Bound on Quantum Computation time: QEC in a critical environment

QEC - 2011

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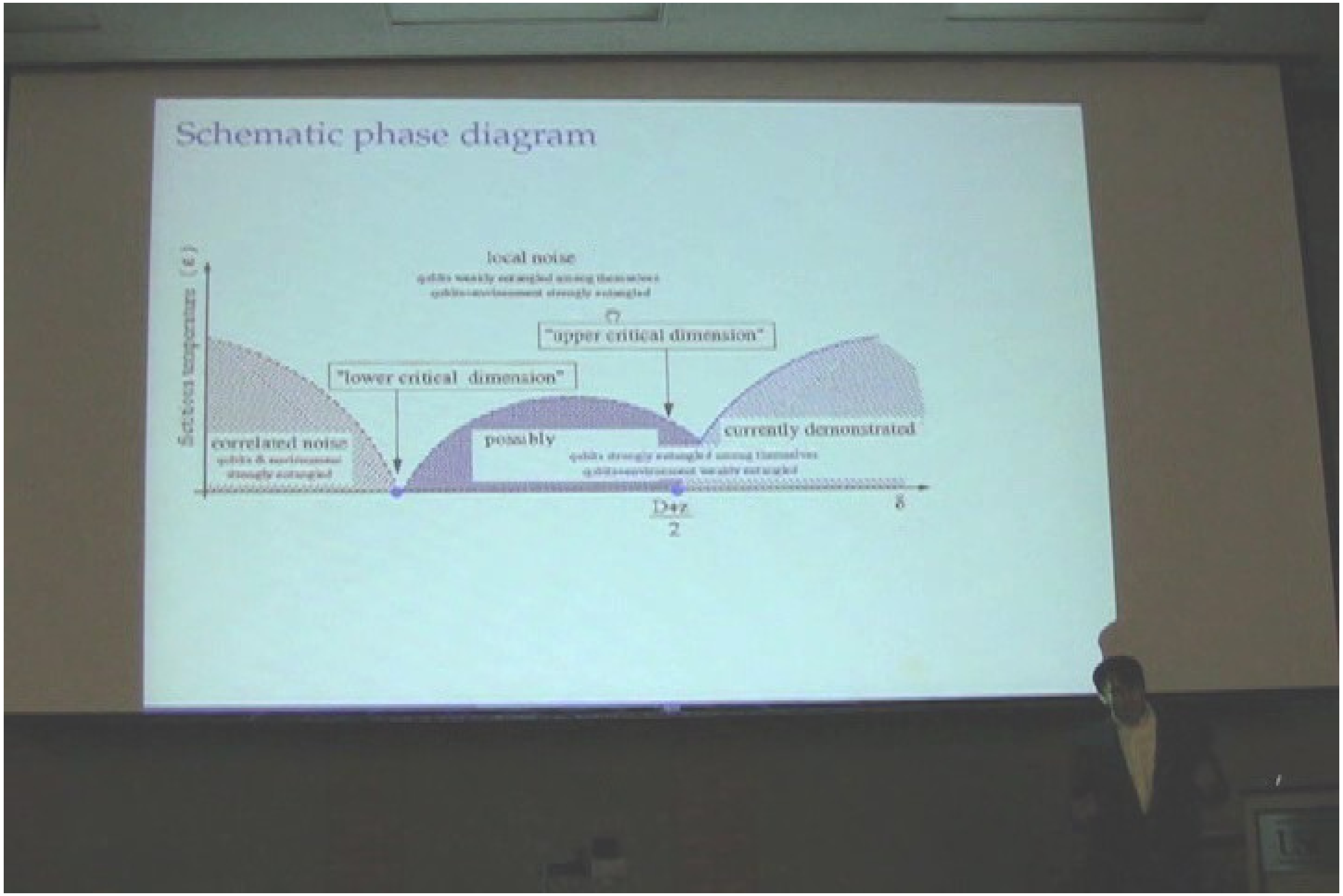
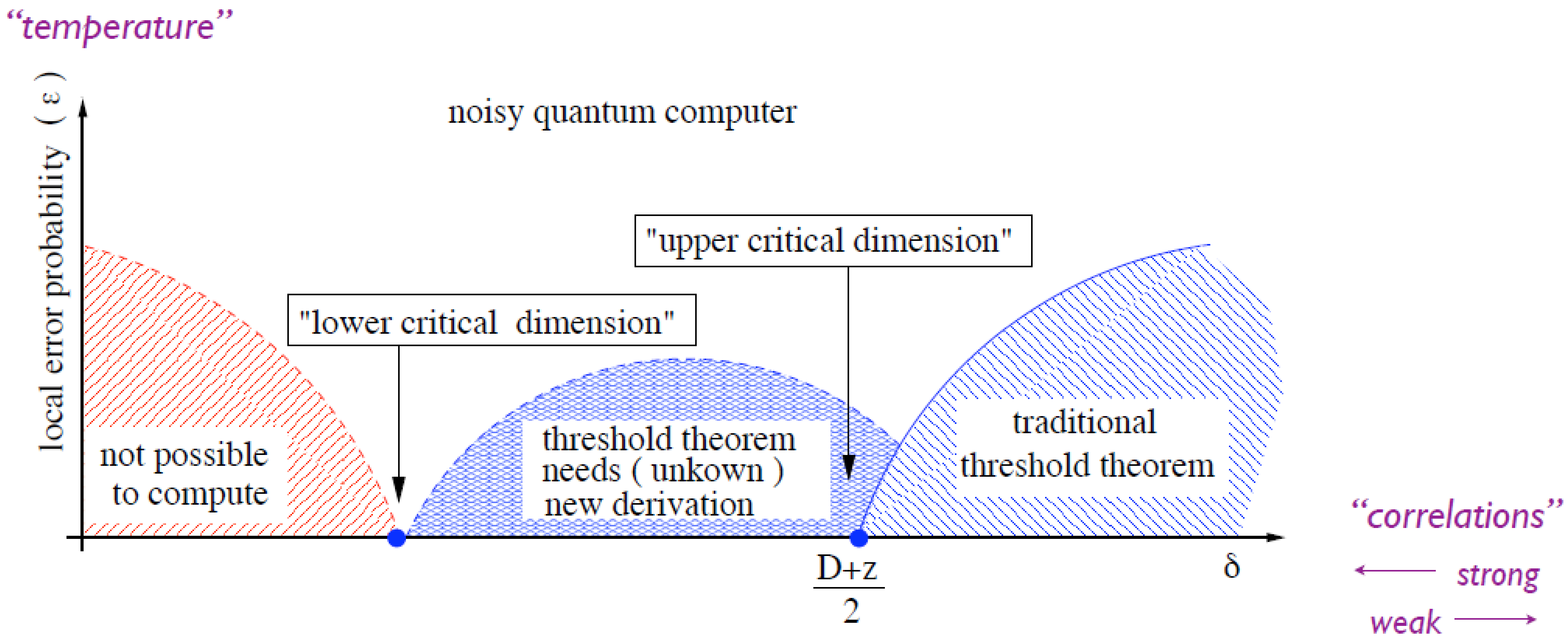
QEC 2007



Schematic phase diagram
local noise
"upper critical dimension"
"lower critical dimension"
correlated noise
possibly
currently demonstrated

Threshold theorem: a quantum phase transition perspective



Motivation: protect quantum information in large quantum computers and long times

- Some strategies:
  - Decoherence free-subspaces
  - Topological systems
  - Dynamical decoupling
  - Quantum error correction

Likely the most versatile and universal

QEC

- “threshold theorem”

  Provided the noise strength is below a critical value, quantum information can be protected for arbitrarily long times. Hence, the computation is said to be fault tolerant or resilient.

QEC

Usual assuptions in the traditional QEC theory:

1- fast measurements(not fundamental-Aliferes-DiVincenzo 07);

2- fast/slow gates (not fundamental – my opinion)

3- error models (add probabilities instead of amplitudes).

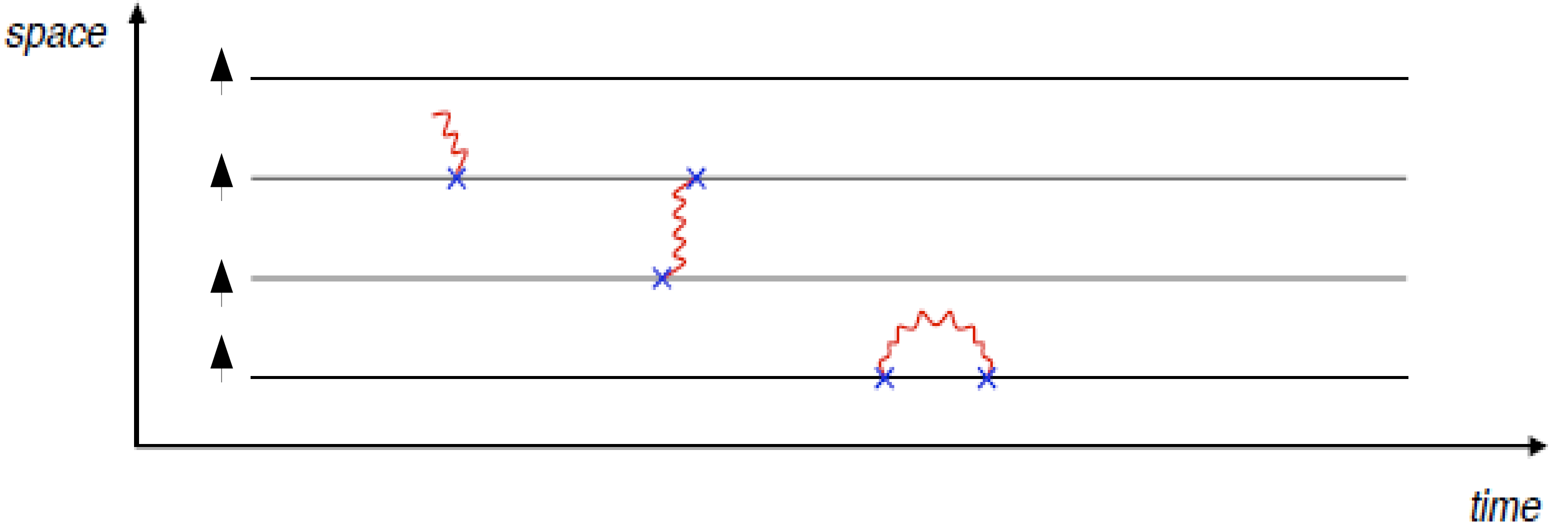
What happens if we start from a Hamiltonian?

R. Alicki, Daniel A. Lidar and Paolo Zanardi, PRA 73 052311 (2006).

I*nternal Consistency of Fault-Tolerant Quantum Error Correction in Light of Rigorous Derivations of the Quantum Markovian Limit.*

“... These assumptions are: fast gates, a constant supply of fresh cold ancillas, and a Markovian bath. We point out that these assumptions may not be mutually consistent in light of rigorous formulations of the Markovian approximation. ...”

What happens if we start from a Hamiltonian?

