Fault-tolerant quantum computation via adiabatic holonomies

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Outline

• Holonomic quantum computation (HQC)
• HQC in subsystems
• Two approaches to fault-tolerant HQC
  1) no extra qubits, but Hamiltonians depend on the error-correcting code
  2) extra (noisy) qubits needed, but Hamiltonians independent of the code
• FTHQC with 2-qubit Hamiltonians
• Related schemes
• Conclusion and outlook
Geometric phases and HQC

Parallel transport of a vector on a curved surface (example):
Geometric phases and HQC

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Parallel transport of a vector on a curved surface (example):

\( \text{holonomy} \)

(equals the flux of the curvature field through the enclosed area)
Holonomic quantum computation

Adiabatic theorem (Kato, 1950): Consider a time-dependent Hamiltonian $H(t/T)$ changing along a curve $H(s), s \in [0, 1]$. Let $\epsilon(s)$ be an eigenvalue with constant degeneracy, whose eigenspace $\mathcal{H}_\epsilon(s)$ has a twice-differentiable projector $\Pi_\epsilon(s)$. 
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Specifically, if \( \{|i, s\}\} \) is an arbitrary (differentiable) basis of \( \mathcal{H}_\epsilon(s) \), the evolution of any initial state is given by [up to an error \( O(T^{-1}) \)]

\[
|\psi(t)\rangle = e^{-i \int_0^t \epsilon(\tau/T) d\tau} \Gamma(t/T) |\psi(0)\rangle ,
\]

where

\[
\Gamma(s) = \lim_{\delta s \to 0} \Pi_\epsilon(s) \Pi_\epsilon(s - \delta s) \ldots \Pi_\epsilon(\delta s) \Pi_\epsilon(0) = \sum_{ij} U_{ij}(s) |i, s\rangle \langle j, 0| .
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$$\uparrow$$

dynamical phase
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$U(s) = \mathcal{P} \exp \int_0^s ds' A(s')$, where $A_{ij}(s) = \langle i, s | \frac{d}{ds} | j, s \rangle$. 
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\mathbf{U}(s) = \mathcal{P} \exp \int_0^s ds' \mathbf{A}(s') , \text{ where } \mathbf{A}_{ij}(s) = \langle i, s \mid \frac{d}{ds} \mid j, s\rangle . \quad \text{(Wilczek-Zee, 1984)}
$$
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is a gauge-invariant quantity (holonomy of the path \( \gamma \) in the Grassmannian).
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The paths we can draw depend on the control parameters of the Hamiltonian.
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Under generic conditions $\Rightarrow$ the holonomies we can generate in the subspace $\mathcal{H}_\epsilon(0)$ suffice for universal quantum computation. (Zanardi and Rasetti, 1999)
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 Appeal: robustness due to adiabaticity and geometric nature of gates
Holonomic quantum computation

However,

- Any system interacts with its environment.

  - HQC in DFSs (Wu, Zanardi, Lidar, 2005), but no symmetry is exact.

- Robustness does not mean flawlessness (control errors are inevitable).

- Scalability of any computational method requires fault tolerance.

  Need for active error correction!

Prospects: combine the inherent resilience of all-geometric control with the software protection of QEC
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Blume-Kohout, Ng, Poulin, Viola, PRL 100, 030501 (2008).
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classical information

quantum information
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**Code space:** $\mathcal{H}^A \otimes \text{Span}\{|0\rangle^B\}$

classical information  quantum information
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\[\text{correctable error}\]  
$\rho^A \otimes \sigma^B$
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**logical information**  
**error syndromes**

**classical information**  
**quantum information**
**Theorem:** Consider a decomposition of the Hilbert space \( \mathcal{H} = \bigoplus_i \mathcal{H}_i^A \otimes \mathcal{H}_i^B \).

Choose a starting Hamiltonian

\[
H(0) = \sum_i I_i^A \otimes H_i^B ,
\]

where \( H_i^B \) and \( H_j^B \) have different eigenvalues for \( i \neq j \). (In the case of \( \mathcal{H} = \mathcal{H}^A \otimes \mathcal{H}^B \) require \( H^B \) to have at least two different eigenvalues.)

By varying this Hamiltonian adiabatically along suitable loops, we can generate

\[
U = \sum_i W_i^A \otimes V_i^B ,
\]

where \( \{ W_i^A \} \) is any desired set of purely geometric (holonomic) operations.

Oreshkov, PRL 103, 090502 (2009).
HQC in subsystems

**Consequence: HQC without initialization**

When \( \mathcal{H} = \mathcal{H}^A \otimes \mathcal{H}^B \), we can implement an arbitrary purely geometric transformation on \( \mathcal{H}^A \) without having to initialize the system in any subspace.

→ Given any system, we can apply to it an arbitrary geometric transformation by appending to it a *noisy* qubit (which absorbs all unwanted dynamical effects).

May be useful where initialization is difficult to perform.
Fault-tolerant quantum computation

For a (stabilizer) QEC code, \( \mathcal{H} = \mathcal{H}^A \otimes \mathcal{H}^B \).

Logical observables are generally highly nonlocal!

