Approximate Operator Quantum Error Correction

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P. Mandayam and H.K. Ng (in prep.)

Talk Outline

- The Transpose Channel and its role in perfect QEC

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- A simple, analytical approach to approximate (subspace) QEC
- Finding good approximate codes using the transpose channel
- From subspace to subsystem codes: Approximate Operator Quantum Error Correction
- Conclusion: A unified framework for approximate QEC via the Transpose Channel

The Transpose Channel

Definition

For a given noise channel $\mathcal{E} \sim \{E_i\}$, and a code $\mathcal{C}$ (with projector P),
Transpose Channel : $\mathcal{R}_{\mathrm{T}} \sim \{R_i\}_{i=1}^{N}$, $R_i \equiv P E_i^{\dagger} \mathcal{E}(P)^{-1/2}$

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  - $\mathcal{E}^{\dagger}$ is the adjoint channel
  - $\mathcal{P}$ is the projection onto $\mathcal{C}$
  - $\mathcal{N}$ is the normalization map $\mathcal{N}(\cdot) = \mathcal{E}(P)^{-1/2}(\cdot)\mathcal{E}(P)^{-1/2}$

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  $\Rightarrow \mathcal{R}_{\rm T}$ Independent of the Kraus representation.
- $\mathcal{R}_{\rm T}$ is trace-preserving on the support of $\mathcal{E}(P)$.
- If $\mathcal{E}$ is perfectly correctable on $\mathcal{C}$, $\mathcal{R}_{\rm T}$ is the recovery map that recovers the information encoded in $\mathcal{C}$.

Role of $\mathcal{R}_{\mathrm{T}}$ in perfect QEC

Lemma (Alternative form of perfect QEC conditions[1])

A channel $\mathcal{E} \sim \{E_i\}_{i=1}^{N}$ satisfies the Knill-Laflamme conditions for a code $\mathcal{C}$ iff

$$PE_i^{\dagger}\mathcal{E}(P)^{-1/2}E_jP = \beta_{ij}P, \ \ \forall i,j = 1,...,N$$

for some positive matrix β.

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- Can be perturbed to obtain sufficient conditions for approximate QEC. Size of the perturbation is directly related to the fidelity due to $\mathcal{R}_{\mathrm{T}}$.

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Approximate Quantum Error Correction

- Worst-case fidelity: For a codespace $\mathcal{C}$, under the action of the noise channel $\mathcal{E}$ and recovery $\mathcal{R}$,

$$F^2_{\min}[\mathcal{C}, \mathcal{R}\circ\mathcal{E}] = \min_{|\psi\rangle\ \in\mathcal{C}} F^2[|\psi\rangle, \mathcal{R}\circ\mathcal{E}(|\psi\rangle\langle\psi|)]\ ,\ \left(F^2[|\psi\rangle, \sigma] \equiv \langle\psi|\sigma|\psi\rangle\right).$$

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- Channel $\mathcal{E}$ is *approximately correctable* on code space $\mathcal{C}$ if $\exists$ a physical (CPTP) map $\mathcal{R}$ such that $F^2_{\min}[\mathcal{C}, \mathcal{R}\circ\mathcal{E}] \approx 1$.

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- Finding the optimal recovery for worst-case fidelity is not a convex-optimization problem!
  - Optimizing for entanglement fidelity is tractable via SDP, convex-optimization[2] .
  - Channel similar to $\mathcal{R}_T$ is close to optimal for entanglement fidelity[3].
  - Analytically, close-to-optimal recovery maps have been constructed for worst-case entanglement fidelity[4].

[2] Fletcher et al. PRA, **75**, 021338 (2007), Kosut et al. PRL, **100**, 020502 (2008)
[3] Barnum and Knill, JMP, **43**, 2097 (2002)
[4] Beny and Oreshkov, PRL, **104**, 120501 (2010)

A simple, analytical approach to approximate QEC[5]

Theorem (Near-optimality of the transpose channel)

Given a code space $\mathcal{C}$ *of dimension* d *and optimal recovery map* $\mathcal{R}_{\rm op}$ *with optimal fidelity loss* η_{op}*, such that* $F^2[|\psi\rangle, (\mathcal{R}_{op} \circ \mathcal{E})(|\psi\rangle\langle\psi|)] \geq 1 - \eta_{\rm op}$*, then,*

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- **Corollary**: $\eta_{\rm op} \leq \eta_T \leq \eta_{\rm op}[(d+1) + O(\eta_{\rm op})]$.