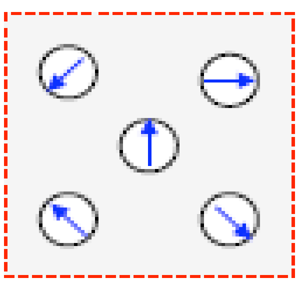


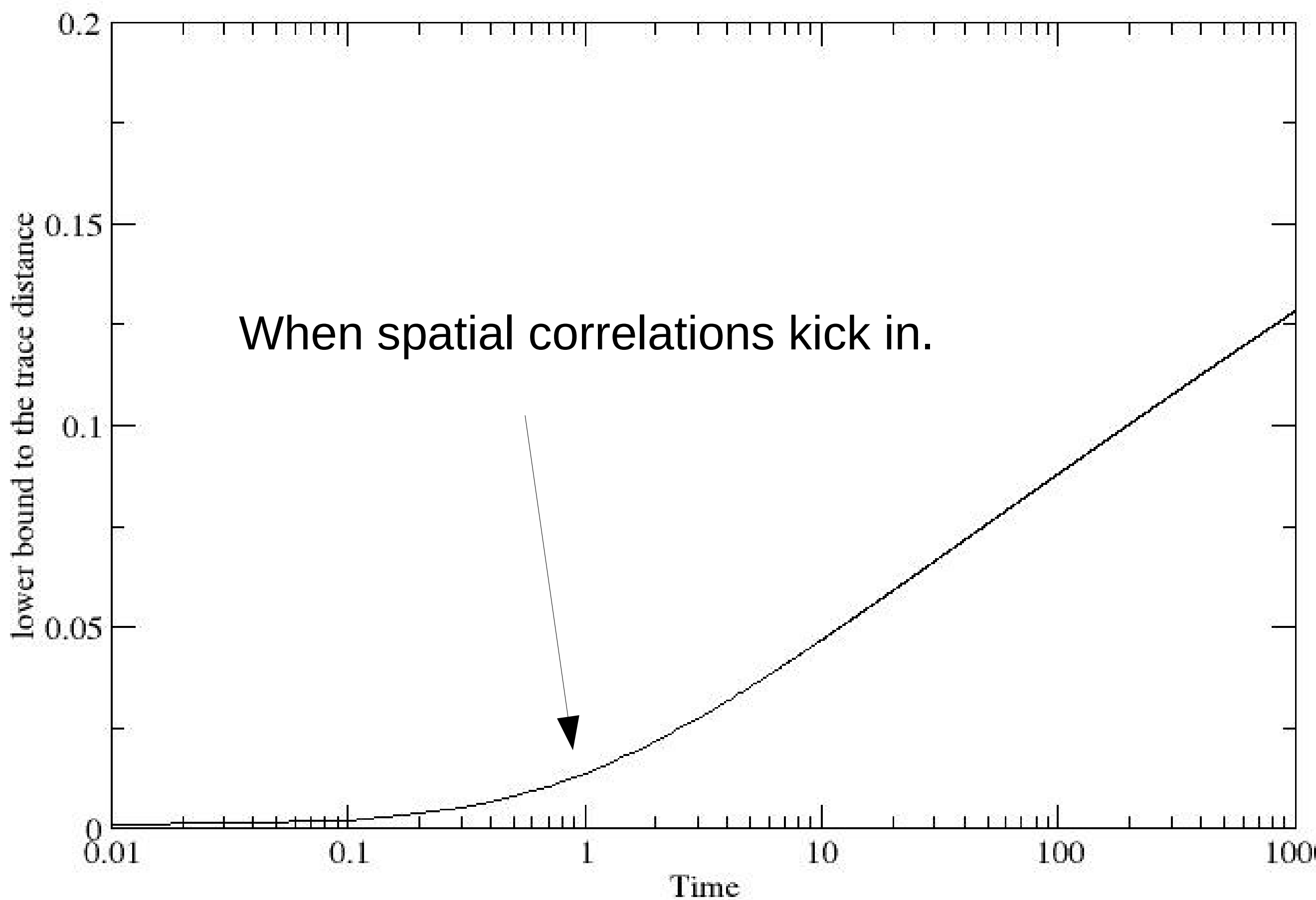
Real systems are likely to have correlations in space and time

Can correlations be so bad? (a back of the envelope calculation)

Consider a pure dephasing bath with ohmic spectrum acting on a logical qubit.

Calculate the trace distance between the logical qubit and the idle evolution.





We do not want to assume Born-Markov approximation.

Some references that also do not assume B-M:

Knill, Laflamme, Viola, PRL 84, 2525 (2000),

Terhal and Burkard, PRA 71, 012336 (2005),

Aliferis, Gottesman, Preskill, QIC 6, 97 (2006),

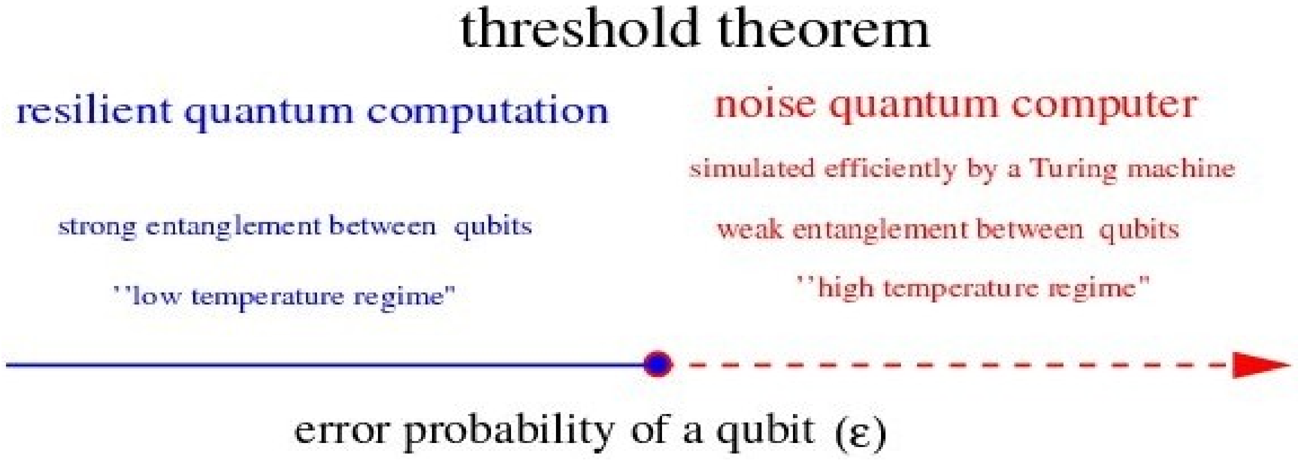
Aharonov, Kitaev, Preskill, PRL 96, 050504 (2006),

Hui Khoon Ng, Preskill, PRA 79, 032318 (2009),

Etc...

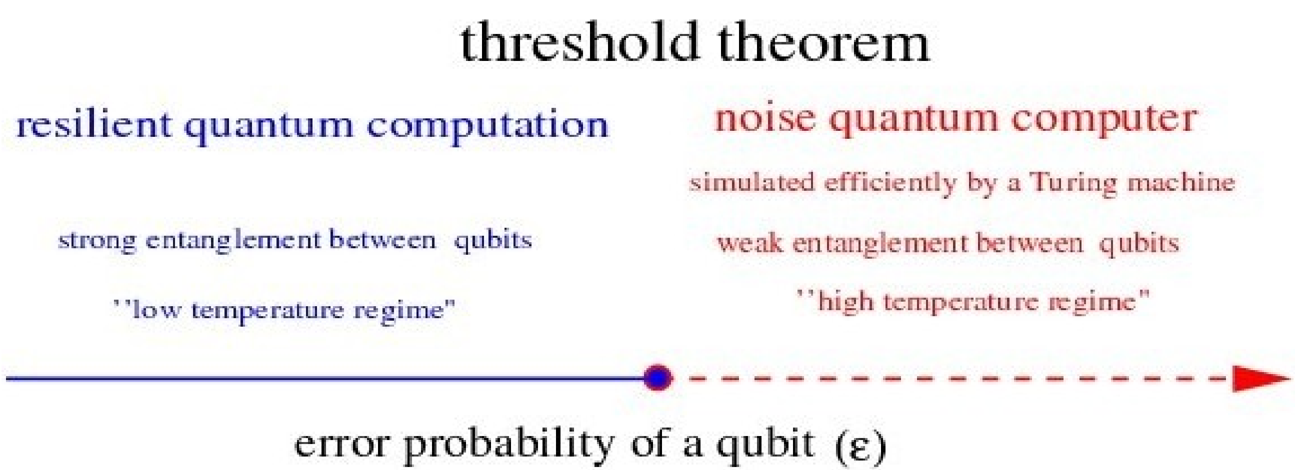
QEC: quantum-classical transition

- Dorit Aharonov, Phys. Rev. A 062311 (2000). Quantum to classical phase transition in a noisy QC.



QEC: quantum-classical transition

- Is there a quantum-quantum transition?



QEC: quantum-quantum transition

- It certainly makes sense.

  - A quantum phase transition is defined by a qualitative change in the ground state wave function of a quantum system as a function of a parameter in the Hamiltonian of the model.

  - In QEC there is no “Hamiltonian”, but we are forcing the system to be in a particular state. In this sense, we are defining a quantum phase and exploring its stability with respect to perturbations due to the environment.

The model (gaussian noise)

$$H = H_{\text{computer}} + H_{\text{bath}} + V \longleftarrow$$

qubit-bath interaction: $V = \sum_{\mathbf{x},\alpha} \lambda_\alpha \, f_\alpha(\mathbf{x}) \, \sigma_\alpha(\mathbf{x})$

qubits

environment's "free" Hamiltonian (gapless)

environmental degrees of freedom

$$\langle f_\alpha(\mathbf{x}_1, t_1) \, f_\alpha(\mathbf{x}_2, t_2) \rangle_{\text{env}} \sim O\left(\frac{1}{|\mathbf{x}_1 - \mathbf{x}_2|^{2\delta_\alpha}}, \frac{1}{|t_1 - t_2|^{2\delta_\alpha / z}} \right)$$

- spatial dimension D
- bath mode velocity c
- dynamical exponent z

Gaussian noise: n-correlations can be factorized using Wick's theorem.

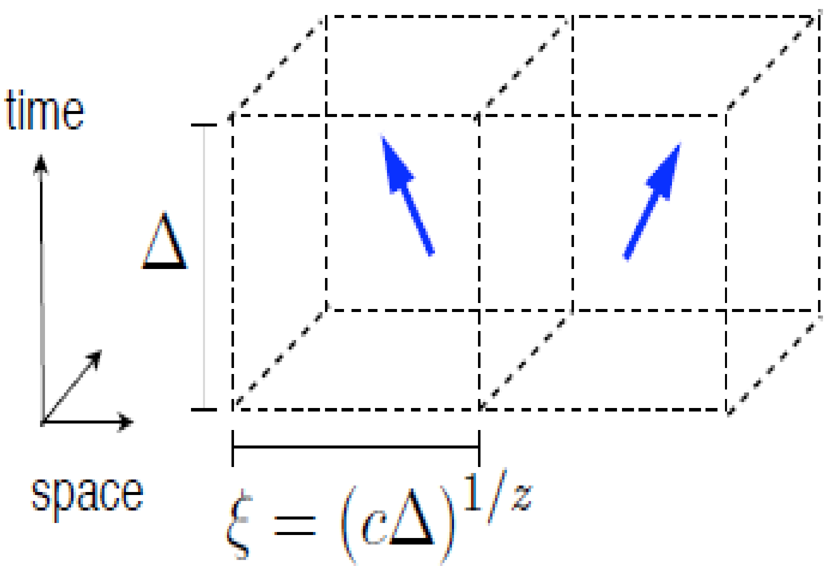
How general is the model?

- It encompass:
  - EM fluctuaions,
  - phonons,
  - charge fluctuations,
  - etc...
- It does not cover a spin-bath.

How general is the model?