→ to perform computation in \( \mathcal{H}^A \) (and error correction in \( \mathcal{H}^B \)) using local interactions, the logical states will have to go outside of \( \mathcal{H}^A \) during the implementation, i.e., the logical subsystem has to be moved around:

\[
\mathcal{H} = \mathcal{H}^A(t) \otimes \mathcal{H}^B(t) = U(t)\mathcal{H}^A \otimes \mathcal{H}^B.
\]

Can it be done so that the logical information is not exposed to further danger?
Fault-tolerant computation

Shor, DiVincenzo, Knill, Laflamme, Zurek, Aharonov, Ben-Or, Kitaev, Gottesman…. (1996 - 1997)

**Definition (fault tolerance):** a QEC circuit is fault-tolerant if an error occurring *during* its implementation renders the result correctable.

**Threshold theorem:** If the probability for an error per elementary information carrier (e.g., qubit) per gate is below some value $p$, an arbitrarily long computation can be implemented reliably with a polylogarithmic computational overhead.
Fault-tolerant computation

Consider a stabilizer QEC code for the correction of single-qubit errors.

**Building blocks of a dynamical fault-tolerant scheme:**

- **Transversal unitary operations:**
  - single-qubit unitaries
  - transversal C-NOT

- **Preparation and use of a ‘cat’ state** \((|0...0\rangle + |1...1\rangle)/\sqrt{2}\):
  - preparation
  - verification (measurement of the parity of the state)
  - transversal C-NOT gates from logical states to the cat state

- **Single-qubit measurements** in the computational basis.
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- Single-qubit measurements in the computational basis.

These procedures prescribe how to move the logical subsystem $\mathcal{H}^A(t)$!
Fault-tolerant HQC

Approach 1: (no additional qubits)

Consider a $[[n,1,r,3]]$ stabilizer code, $\mathcal{G} = \langle i, S_1, \ldots, S_s, X^{g_1}, Z^{g_1}, \ldots, X^{g_r}, Z^{g_r} \rangle$.

$$\mathcal{H} = \mathcal{H}^A \otimes \mathcal{H}^B$$

logical qubits syndrome, gauge, ancilla qubits

The starting Hamiltonian must have the form $H(0) = I^A \otimes H^B$.

$\Rightarrow H(0)$ a linear combination of elements of $\mathcal{G}$.

During the evolution, $\mathcal{H} = \mathcal{H}^A(t) \otimes \mathcal{H}^B(t) = U(t)\mathcal{H}^A \otimes \mathcal{H}^B \Rightarrow$

$H(t) = I^{A(t)} \otimes H^{B(t)}$ [a linear combination of elements of $\mathcal{G}(t)$].

Operators in $\mathcal{G}$ couple qubits in the same block $\Rightarrow$ **transversality impossible!**

Oreshkov, Brun, Lidar, PRL 102, 070502 (2009) ; PRA 80, 022325 (2009)
Fault-tolerant HQC
Approach 1: (no additional qubits)

Transversal unitaries are not the only ones that prevent propagation of errors:

A transversal unitary followed by a gauge transformation is also fault-tolerant!

The main idea:

1) Adiabatically drag the logical subsystem along a sequence of paths segments, such that during each segment the unitary that we generate in the full Hilbert space is transversal up to a gauge transformation (see next slide).

2) Follow a sequence of transversal operations just like in a dynamical FT scheme. The result is the desired operation followed by a gauge transformation.

3) When we complete each operation, the logical system has been taken around a loop whose associated holonomy is the desired logical gate.

1) & 2) \(\rightarrow\) fault-tolerance; 3) \(\rightarrow\) the computation is purely geometric

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How we do it…

**Proposition:** Take a starting Hamiltonian \( H(0) = Z^1 \otimes \tilde{G} \), where \( \tilde{G} \) is a Pauli operator on the rest of the qubits. Change the Hamiltonian adiabatically as

\[
H(t) = H^1(t) \otimes \tilde{G} , \quad \text{Tr} H^1(t) = 0.
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\[\Rightarrow\] The resultant unitary is \( U(t) \approx U^1(t) \otimes \tilde{I} \) (up to a gauge transformation).

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**Example**

Single-qubit X gate: $Z \otimes \tilde{G} \rightarrow Y \otimes \tilde{G} \rightarrow -Z \otimes \tilde{G}$

… In a similar way we can generate all necessary elementary operations.

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Properties of the scheme:

- **The threshold** (error per qubit per gate) **is the same** as for a dynamical scheme.

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Properties of the scheme:

- **The threshold** (error per qubit per gate) **is the same** as for a dynamical scheme.

But the adiabatic approximation requires slow evolution: for error $10^{-4}$, holonomic gates may need to be 10 – 100 times slower than dynamical ones.  
→ environmental noise has to be weaker!

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  HQC would be advantageous if it leads to an increase in control precision that outweighs the increase of environment errors due to the slowdown.

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• Universal FTHQC with this method **requires at least 3-local Hamiltonians**. This is achievable with suitable codes (e.g., Bacon-Shor code).

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Can we reduce to 2-local Hamiltonians with perturbative techniques?

