivation

$H_B$

$H_{SB}$

QEC + FT

$qMac$

$H_S$
Motivation

Dynamical Decoupling

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Dynamical Decoupling

QEC + FT

qMac

$H_B$

$H_{SB}$

$H'_{SB}$

$H_{SB}$

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\[ \bar{\eta}_0 = \|H_{SB}\| \tau_{\text{min}} \]

\[ \{ \text{QEC code} \} \]
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\[ \text{Ng,Lidar,Preskill PRA 84, 012305(2011)} \]

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\textit{n-qubit Pauli basis as decoupling group} \quad \Rightarrow \quad \text{No ‘local bath assumption’}

\begin{itemize}
  \item Length of sequence exponential in \(2n\)
  \item Pulses look like errors to the code \Rightarrow limits possible integration with other schemes
\end{itemize}
Desiderata for DD +QEC:

I. No extra locality assumptions

II. Pulses in the code

III. Shorter sequences than full decoupling approach.

\[ \eta_{DD} < \eta_0 \]
The magic is in the decoupling group
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Too small → No arbitrary order decoupling
   → No general Hamiltonians

Too large → Overkill
   → Shorter sequences are better
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- **Mutually Orthogonal Operator (generator) Set** = \{Ω_i\}_{i=1,...,K}

\[(Ω_i)^2 = I\]

Ω_ι Ω_j = \((-1)^{f(i,j)}\) Ω_j Ω_ι; \quad f(i, j) = \{0, 1\}

Ω_ι Ω_j ≠ Ω_k
The magic is in the decoupling group

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  \[ (\Omega_i)^2 = I \]
  
  \[ \Omega_i \Omega_j = (-1)^{f(i,j)} \Omega_j \Omega_i; \quad f(i,j) = \{0,1\} \]
  
  \[ \Omega_i \Omega_j \neq \Omega_k \]

Concatenated Dynamical Decoupling (CDD)
[Khodjasteh and Lidar, Phys. Rev. Lett. 95, 180501 (2005)]

Nested Uhrig Dynamical Decoupling (NUDD)
What we propose...

- Stabilizer generators = \{S_i\}_{i=1,...,Q}

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- Logical operators (Pauli basis): $\{X_i^{(L)}, Z_i^{(L)}\}_{i=1,...,k}$

MOOS = $\{S_i\}_{i=1,...,Q} \cup \{X_i^{(L)}, Z_i^{(L)}\}_{i=1,...,k}$
What we propose...

- Stabilizer generators = \{S_i\}_{i=1,...,Q}
- Logical operators (Pauli basis) = \{X_i^{(L)}, Z_i^{(L)}\}_{i=1,...,k}

\[ \text{MOOS} = \{S_i\}_{i=1,...,Q} \cup \{X_i^{(L)}, Z_i^{(L)}\}_{i=1,...,k} \]

\[ U_{DD}(T) = e^{-i(H_{\phi,\text{eff}}O(T)+H_{SB,\text{eff}}O(T^{N+1}))} \]

\[ H_{\phi,\text{eff}} \propto \{S_i\}_{i=1,...,Q} \]

Contains no physical or logical errors! Only harmless terms!

Even if \(H_{SB}\) is a logical error!
What do we gain?

✓ No extra locality assumptions: The DD group is powerful enough.

CDD: → NO higher order Magnus term is UNDECOUPLABLE and HARMFUL
      → The next level of concatenation can deal with it

NUDD: (See proof in W.-J. Kuo, GAPS, G. Quiroz, D. Lidar in preparation)

\[1 - 0 \quad H_{SB}\]
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  NUDD: (See proof in W.-J. Kuo, GAPS, G. Quiroz, D. Lidar in preparation)

- DD Pulses are bitwise / transversal in the code

  Pulses do not look like errors to the code
  \( \rightarrow \) Allows interaction with other protection schemes.
What else do we gain?