- A qualitative argument:
  - After doing all possible hardware methods to reduce decoherence, the qubits will still see an effective environment.
  - By hypothesis this environment still has many more degrees of freedom than the computer.
  - The environment will be in a minimum of its energy landscape.
    - Assume an harmonic approximation for the environment
    - Assume linear response for the computer+environment interaction.

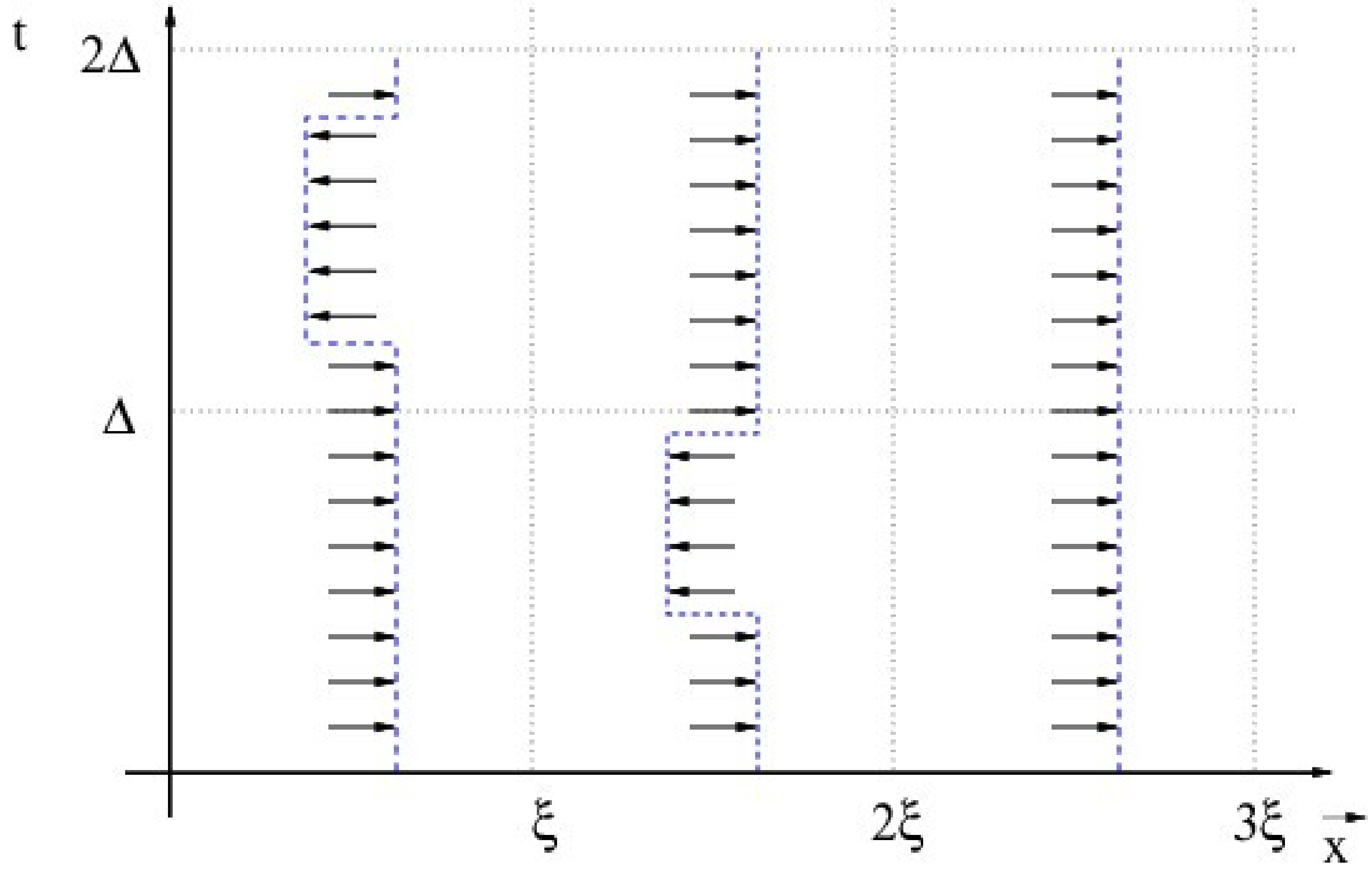
The basic assumption to help in organizing the calculation

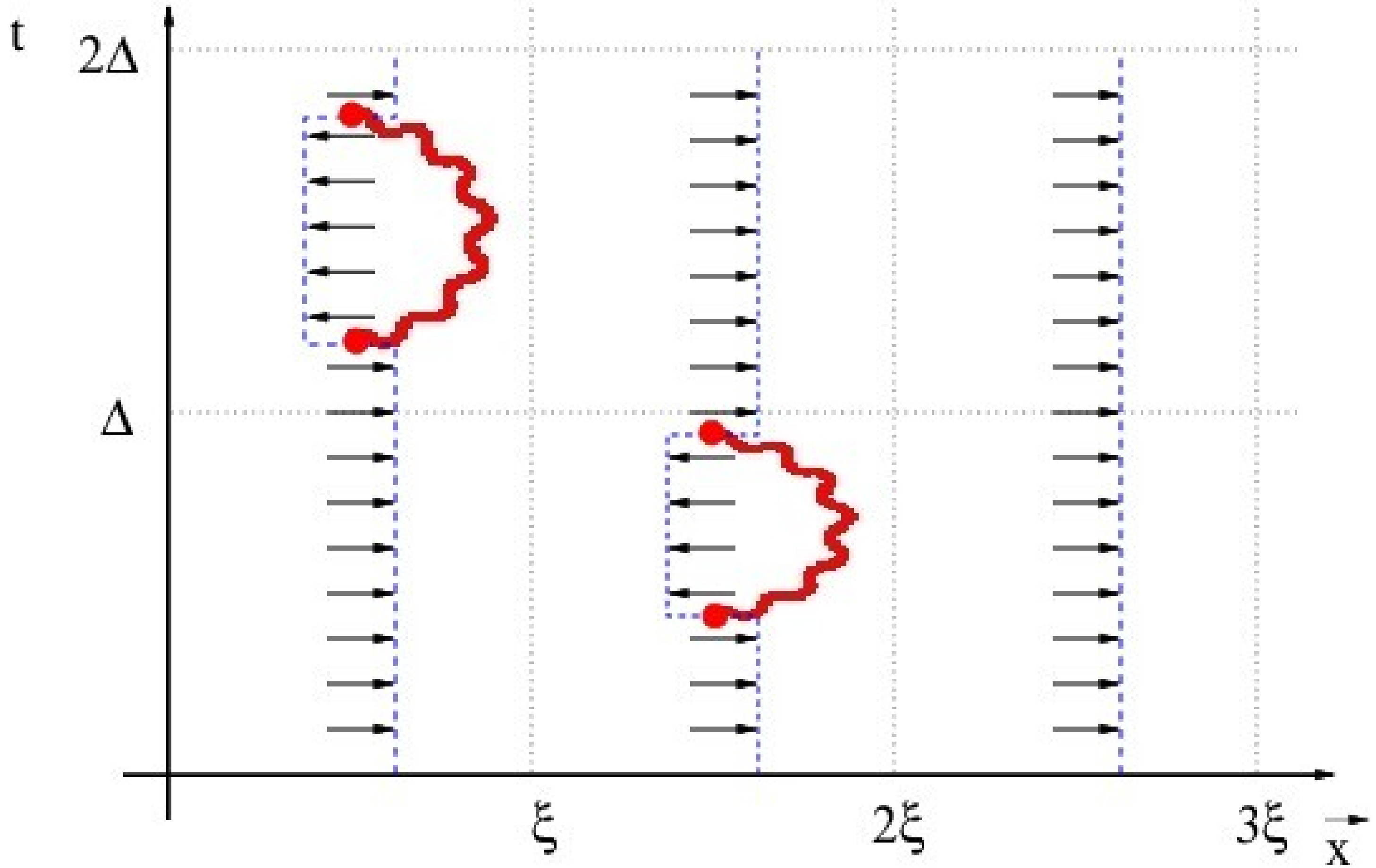


- defines a single-qubit error probability ϵ

- maps into an impurity problem for short times.