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Fault-tolerant HQC
Approach 2: (extra gauge qubits)

An alternative (and conceptually simpler) scheme:

HQC is performed on the entire system’s Hilbert space (by coupling each qubit or pair of qubits to an ‘*external*’ gauge qubit).

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Key features:

- Qubits in the same block do not interact by construction.
- Extra qubits → lower threshold (~1.5 times). [but no need to initialize them]
- The Hamiltonians are independent of the code.

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- Qubits in the same block do not interact by construction.
- Extra qubits $\rightarrow$ lower threshold ($\sim 1.5$ times). [but no need to initialize them]
- The Hamiltonians are independent of the code. [3-local Hamiltonian required]
Fault-tolerant HQC with 2-qubit Hamiltonians

Perturbative gadgets:


Example: \[ H(t) = f(t) I^1 \otimes Y^2 \otimes Z^3 + g(t) Z^1 \otimes Z^2 \otimes Z^3 \]
Fault-tolerant HQC with 2-qubit Hamiltonians

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\[ (|000\rangle + |111\rangle)/\sqrt{2} \]
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\[ H_{\text{gad}}(t) = \sum_{s=1}^{2} H_{s}^{\text{anc}} + \lambda \sum_{s=1}^{2} V_{s}(t) \]

(|000⟩ + |111⟩)/√2
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\[ H_{s}^{\text{anc}} = \sum_{1 \leq i < j \leq 3} \frac{1}{2}(I - Z_{s,i}Z_{s,j}) \]

\[ V_{1}(t) = \sqrt[3]{f(t)}I^1 \otimes X_{1,1} + \sqrt[3]{f(t)}Y^2 \otimes X_{1,2} + \sqrt[3]{f(t)}Z^3 \otimes X_{1,3} \]

\[ V_{2}(t) = \sqrt[3]{g(t)}Z^1 \otimes X_{2,1} + \sqrt[3]{g(t)}Z^2 \otimes X_{2,2} + \sqrt[3]{g(t)}Z^3 \otimes X_{2,3} \]
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\[ H_{\text{gad}}(t) = \sum_{s=1}^{2} H_{s}^{\text{anc}} + \lambda \sum_{s=1}^{2} V_{s}(t) \]

Effective Hamiltonian: \[ H_{\text{eff}} = \frac{3\lambda^3}{2} H(t) \otimes P_{\text{cat}} + O(\lambda^4) \]
Fault-tolerant HQC with 2-qubit Hamiltonians

Drawbacks of the gadgets:

• An error on any of the 9 qubits can result in an error on the 2 system qubits $\Rightarrow$ the threshold is $9/2$ times smaller.

• If for adiabatic precision $1 - \delta$ with the Hamiltonian $H(t)$ we need time $T$, with the effective Hamiltonian $\frac{3\lambda^3}{2} H(t)$ we need time $T' = \frac{2}{3\lambda^3} T$.

But $O(\lambda^4 T') = O(\lambda T) = O(\delta) \Rightarrow T' = O\left(\frac{T^4}{\delta^3}\right)$.

$\Rightarrow$ We need $O\left(\frac{T^3}{\delta^3}\right)$ times longer evolution! (very large slow-down)
Related schemes

• Adiabatic gate teleportation:  
  Bacon and Flammia,  
  PRL 103, 120504 (2009)
  - Each gate is implemented by dragging the logical system only along a single line segment! (could be viewed as open-path HQC)
  - Compatible with fault-tolerant techniques and perturbative gadgets

• Cluster state adiabatic computation:  
  Bacon and Flammia,  
  PRA 82, 030303(R) (2010)
  - One-way computing without measurements

• HQC in ground states of spin chains protected by topological order:  
  Renes, Miyake, Brennen,  
  Bartlett, arXiv: 1103.5076
  - Based on 2-local Hamiltonians without gadgets
Related schemes

- **HQC by dissipation** *(Adiabatic Markovian Dynamics)*

\[
\frac{d\rho}{dt} = -i[H(t), \rho] + \sum_i \left( L_i(t)\rho L_i(t)^\dagger - \frac{1}{2} L_i^\dagger(t) L_i(t)\rho - \frac{1}{2} \rho L_i^\dagger(t) L_i(t) \right)
\]

Oreshkov and Calsamiglia, PRL 105, 050503 (2010)

- A noiseless subsystem is dragged by a slowly varying Lindblad generator.

(Compatible with fault-tolerance Approach 2)

A new type of geometric phase: generalizes the Wilczek-Zee and Uhlmann holonomies.

Oreshkov and Sjoqvist, in preparation
Conclusion and outlook

• HQC can be done in subsystems (rather than subspaces).

• Adiabaic geometric control is compatible with the techniques for fault-tolerant computation on stabilizer codes.

  → HQC is in principle scalable.

  → the software protection of QEC could be aided by the robustness of HQC.

• FTHQC is possible with 2-qubit Hamiltonians (but the gadgets are inefficient).

• Is it possible to find a non-perturbative realization with 2-qubit interactions?

• Physical implementations?

• Can some of these ideas be useful for fault-tolerant adiabatic QC?