Quantum Double Subsystem Codes

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Overview

In a Nutshell
Introduce a new quantum error correcting codes which combine the useful properties of two interesting classes of codes: the Bacon-Shor codes and the quantum double models.
Background

- Bacon-Shor code
- Quantum double models
  - Toric code
  - Non-Abelian quantum double
Bacon-Shor code

One particular code which has been shown to have a high error threshold is the Bacon-Shor code, which encodes one logical qubit.

Bacon-Shor stabilizers

This is because we can infer syndromes indirectly.
Syndrome measurement in Bacon-Shor

We can infer error syndromes using two-body measurements

Example (Bacon-Shor gauge operators)

These are known as gauge operators.
Bacon-Shor gauge decomposition

By taking products of gauge operators, we can obtain the syndrome associated with a stabilizer

\[ \text{Diagram} \]

This is good because

- Lower-weight operators ⇒ lower probability of errors
- Can parallelise measurement ⇒ advantage in speed means fewer errors
- Simple fault-tolerant gate set to measure these operators

(1)
Error correction in Bacon-Shor

Error correction takes an especially simple form.

Example
Say we have an $Z$ error on a qubit. This will anticommute with this stabilizer

$$\begin{array}{ccc}
\begin{array}{cccc}
\otimes \quad \otimes \\
\otimes & \otimes & \otimes & \otimes \\
\otimes & \otimes & \otimes & \otimes \\
\otimes & \otimes & \otimes & \otimes \\
\end{array} & = & 
\begin{array}{cccc}
\otimes \quad \otimes \\
\otimes & \otimes & \otimes & \otimes \\
\otimes & \otimes & \otimes & \otimes \\
\otimes & \otimes & \otimes & \otimes \\
\end{array}
\end{array} \tag{2}$$

But if we apply a second $Z$ on the qubit shown, we have a gauge operator

$$\begin{array}{ccc}
\begin{array}{cccc}
\otimes \quad \otimes \\
\otimes & \otimes & \otimes & \otimes \\
\otimes & \otimes & \otimes & \otimes \\
\otimes & \otimes & \otimes & \otimes \\
\end{array} & = & 
\begin{array}{cccc}
\otimes \quad \otimes \\
\otimes & \otimes & \otimes & \otimes \\
\otimes & \otimes & \otimes & \otimes \\
\otimes & \otimes & \otimes & \otimes \\
\end{array}
\end{array} \tag{3}$$
Error correction in Bacon-Shor

Example

The operator overlaps with in two places.

Since $-1 \times -1 = 1$ gauge operators commute with stabilizers

$\Rightarrow$ we have corrected the error!

- We call them gauge operators because they do not affect logical information
- $\mathcal{H}_T = \mathcal{H}_G \otimes \mathcal{H}_L$
- They do affect gauge qubits, but we don’t store useful information in these
Research question

Motivation ➤ Can we generalise the Bacon-Shor code?

Requirements ➤ We want our generalisations keep all the nice properties of the Bacon-Shor code
  1. Two-body gauge decomposition
  2. Error correction procedure

What’s next? ➤ We want to consider generalisations using the quantum double models
Toric code

- Defined on a lattice embedded in a torus
- Associate a qubit with each edge
- Encodes two logical qubits

- $Z$ stabilizers are plaquettes; $X$ stabilizers are vertices
- Plaquettes and vertices commute
- Products of stabilizers are also stabilizers
  - This gives us a link with the Bacon-Shor code
  - Toric code stabilizers are Bacon-Shor stabilizers
Bacon-Shor decomposition of the toric code

We can multiply adjacent plaquettes and vertices

The overlapping edges cancel, since $ZZ = XX = I$, producing Bacon strips. We do the same for the vertical direction as well.
Bacon-Shor decomposition of the toric code

We end up with two one-qubit codes

- Each on only horizontal or vertical edges

Example (Toric Bacon strips)

We can decompose these Bacon strips into gauge operators as with the regular Bacon-Shor code.
Non-Abelian quantum double

This is a more complex, non-stabilizer, generalisation

▶ We place qudits on edges, but these are labeled by elements of a non-Abelian finite group
▶ We can think of these codes using an electromagnetic analogy

\[ v \]  

\[ \text{‘Electric’ operators} \quad = A^g(v) \quad (4) \]

\[ v \]  

\[ \text{‘Magnetic’ operators} \quad = B^h(v, p) \quad (5) \]

▶ These operators are labeled by an element of the group
▶ Note that these operators are not analogous to stabilizers as they don’t directly detect errors
Non-Abelian quantum double

- Errors can be thought of as excitations called anyons
  - Anyons are quasiparticles which occur in 2D systems
- The error syndrome corresponds to the ‘charge’ carried by the anyon
- Anyons with electric and magnetic charge are called dyons
- In order to fully describe the model we need operators on a pair of adjacent vertices and plaquettes; a site

\[
D^{h,g} = v_1 p_1 = B^h A^g
\]  

- These operators generate the quantum double algebra
  - Consists of linear combinations of \( D^{h,g} \) operators
In order to detect errors, we need projectors onto anyon charge states.