Δ = period of QEC cycle


t
2Δ
Δ
ξ
2ξ
3ξ
x



t
2Δ
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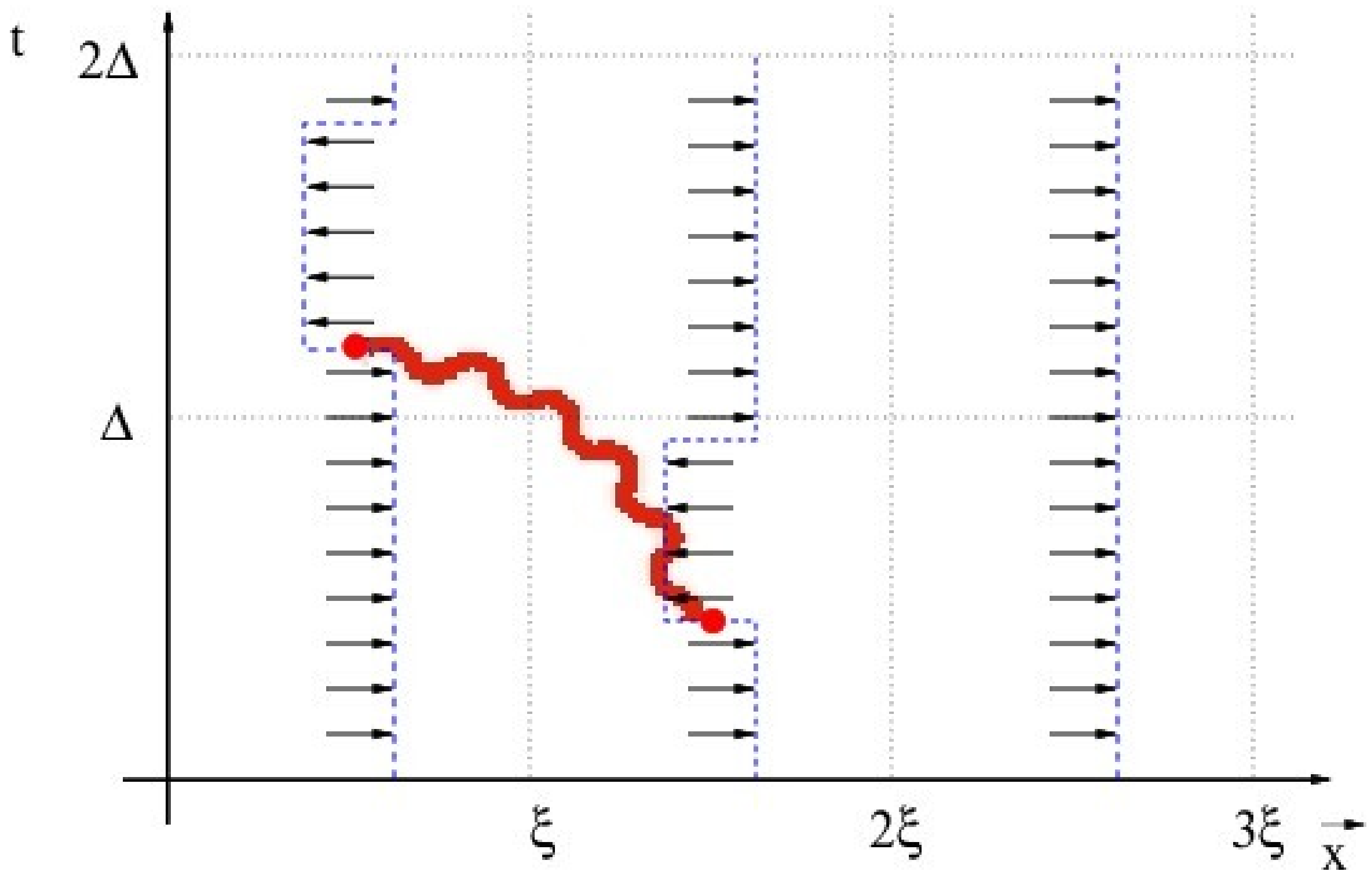
t
2Δ
Δ
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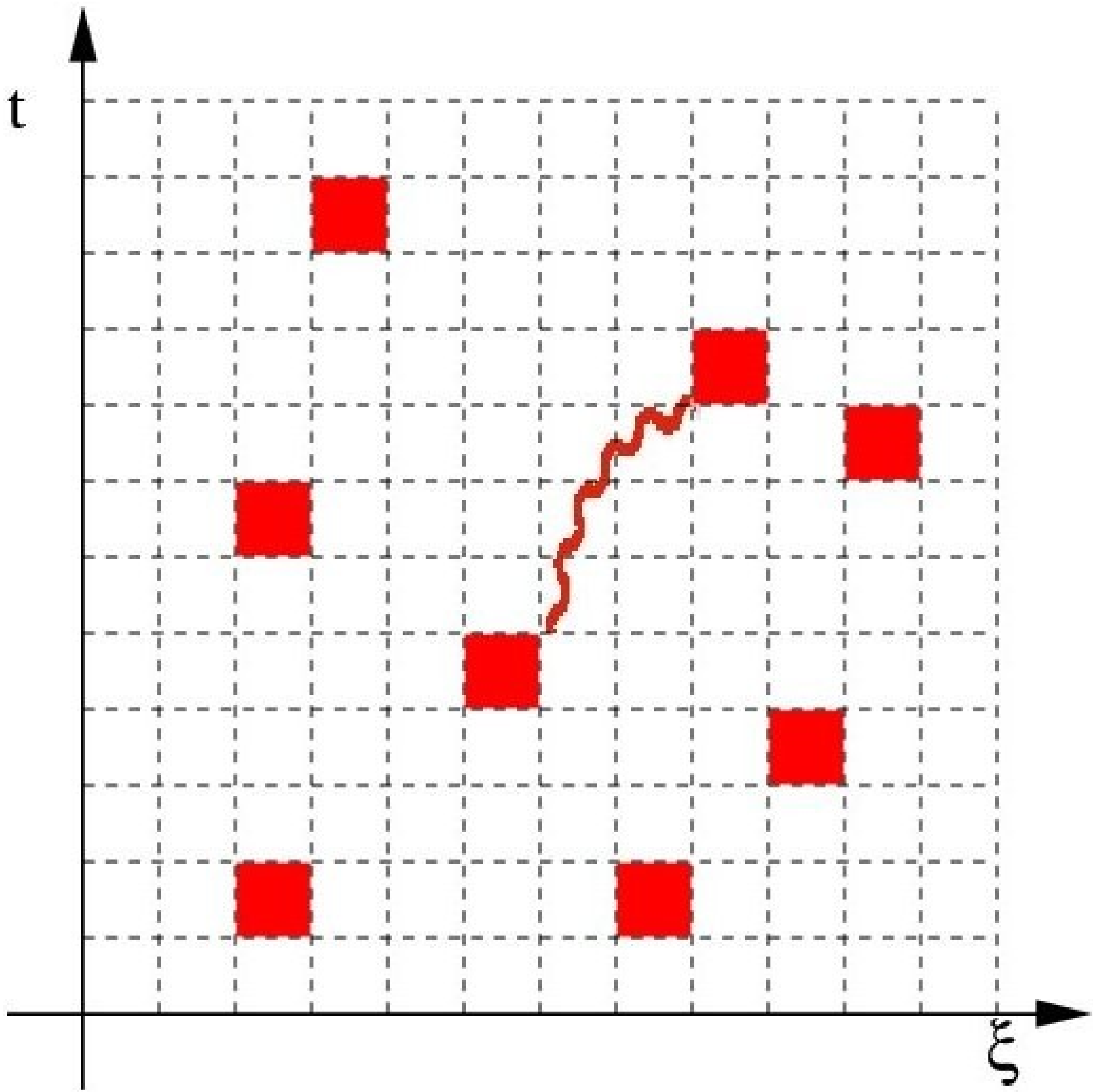
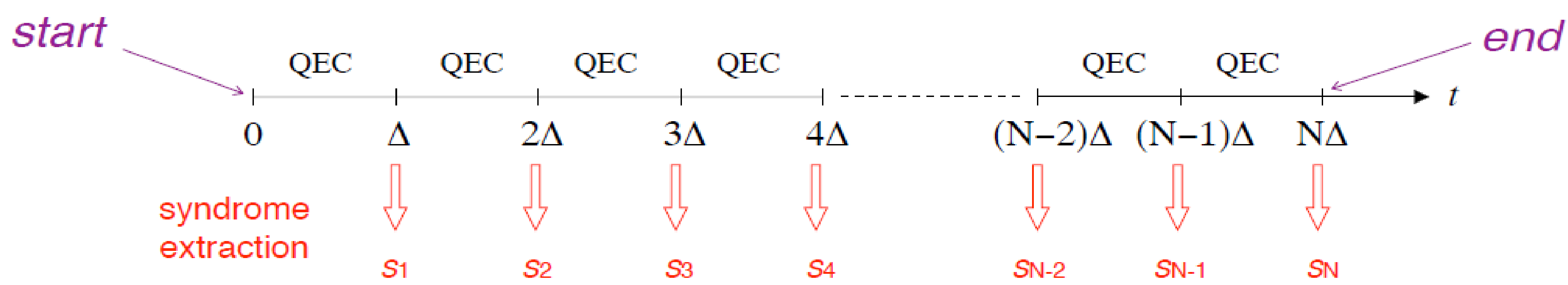
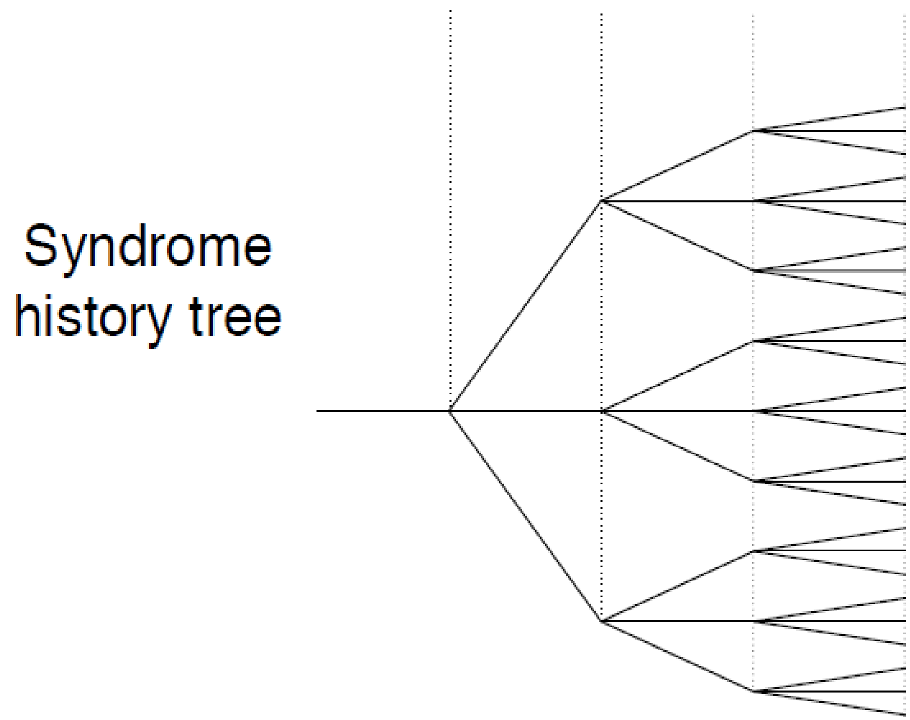
$\vec{x}$


t
ξ

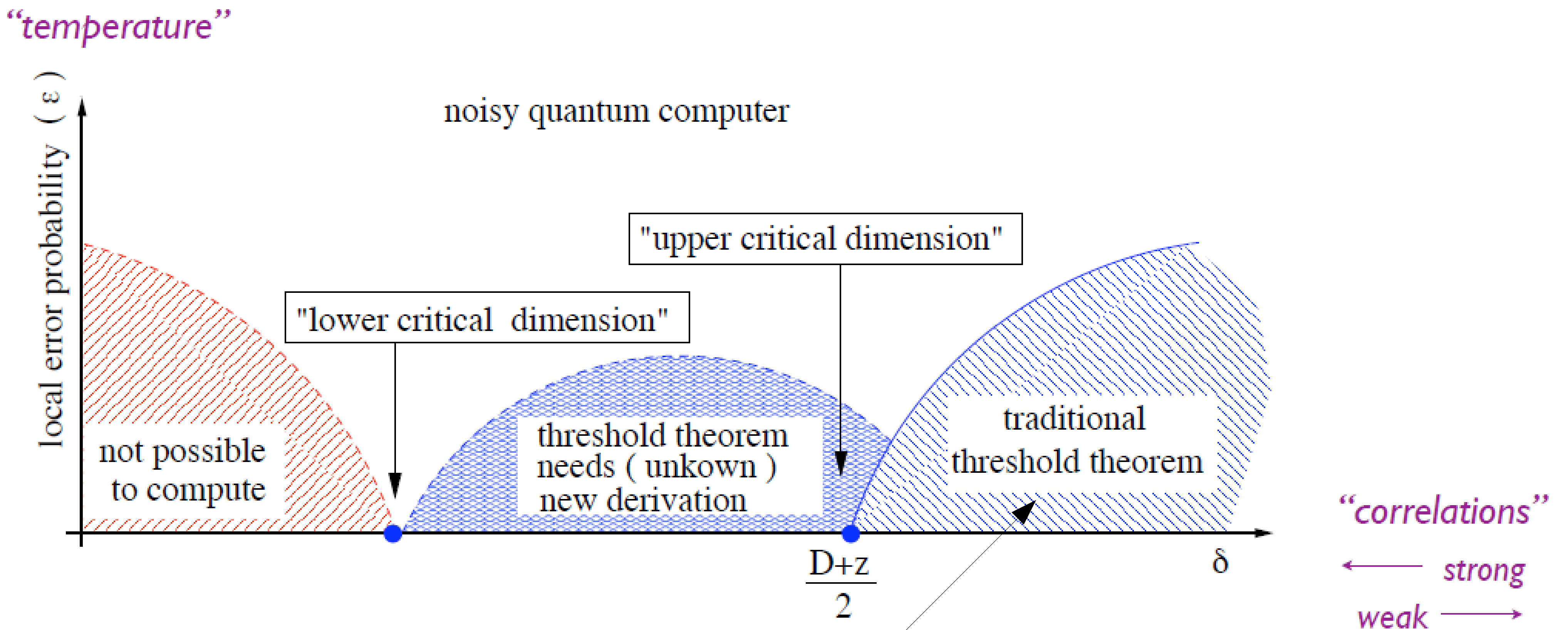
The calculation

1- to develop a systematic expantion to include correlations.
2- to study the stability of this expansion.