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- Sufficient and necessary conditions for approximate subspace QEC .

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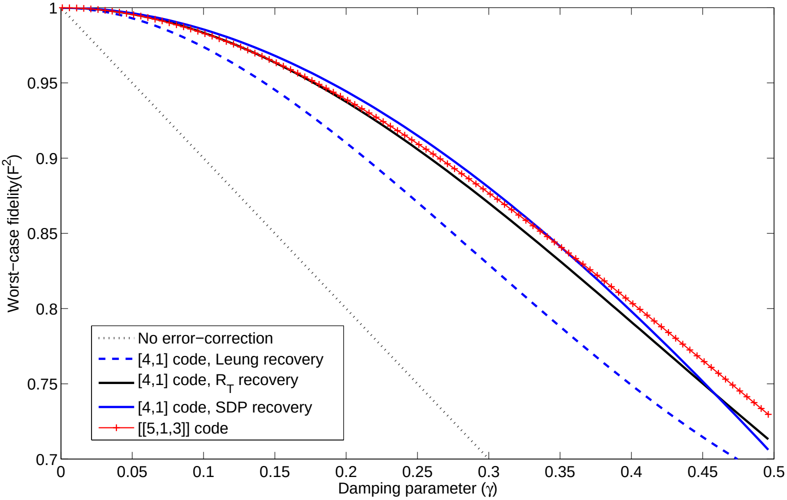
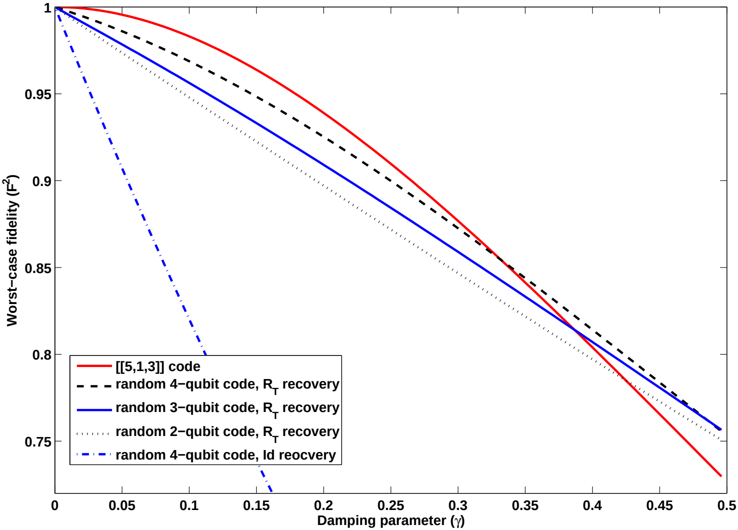
Examples: Codes for the Amplitude Damping channel



- $[4, 1]$ code: Leung et al. PRA **56**, 2567 (1997)

Examples

Random 2-,3-,4-qubit AQEC codes for the Amplitude Damping channel



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  A fixed partition of the physical (system) Hilbert space: $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \oplus \mathcal{K}$.
  Information is encoded only in subsystem $\mathcal{H}_A$.

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Theorem (Alternate conditions for perfect OQEC[7])

A code $\mathcal{C} \equiv \mathcal{H}_A \otimes \mathcal{H}_B$ with projector $P \equiv P_A \otimes P_B$ is perfectly correctable on subsystem $\mathcal{H}_A$ under $\mathcal{E} \sim \{E_i\}$ iff $\exists$ operators B'_{ij} on $\mathcal{H}_B$ such that,

$$PE_i^\dagger \mathcal{E}(P)^{-1/2} E_j P = P_A \otimes B'_{ij}, \qquad \forall i, j,$$

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[7] P. Mandayam and H.K. Ng, in preparation.

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- $\mathcal{R}_{\mathrm{T}}$ is the appropriate recovery map: Subsystem $\mathcal{H}_A$ is perfectly correctable under $\mathcal{E}$ iff $\mathcal{R}_{\mathrm{T}} \circ \mathcal{E}$ is the identity operation on $\mathcal{H}_A$.