- Shorter sequences than full decoupling approach:

For stabilizer/subsystem codes: MOOS: \( n-k-g \) stabilizers + \( 2k \) logical operators

\[
\begin{align*}
CDD(<\Omega_i>,N) & \rightarrow 2^{(n+k-g)N} < 2^{2nN} \\
NUDD(\{\Omega_i,N\}) & \rightarrow (N + 1)^{n+k-g} < (N + 1)^{2n}
\end{align*}
\]
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For stabilizer/subsystem codes: MOOS : $n-k-g$ stabilizers + $2k$ logical operators

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$$NUDD(\{\Omega_i, N\}) \rightarrow (N + 1)^{n+k-g} < (N + 1)^{2n}$$

e.g. $[[ n^2, 1, n ]]$ Bacon Shor code:

Stabilizer generators: $2(n-1)$
Logical generators: 2

<table>
<thead>
<tr>
<th></th>
<th>SXDD</th>
<th>Full decoupling</th>
</tr>
</thead>
<tbody>
<tr>
<td>D(MOOS)</td>
<td>$2n$</td>
<td>$2n^2$</td>
</tr>
<tr>
<td>NUDD</td>
<td>$(N + 1)^{2n}$</td>
<td>$(N + 1)^{2n^2}$</td>
</tr>
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</table>

$$3 - 0 \quad H_{SB}$$
\[ \eta_{DD} < \eta_0 \ ? \]

- Recall our (effective) noise rates:

\[
\eta_0 = \| H_{SB} \tau_{min} \|
\]
\[
\eta_{DD}(N) = \| H_{SB,eff} O(T^{N+1}) \|
\]

- Are an overestimation:
  - bounds obtained without using the QEC code structure. (work in progress)
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How to compute \(\|H_{SB,eff} O(T^{N+1})\|\)?

\[\rightarrow \text{NLP results (Eqs. 152-164)}\]

Recursive relations for \(\|H_{SB,eff}(q)\|\) and \(\|H_{\emptyset,eff}(q)\|\) at every degree of concatenation \(q\).

\[
\|H_{SB,eff}(q)\| \leq R^{q(q+3)/2} (\bar{c} \|H_{SB} + H_B\|\tau_0)^{q-1}\|H_{SB}\|
\]

\[T(q) = R^q \tau_{min}\]

where \(R = 2^{D(MOOS)}\) and \(\bar{c} \sim 1\)
\[ \overline{\eta_{DD}} < \overline{\eta_0} \]

\[ \log_{10}[J_{SB}\tau_{\text{min}}] \]

\[ \log_{10}[\frac{\eta_{DD}}{\eta_0}] \]

\[ \log_{10}[J_0\tau_{\text{min}}] \]

\[ \|I \otimes H_B\| = J_0 \]

\[ \|H_{SB}\| = J_{SB} \]

\[ [[9,1,3]] - \text{BS code: } \tau_{\text{min}} = 1; D(MOOS) = 4 + 2 \]
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\[ \log_{10}[\eta_{DD}/\eta_0] \]

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\[ [[9,1,3]] – \text{BS code: } \tau_{\text{min}} = 1 ; D(\text{MOOS}) = 4 + 2 \]
Beyond $H_S = 0$

- DD-based methods for fidelity enhanced gates can be directly ported:
  - Dynamically protected gates: works for both CDD and NUDD
    Append SXDD sequence to a gate.
    \[ [\text{NLP, PRA 84, 012305(2011)}] \]
  - (Concatenated) Dynamically corrected gates: based on CDD
    Eulerian cycle on the Caley graph of DD group
    \[ [\text{Khodjasteh and Viola, PRL 102, 080501 (2009)}] \]
    \[ [\text{Khodjasteh, Lidar, Viola, PRL 104, 090501 (2010)}] \]

$$5 - 1 \quad H_{SB}$$
Conclusions

• We have shown how to integrate dynamical decoupling and quantum error correction codes in a ‘natural’ way.
  ✓ No extra locality conditions
  ✓ Pulses in the code.
  ✓ Shorter sequences than full decoupling approach.
  ✓ Improve effective error rates AND deal where Hamiltonians QEC fail.

• The pulses of the DD are bitwise and therefore EXPERIMENTALLY/fault-tolerance friendly
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THANKS! QUESTIONS?