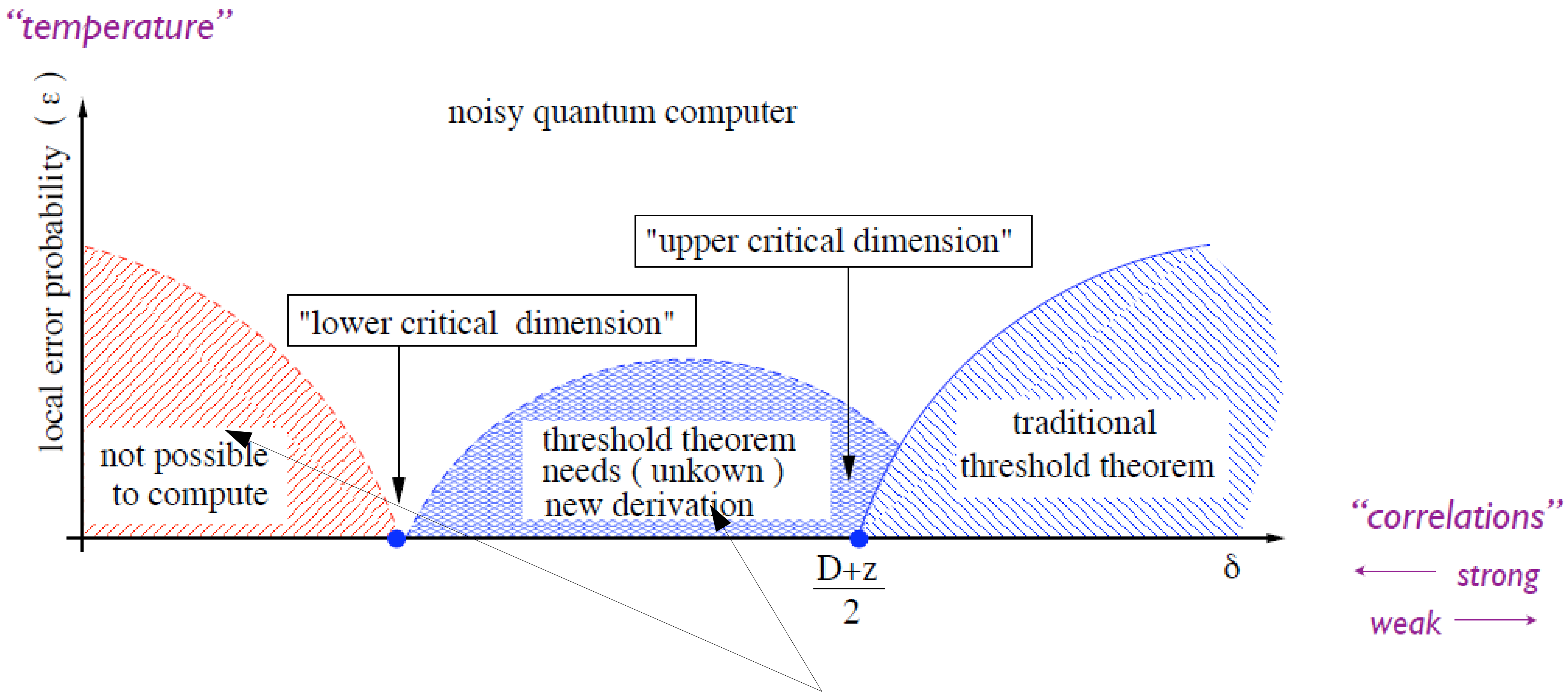


Threshold theorem with correlated (gaussian) noise ...



The expansion is well behaved.

Threshold theorem with correlated (gaussian) noise ...



What are these?

Questions we would like to answer:

- What are these other phases?
- Do they mean something?
- How to consider a “dense” set of physical qubits.

To proceed we had to change the question.

Given a desired error tolerance:

for how long can we compute using QEC?

"friendly" assumptions

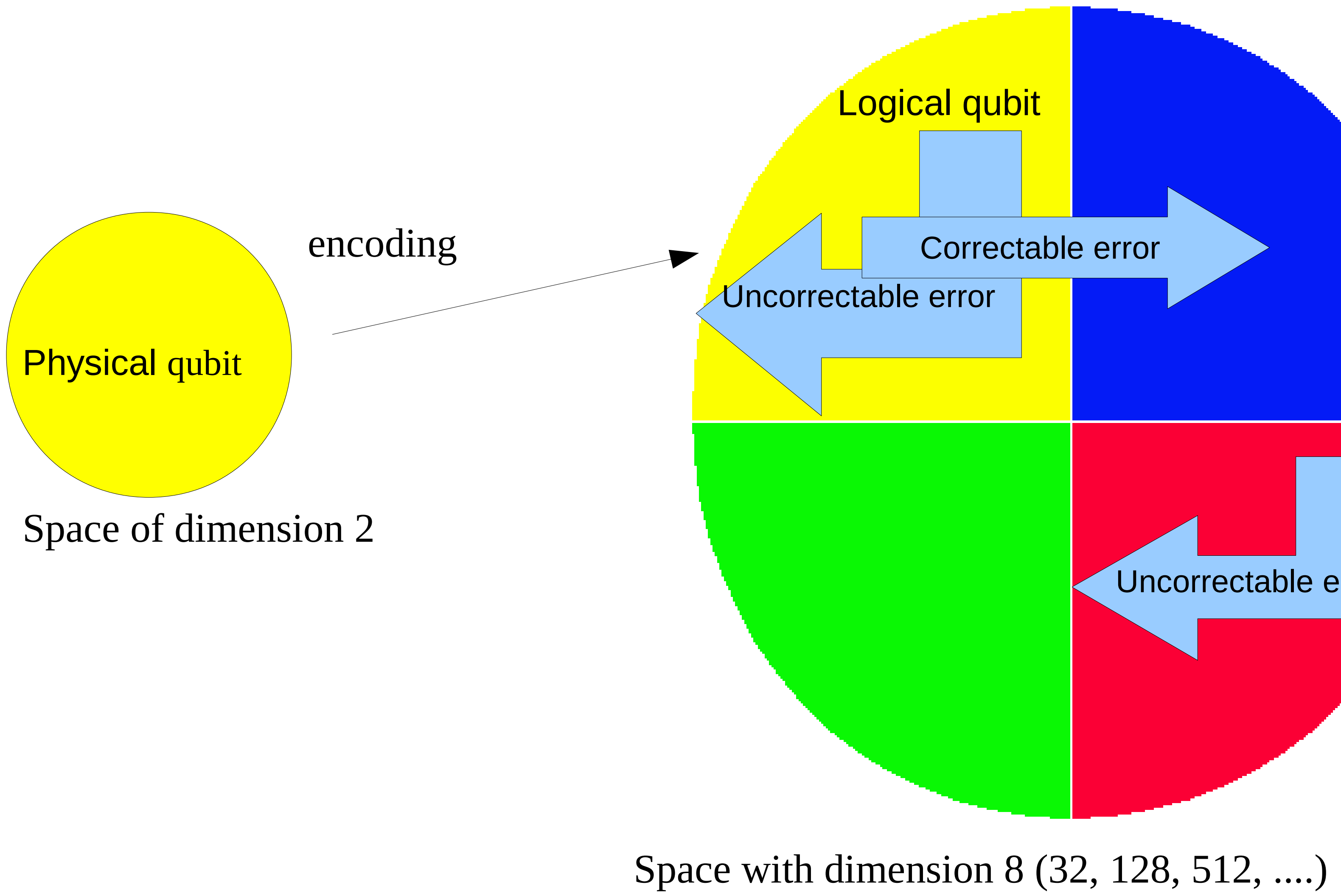
- standard quantum error correction code (circuit model QC).
- state preparation, gates, measurements all done perfectly.
- quantum evolution with only non-error syndromes (for upper bound).

"unfriendly" assumptions

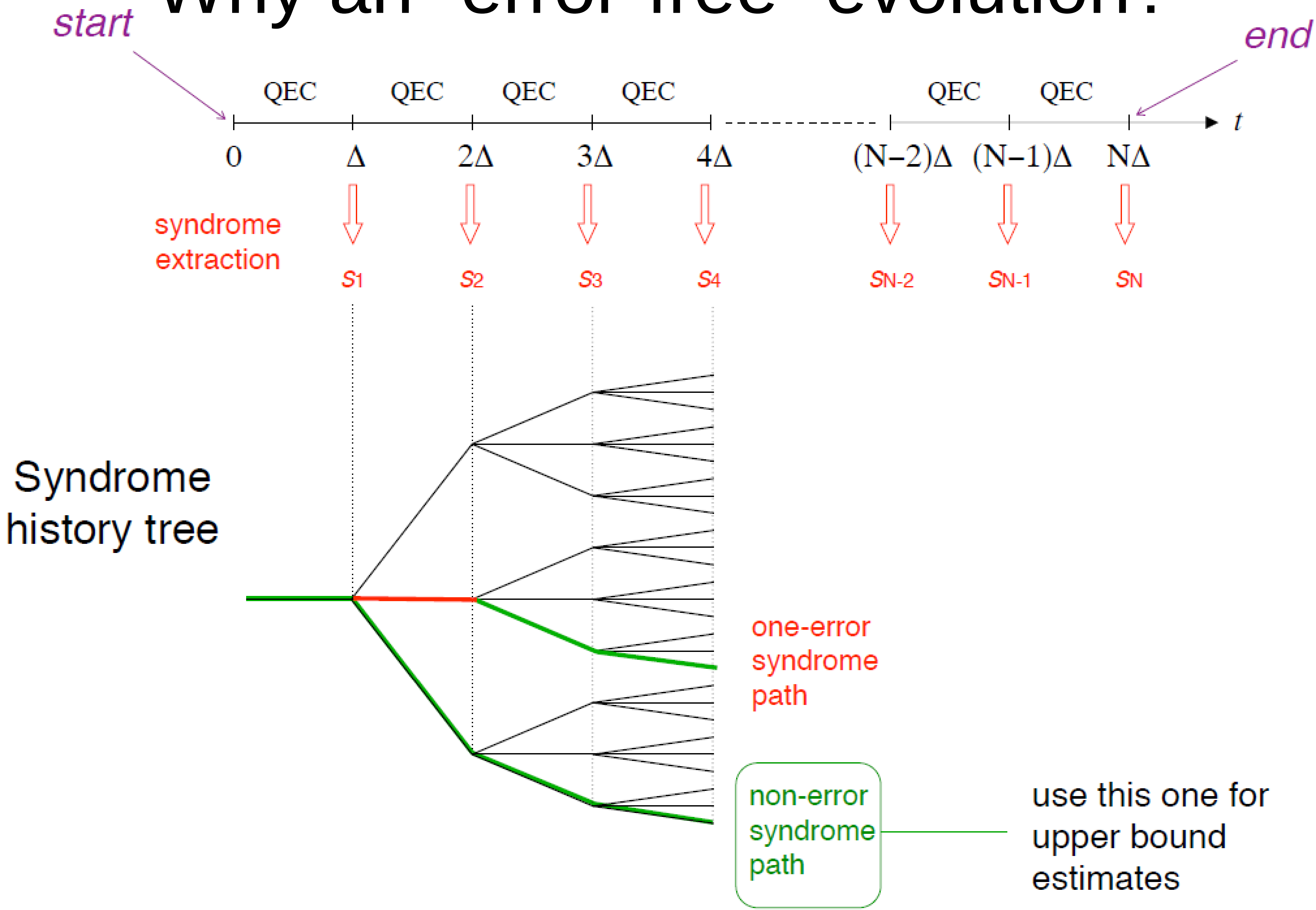
- gapless bath: power-law correlations in time and space (non exponential)

(Examples: phonons, EM fluctuations, etc.)

QEC


Logical qubit
encoding
Correctable error
Uncorrectable error
Physical qubit
Uncorrectable error
Space of dimension 2
Space with dimension 8 (32, 128, 512, ...)

Why an "error-free" evolution?