A bit of group theory

A representation of a group is a set of matrices which obey the multiplication rules of the group. Can think of a set of all possible representations of $G$, with elements labeled by $\Gamma$.

A conjugacy class is the set of all elements $h \in C$ such that $C(h) = g_i h \bar{g}_i$ for all $g_i \in G$. Can think of a set of all conjugacy classes of $G$ labeled by $C$. 
In order to detect errors, we need projectors onto anyon charge states. Electric charge is labelled by $\Gamma$ and magnetic charge is labelled by $C$.

\[
A^\Gamma = \sum_i \frac{d_i}{|G|} \sum_{g \in G} [\Gamma(g)]_{ii} A^g(v)
\]

\[
B_C = \frac{1}{|C|} \sum_{h \in C} B^h(v, p)
\]

We denote projectors onto electric and magnetic charge (dyons) by $D^{\Gamma, C}$.

Projectors are akin to stabilizers (insofar as they detect errors).
Non-Abelian Bacon-Shor quantum double

We have a problem here
We need Bacon strip projectors to detect anyonic charge (errors)

▶ We cannot construct Bacon strips out of products of plaquette and vertex operators as before
▶ Non-Abelian models have much more structure so our construction for the qudit toric code fails

The solution?
A Bacon strip, such as this, is a closed, non-contractible loop.

▶ We need operators like this
Non-Abelian Bacon-Shor quantum double

- We need ribbon operators
- These control the transport of anyons around the lattice
- Dual & Direct

\[
L^g_+ = \sum_{h \in G} |gh\rangle\langle h| \quad T^h_+ = |h\rangle\langle h|
\]

\[
F_{\tau, g}^{h,g} = T_{\tau}^{\bar{g}} = \quad F_{\tau', g}^{h,g} = \delta_{1,g} L_{\tau'}^h
\]

- Dual ribbons transport magnetic charge
- Direct ribbons transport electric charge
Non-Abelian Bacon-Shor quantum double

Topological Bacon strips

- We can construct analogues of the Bacon strips using ribbon operators.

- Note that these operators are not simply products of vertices and plaquettes.
Non-Abelian Bacon-Shor quantum double

Topological Bacon strips

- Note the similarity here to the qudit Toric Bacon strips

Vertical X strip

Horizontal Z strip
Non-Abelian Bacon-Shor quantum double

Projective Bacon strips

▶ We want to construct operators to detect charge (errors) within the code
  ▶ We can take linear combinations of the Bacon strip ribbons to create electric and magnetic projectors
▶ We are really interested in projectors onto dyons
  ▶ Need to project onto electric and magnetic charge
  ▶ But Bacon strips consist of only one kind of charge (colour)
  ▶ This means that they transport or create charge along their length
Non-Abelian Bacon-Shor quantum double

Closed ribbons

▶ We need to use combined direct-dual ribbons

▶ This operator does not affect charge along the strip
▶ This is like a combination of two Bacon strips from the horizontal and vertical codes in the qudit Toric code
  ▶ This is one two qudit code
▶ These are similar to site $D^{h,g}$ operators
Non-Abelian Bacon-Shor quantum double

Next, we want to consider projective versions of these ‘combined’ ribbon operators

Unfortunately . . .

The equation for these projectors is a horrendous mess

$$\mathcal{F}^{\Gamma C}_{\rho_{BS}} = \frac{d_{\Gamma}}{|Z_k|} \sum_{D \in (Z_k)_{c_j}} \chi_{\Gamma}(D) \sum_{\{\prod_k g_k \prod_j g_j = g; \ i,j,k,l\}} \bigotimes_{qg_k \bar{q} \in G; \ \ d \in D; \ \ q \in Q_k}$$

There is no (straightforward) way to decompose this into gauge terms
Non-Abelian Bacon-Shor quantum double

Gauge operators

- ‘Bacon strip’ ribbon operators define a new code
- The Bacon strip ribbon operators are equivalent to the vertex and plaquette operators
- This code has an excitation space which is a subset of that of the original code
  - This is a gauge subspace
- Gauge operators satisfy \([F_{\rho_G}^{h,g}, \mathbb{F}_{\rho_{BS}}^\Gamma C] = 0\)
- Can get two-body gauge operators

\[
F_{\rho_G}^{h,g} = \sum_{g_1 g_2 = g} T_{\tau_1}^{g_1} T_{\tau_2}^{g_2} = \delta_{e,g} L_{\tau_1}^h L_{\tau_2}^{h'}
\]
Summary

- Generalised Bacon-Shor using the qudit toric code.
- Outlined a non-Abelian Bacon-Shor quantum double.