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Theorem (Sufficient Conditions for Approximate OQEC)

A subsystem code $\mathcal{C}$ is ϵ-correctable on $\mathcal{H}_A$ under the action of $\mathcal{E}$ if $\exists$ operators B'_{ij} on B such that

$$PE_i^{\dagger}\mathcal{E}(P)^{-1/2}E_jP = P_A \otimes B'_{ij} + \Delta_{ij}, \qquad \forall i,j,$$

where Δ_{ij} is an operator on $\mathcal{H}_A \otimes \mathcal{H}_B$ such that $\sum_{i,j} \|\Delta_{ij}\|_{tr} \leq \epsilon$.

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- $\sum_{i,j} \|\Delta_{ij}\|_{tr}$ is directly related to the worst-case fidelity using the transpose channel : $\min_{|\psi\rangle_A \in \mathcal{H}_A} F^2[|\psi\rangle_A, \mathrm{tr}_B\{(\mathcal{R}_{\mathrm{T}} \circ \mathcal{E})(|\psi\rangle_A\langle\psi| \otimes \rho_B)\}]$.

Transpose channel recovery for approximate OQEC?

- We prove $\mathcal{R}_{\mathrm{T}}$ is near-optimal when
  - $\mathcal{E} = \mathcal{E}_A \otimes \mathcal{E}_B$, and $\mathcal{H}_A$ is approximately correctable under $\mathcal{E}_A$

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  - $\mathcal{E} = \mathcal{E}_A \otimes \mathcal{E}_B$, and $\mathcal{H}_A$ is approximately correctable under $\mathcal{E}_A$
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Theorem

Let $\sigma_B^{\max}$ be the maximally-mixed state on the noisy subsystem $\mathcal{H}_B$. Then,

$$\mathcal{F}^2\left[|\psi\rangle_A, \mathrm{tr}_B\left\{(\mathcal{R}_{\mathrm{T}} \circ \mathcal{E})(|\psi\rangle_A\langle\psi| \otimes \sigma_B^{\max})\right\}\right] \geq 1 - (d_A + 1)\eta_{op},$$

for all $|\psi\rangle_A \in \mathcal{H}_A$.

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for all $|\psi\rangle_A \in \mathcal{H}_A$.

- Special case : For qubit codes in an independent error model, where individual error operators on $\mathcal{H}_B$ are scaled Paulis, $\mathcal{F}^2\left[|\psi\rangle_A, {\rm tr}_B\left\{(\mathcal{R}_{\rm T}\circ\mathcal{E})(|\psi\rangle_A\langle\psi|\otimes\rho_B)\right\}\right]$ is independent of ρ_B.

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For any pure state $|\psi\rangle_A \in \mathcal{H}_A$ *and any state* $\rho_B \in \mathcal{H}_B$

$$\mathcal{F}^2\left[|\psi\rangle_A, \mathrm{tr}_B\left\{(\mathcal{R}_{\mathrm{T}} \circ \mathcal{E})(|\psi\rangle_A\langle\psi| \otimes \rho_B)\right\}\right] \geq 1 - (d_A + 1)\eta_{op} - 3\delta,$$

where $0 \leq \delta < 1$ *is the smallest possible value such that*

$$\left\|\mathcal{E}\left[|\psi\rangle_A\langle\psi| \otimes (\rho_B - \sigma_B^{\max})\right]\right\|_{tr} \leq \delta\|\rho_B - \sigma_B^{\max}\|_{tr}$$

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$$\left\|\mathcal{E}\left[|\psi\rangle_A\langle\psi| \otimes (\rho_B - \sigma_B^{\max})\right]\right\|_{tr} \leq \delta \|\rho_B - \sigma_B^{\max}\|_{tr}$$

- For example, if $\mathcal{E}$ is strictly-contractive on $\mathcal{H}_B$ with $\delta << 1$, the transpose channel can recover with high fidelity for any state on $\mathcal{H}_B$.
- If $\mathcal{E}$ nearly destroys all information in $\mathcal{H}_B$, it is close to being completely disentangling on $\mathcal{H}_A$ and $\mathcal{H}_B$.

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  Established for subsystem codes, when:
  - Noisy subsystem $\mathcal{H}_B$ starts in a maximally mixed state;
  - Qubit codes, independent error model with scaled Pauli error operators on $\mathcal{H}_B$;
  - Noise channel nearly destroys *all* information in the noisy subsystem $\mathcal{H}_B$.

Some Open Questions...

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- Approximate error correction as the first step of encoding in a fault-tolerant architecture?