An example: the 5 qubit code

$$|\bar{0}\rangle = \frac{1}{4}\big[|00000\rangle + |10010\rangle + |01001\rangle + |10100\rangle + |01010\rangle - |11011\rangle - |00110\rangle - |11000\rangle$$
$$-|11101\rangle - |00011\rangle - |11110\rangle - |01111\rangle - |10001\rangle - |01100\rangle - |10111\rangle + |00101\rangle\big]$$

$$|\bar{1}\rangle = \frac{1}{4}\big[|11111\rangle + |01101\rangle + |10110\rangle + |01011\rangle + |10101\rangle - |00100\rangle - |11001\rangle - |00111\rangle$$
$$-|00010\rangle - |11100\rangle - |00001\rangle - |10000\rangle - |01110\rangle - |10011\rangle - |01000\rangle + |11010\rangle\big]$$

stabilizers and logical operations

$$g_1 = xzzx1$$
$$g_2 = 1xzzx$$
$$g_3 = x1xzz$$
$$g_4 = zx1xz$$

$$\bar{X} = xxxxx$$
$$\bar{Y} = yyyyy$$
$$\bar{Z} = zzzzz$$

$4^5 = 1024$ error operators

16 possible syndromes

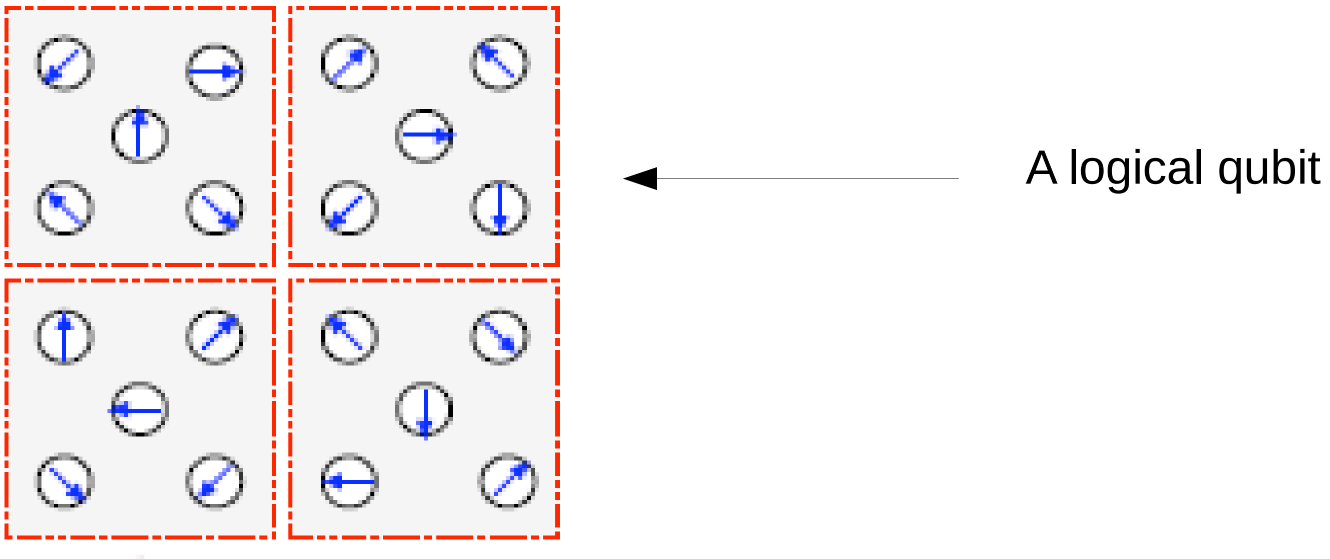
$\downarrow$

but only 64 error operators can lead to a <u>non-error syndrome</u>

$\longrightarrow$

	1	X	Y	Z
$\bar{1}$	11111	$xxxxx$	$yyyyy$	$zzzzz$
g_1	$xzzx1$	$1yy1x$	$zxxzy$	$y11yz$
g_2	$1xzzx$	$x1yy1$	$yzxxz$	$zy11y$
g_3	$x1xzz$	$1x1yy$	$zyzxx$	$yzy11$
g_4	$zx1xz$	$y1x1y$	$xzyzx$	$1yzy1$
h_1	$xy1yx$	$1zxz1$	$z1y1z$	$yxzxy$
h_2	$1zyyz$	$xyzzy$	$yx11x$	$z1xx1$
h_3	$yyz1z$	$zzyxy$	$11xyx$	$xx1z1$
h_4	$xxy1y$	$11zxz$	$zz1y1$	$yyxzx$
h_5	$z1zyy$	$yxyzz$	$xyx11$	$1z1xx$
h_6	$yxxy1$	$z11zx$	$1zz1y$	$xyyxz$
h_7	$1yxxy$	$xz11z$	$y1zz1$	$zxyyx$
h_8	$zyyz1$	$yzzyx$	$x11xy$	$1xx1z$
h_9	$y1yxx$	$zxz11$	$1y1zz$	$xzxyy$
h_{10}	$yz1zy$	$zyxyz$	$1xyx1$	$x1z1x$
h_{11}	$zzx1x$	$yy1x1$	$xxzyz$	$11yzy$

How the qubits are organized?



Spatial Locality:

1- Not very fundamental, but helps in organizing the calculation.

2- It is physical: measurements and gates are hard to do.

Time evolution in the interaction picture

$$U_I(\Delta,0) = T_t \exp\left[-i\int_0^\Delta dt\, H_I(t)\right]$$

We assume that lowest order perturbation theory is OK for “short” times

$$U_I(\Delta,0) \approx 1 - i \sum_{\alpha=\{x,z\}} \sum_{\mathbf{x}} \lambda_\alpha \Delta : f^\alpha(\mathbf{x},0) : \sigma^\alpha_{\mathbf{x}}.$$

Is the expansion parameter.

An evolution with “no-errors”

An example with the 5-qubits code:

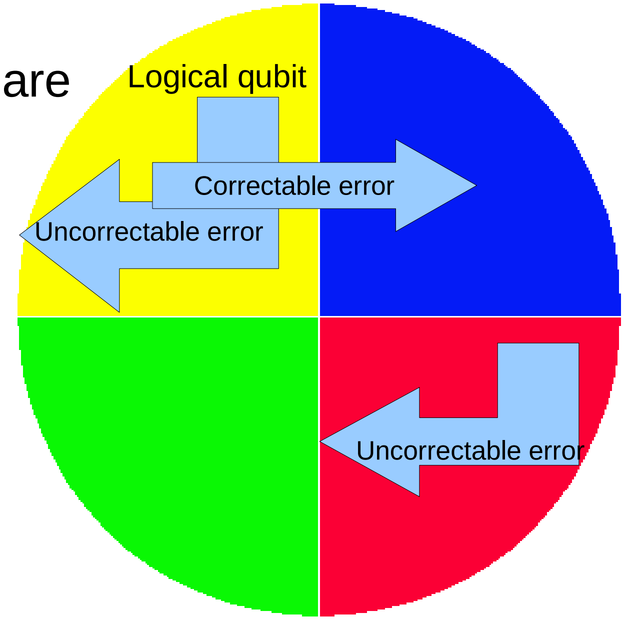
$$v_0(\Delta,0) \;\approx\; \bar{1} + i\Delta^3 \sum_{\mathbf{x},\alpha,\beta,i,j,k} \eta^{\alpha\beta}_{ijk}\lambda_\alpha\lambda^2_\beta$$

$$\times :f^\alpha(\mathbf{x}_i,0)::f^\beta(\mathbf{x}_j,0)::f^\beta(\mathbf{x}_k,0):\bar{\sigma}^\alpha_{\mathbf{x}},$$

Logical qubits

Code dependent constants (all the others are zero in this case):

$$\begin{aligned} \eta^{xz}_{324} &= \eta^{xz}_{435} = \eta^{xz}_{514} \\ = \eta^{xz}_{125} &= \eta^{xz}_{213} = \eta^{zx}_{134} \\ = \eta^{zx}_{412} &= \eta^{zx}_{245} = \eta^{zx}_{523} \\ = \eta^{zx}_{315} &= 1 \end{aligned}$$



An evolution with "no-errors"

After normal ordering, we can in general write:

$$\Gamma_\alpha(\mathbf{x},0) \equiv \lambda_\alpha \sum_{\beta,i,j,k} \eta^{\alpha\beta}_{ijk} (\lambda_\beta \Delta)^2 \, : f^\beta(\mathbf{x}_j,0) f^\beta(\mathbf{x}_k,0) :$$

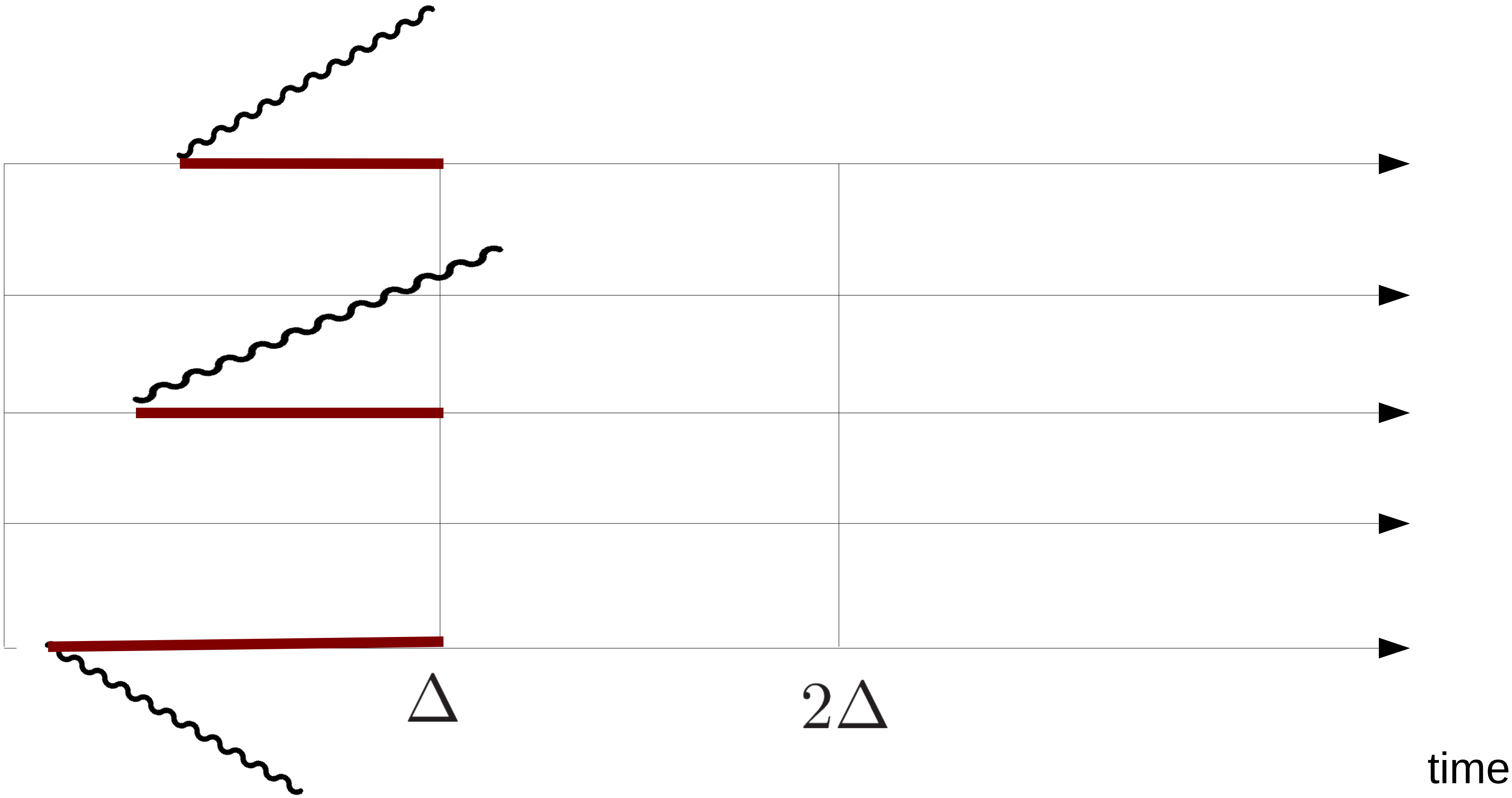
"Higher order" correlations

$$v_0(\Delta,0) \approx \bar{1} + i\Delta \sum_{\mathbf{x},\alpha=\{x,z\}} (\lambda^*_\alpha + \Gamma_\alpha) : f^\alpha(\mathbf{x},0) : \, \bar{\sigma}^\alpha_{\mathbf{x}},$$

Effective coupling constant

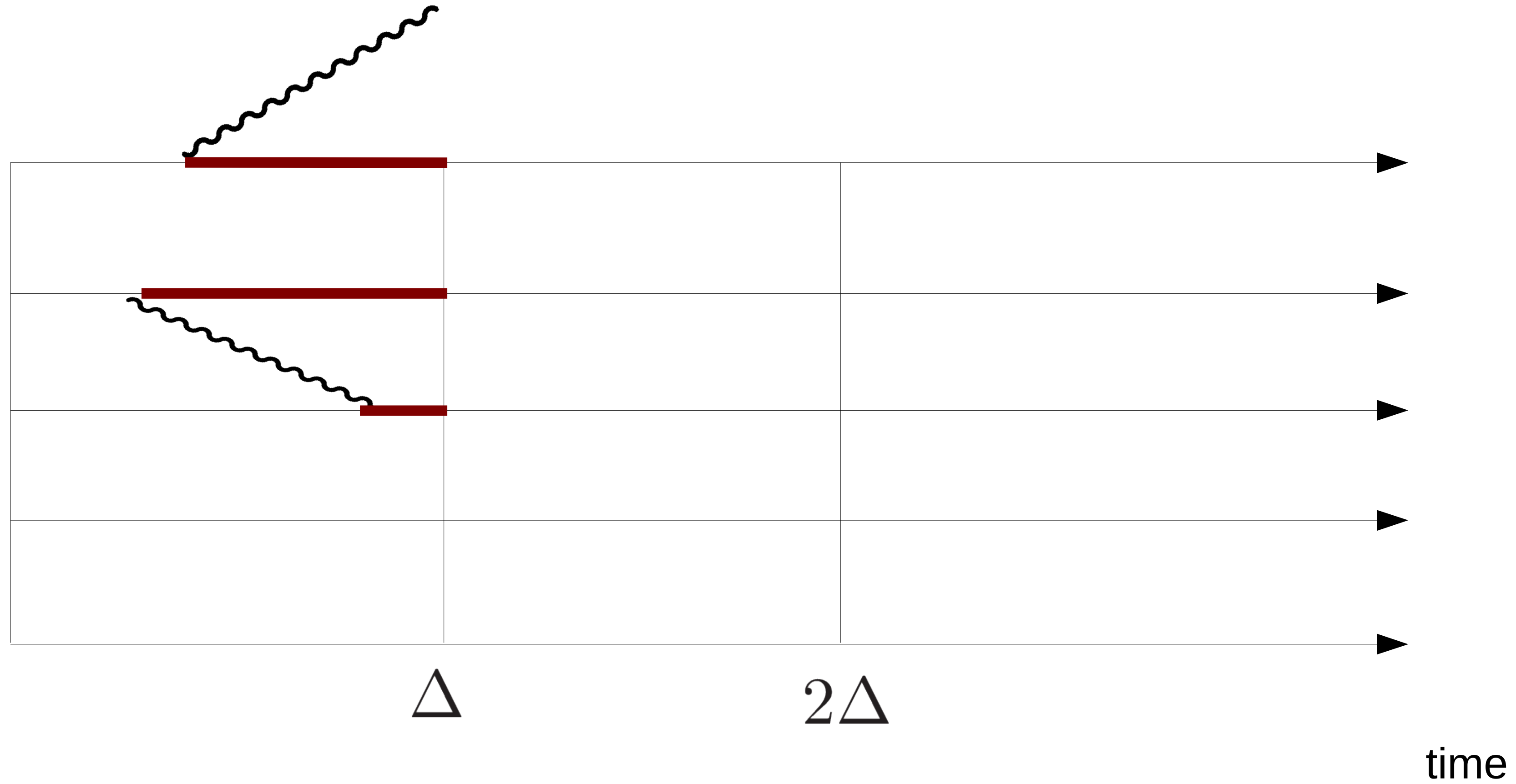
$$\lambda^*_\alpha \equiv \lambda_\alpha \sum_{\beta,i,j,k} \eta^{\alpha\beta}_{ijk} (\lambda_\alpha \Delta)^2 \sum_{\mathbf{k}\neq 0} |u_{\alpha,k}|^2 \exp\left[-i\mathbf{k}\cdot(\mathbf{x}_i - \mathbf{x}_j)\right]$$

Higher order correlation



Using spatial locality of the qubits they are less relevant than the other terms.

Renormalized coupling constant



An evolution with “no-errors”

After normal ordering, we can in general write:

$$\Gamma_\alpha(\mathbf{x},0) \equiv \lambda_\alpha \sum_{\beta,i,j,k} \eta_{ijk}^{\alpha\beta} (\lambda_\beta \Delta)^2 \; : f^\beta(\mathbf{x}_j,0) f^\beta(\mathbf{x}_k,0) :$$

“Higher order” correlations

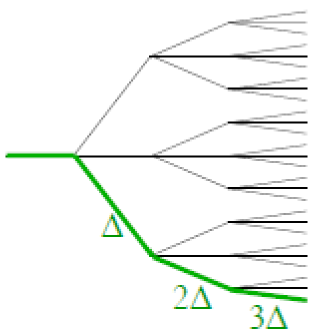
$$v_0(\Delta,0) \approx \bar{1} + i\Delta \sum_{\mathbf{x},\alpha=\{x,z\}} (\lambda_\alpha^* + \Gamma_\alpha) : f^\alpha(\mathbf{x},0) : \; \bar{\sigma}_\mathbf{x}^\alpha,$$

Effective coupling constant

$$\lambda_\alpha^* \equiv \lambda_\alpha \sum_{\beta,i,j,k} \eta_{ijk}^{\alpha\beta} (\lambda_\alpha \Delta)^2 \sum_{\mathbf{k}\neq 0} |u_{\alpha,k}|^2 \exp\left[-i\mathbf{k}\cdot(\mathbf{x}_i - \mathbf{x}_j)\right]$$

Quantum evolution for logical qubits with a no-error syndrome

$$\bar{U}_I(T,0) \approx T_t\, e^{i\int_0^T dt \sum_{\mathbf{x},\alpha=\{x,z\}} \lambda_\alpha^* :f^\alpha(\mathbf{x},t): \bar{\sigma}_\mathbf{x}^\alpha},$$



Δ is the new ultraviolet cut-off.

$T = M\Delta$ total computational time.

M is the number of QEC steps performed.

QEC gave us a lot:

1- lower coupling constant,

2- smaller high frequency cut-off, etc..

But, … the evolution has the same form as the unprotected qubit !

Upper-bound to computational time

To quantify the amount of information lost to the environment, we use the trace distance

$$D\big(\rho_R(T), \rho_0\big) = \frac{1}{2}\mathrm{tr}\,|\rho_R(T) - \rho_0|$$

Real density matrix

Ideal density matrix

It tells how hard it is to distinguish two states by performing measurements.

0 for identical states

1 for orthogonal states

Information lost by a single logical qubit

$$D\big(\rho_R(T), \rho_0\big) = \sqrt{|\delta\sigma^+(T)|^2 + [\delta\sigma^z(T)]^2/4}$$

$$\delta\sigma^\alpha(T) = \langle\bar{\sigma}^\alpha(T)\rangle - \langle\bar{\sigma}^\alpha\rangle$$

It is a straightforward calculation. We do it perturbatively in x direction and exact in the z direction.

$$H_I^{\rm rot}(t) = \lambda_x^* \sum_{\alpha=\{\pm\}} : f^\alpha(t) : \exp\left(-2i\alpha : F^z(t) :\right) \bar{\sigma}^\alpha .$$

$$:F^z\big((n+1)\Delta\big): - :F^z(n\Delta): = \lambda_z^* \Delta :f^z(n\Delta):$$

Zeroth order in x

$$\langle \bar{\sigma}^z(T) \rangle = \langle \bar{\sigma}^z \rangle \qquad \langle \bar{\sigma}^+(T) \rangle = e^{-4\gamma_z(T)} \langle \bar{\sigma}^+ \rangle$$

Decoherence function:

$$\gamma_z(T) = (2\pi/L)^D (\lambda_z^*)^2 \sum_{\mathbf{k}\neq 0} \frac{|u_{z,k}|^2}{\omega_{z,k}^2} [1 - \cos(\omega_{z,k} T)]$$

The result

$$D\big(\rho_R(T),\rho_0\big) = \left\langle \bar{\sigma}^+ \right\rangle \left[1 - e^{-4\gamma_z}(T)\right].$$

$$\gamma_z(M\Delta) \propto \begin{cases} (\lambda_z^*/\omega_0)^2 \ (\omega_0\Delta)^{-\zeta_z/z_z}, & \zeta_z < 0, \\ (\lambda_z^*/\omega_0)^2 \ \ln M, & \zeta_z = 0, \\ (\lambda_z^*/\omega_0)^2 (\omega_0\Delta)^{\zeta_z/z_z} \ M^{\zeta_z/z_z}, & 0 < \zeta_z < 2z_z, \\ (\lambda_z^*\Delta)^2 \ (k_0 L/2\pi)^{\zeta_z - 2z_z} \ M^2, & \zeta_z > 2z_z, \end{cases}$$

$$\zeta_z = 2(z_z - s_z) - D$$

The result

$$D\big(\rho_R(T), \rho_0\big) = \left\langle \bar{\sigma}^+ \right\rangle \, \left[1 - e^{-4\gamma_z}(T)\right].$$

$$\gamma_z(M\Delta) \propto \begin{cases} (\lambda_z^*/\omega_0)^2 \; (\omega_0\Delta)^{-\zeta_z/z_z}, & \zeta_z < 0, \\ (\lambda_z^*/\omega_0)^2 \; \ln M, & \zeta_z = 0, \\ (\lambda_z^*/\omega_0)^2 \, (\omega_0\Delta)^{\zeta_z/z_z} \; M^{\zeta_z/z_z}, & 0 < \zeta_z < 2z_z, \\ (\lambda_z^*\Delta)^2 \; (k_0 L/2\pi)^{\zeta_z - 2z_z} \; M^2, & \zeta_z > 2z_z, \end{cases}$$

Diverges with size of the environment.

The result

We define a critical distance $D_{\rm crit}$

and evaluate the maximum time available to compute:

$$M_{\rm max} \propto \begin{cases} \infty, & \zeta_z < 0, \\ \exp\left[c_{D,z}\, D_{\rm crit}\, (\omega_0/\lambda_z^*)^2\right], & \zeta_z = 0, \\ D_{\rm crit}^{z_z/\zeta_z}\, (\omega_0/\lambda_z^*)^{2z_z/\zeta_z}/(\omega_0\Delta), & 0 < \zeta_z < 2z_z, \\ \sqrt{D_{\rm crit}}/(\lambda_z^*\Delta) \to 0, & \zeta_z > 2z_z, \end{cases}$$

Maximum number of QEC steps

What about an array of qubits?

$$D\big(\rho_R(T), \rho_0\big) = \tfrac{1}{2}\mathrm{tr}\,|\rho_R(T) - \rho_0|$$ ← hard problem

$$D_{HS}\big(\rho_R(T), \rho_0\big) \le D\big(\rho_R(T), \rho_0\big) \le 2^{N/2} D_{HS}\big(\rho_R(T), \rho_0\big),$$

Hilbert-Schmidt norm

$$D_{HS}\big(\rho_R(T), \rho_0\big) = \tfrac{1}{2}[\mathrm{tr}|\rho_R(T) - \rho_0|^2]^{1/2}$$

and N is the number of logical qubits.

Hilbert-Schmidt norm

$$D_{HS}\big(\rho_R(T), \rho_0\big) \propto \sqrt{\sum_{\alpha} (\lambda_{\alpha}^{*})^2 \Big| \sum_{\mathbf{x},\mathbf{y}} W_{\mathbf{x},\mathbf{y}}^{\alpha}(T) \Big|^2},$$

$$W_{\mathbf{x},\mathbf{y}}^{\alpha}(T) = \left(\frac{2\pi}{L}\right)^{D} \sum_{\mathbf{k}\neq 0} \frac{\left|u_{\alpha,k}\right|^2}{\omega_{\alpha,k}^2} e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \big(1 - e^{-i\omega_{\alpha,k}T}\big).$$

The result

$$\Big|\sum_{\mathbf{x},\mathbf{y}} W^{\alpha}_{\mathbf{x},\mathbf{y}}(T)\Big| \propto \begin{cases} N\omega_0^{-1}\,(\omega_0\Delta)^{-\zeta_\alpha/z_\alpha}, & \zeta_\alpha < 0, \\ N\omega_0^{-1}\,\ln M, & \zeta_\alpha = 0, \\ N\omega_0^{-1}(\omega_0\Delta\, M)^{\zeta_\alpha/z_\alpha}, & 0 < \zeta_\alpha < z_\alpha, \\ N\Delta\,(k_0L/2\pi)^{\zeta_\alpha - z_\alpha}\, M, & \zeta_\alpha > z_\alpha, \end{cases}$$

Number of dimensions of the bath

self-interacting part: $\zeta_\alpha = 2(z_\alpha - s_\alpha) - D$

correlation part: $\zeta_\alpha = 2(z_\alpha - s_\alpha) + D_x - D$

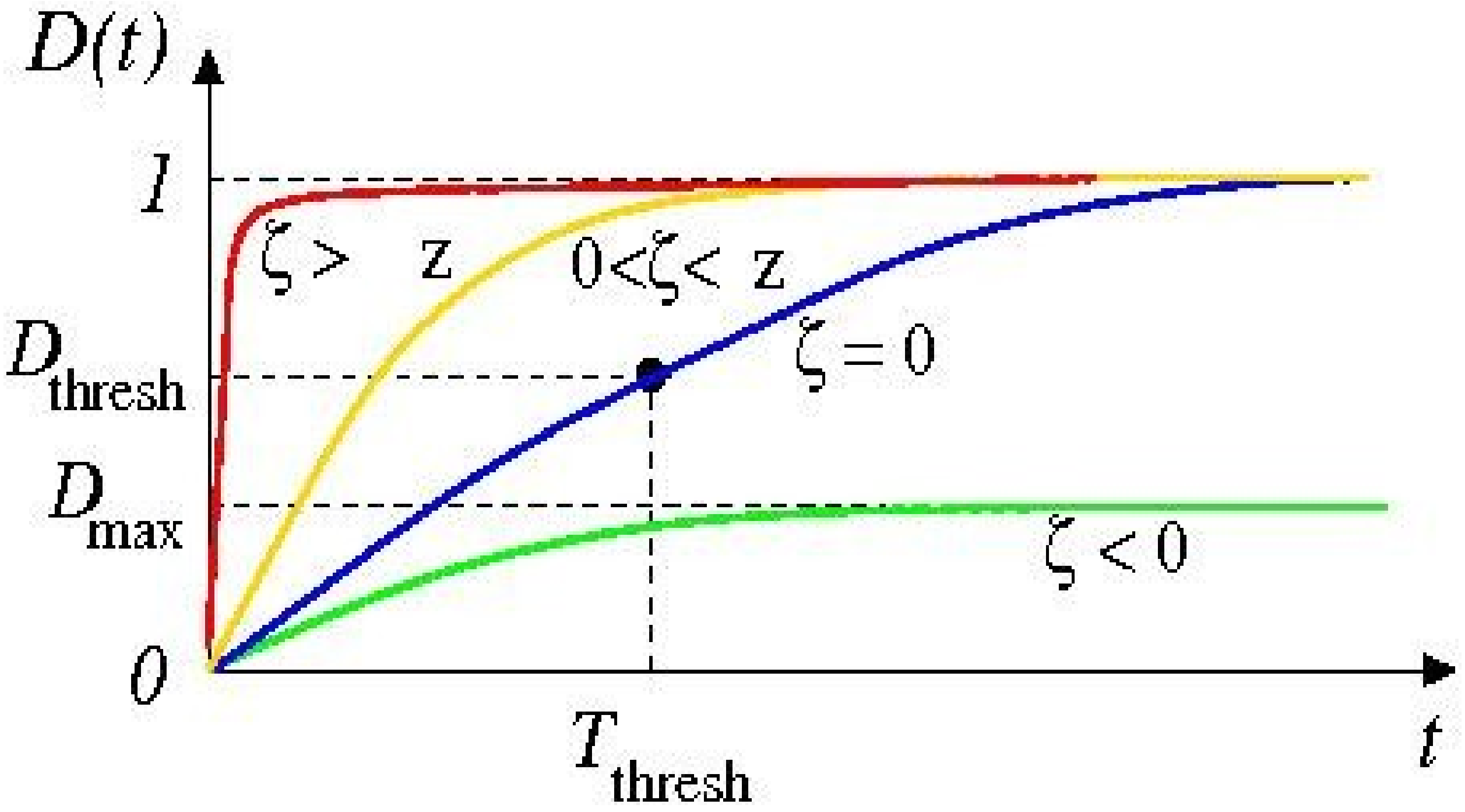
Number of spatial dimensions of the computer

For how long is it possible to quantum compute?

$$M_{\max} = \begin{cases} \infty, & \zeta_\alpha < 0, \\ \exp\left[\dfrac{b_{D,\alpha}\, D_{\text{crit}}}{N(\lambda_z^*/\omega_0)}\right], & \zeta_\alpha = 0, \\ (\omega_0 \Delta)^{-1} \left[\dfrac{D_{\text{crit}}}{N(\lambda_z^*/\omega_0)}\right]^{z_\alpha/\zeta_\alpha}, & 0 < \zeta_\alpha < z_\alpha, \\ (2\pi/k_0 L)^{\zeta_\alpha - z_\alpha}\ \dfrac{D_{\text{crit}}}{N(\lambda_z^* \Delta)} \to 0, & \zeta_\alpha > z_\alpha, \end{cases}$$

$$\zeta_\alpha = 2(z_\alpha - s_\alpha) - D \qquad \zeta_\alpha = 2(z_\alpha - s_\alpha) + D_x - D$$

For how long is it possible to quantum compute?



$$\zeta_\alpha = 2(z_\alpha - s_\alpha) - D \qquad \zeta_\alpha = 2(z_\alpha - s_\alpha) + D_x - D$$

Conclusions

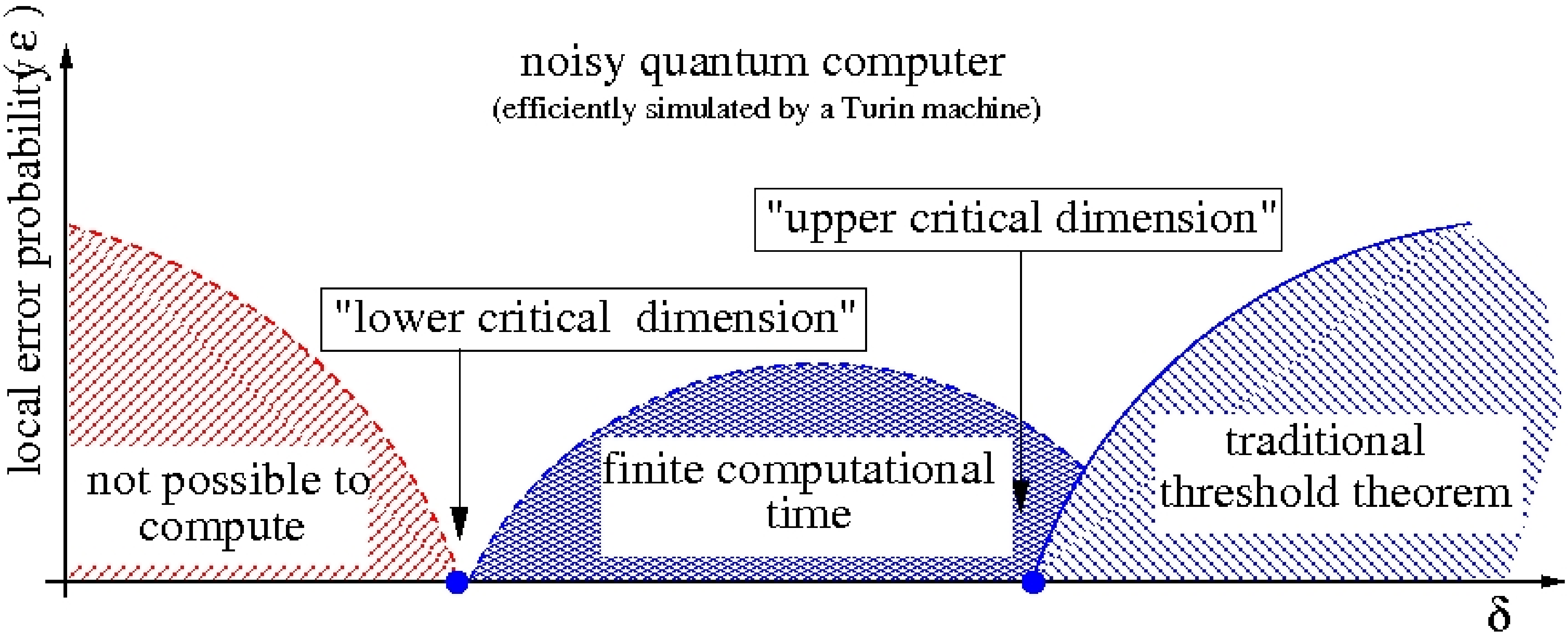
- there are adverse environments to QEC (where there is no threshold that allows computation);

- in situations for which it is possible to compute, there are microscopic parameters that must be factored into the choice of code, concatenation level, position of the physical qubits, etc.

- in all cases, the total number of logical qubits appears in the result for the maximum available time, even in the most benevolent environment.

- The three regimes that we found nicely fitted the qualitative interpretation of resilience as a “dynamical” quantum phase transition.

QEC and “Quantum Phase Transitions”



- Phys. Rev. Lett. 97, 040501 (2006).
- Phys. Rev. Lett. 98, 040501 (2007).
- Phys. Rev. A 78, 012314 (2008).
- Phys. Rev. A 80, 020303R (2010).