Quantum Double Subsystem Codes

Prashant Kumar

University of Sydney

p.kumar@physics.usyd.edu.au

November 27, 2011

Overview

In a Nutshell

Introduce a new quantum error correcting codes which combine the useful properties of two interesting classes of codes: the Bacon-Shor codes and the quantum double models.

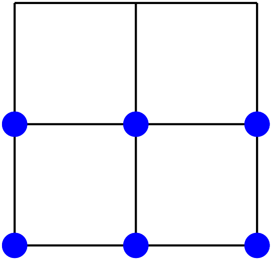
Background

- Bacon-Shor code
- Quantum double models
  - Toric code
  - Non-Abelian quantum double

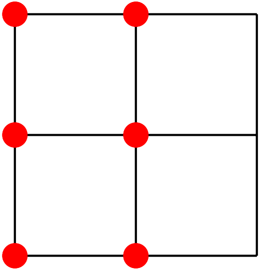
Bacon-Shor code

One particular code which has been shown to have a high error threshold is the Bacon-Shor code, which encodes one logical qubit.

Bacon-Shor stabilizers



X stabilizer



Z stabilizer

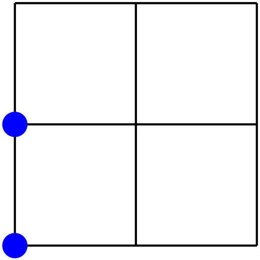
Bacon strips

This is because we can infer syndromes indirectly

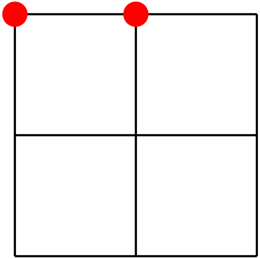
Syndrome measurement in Bacon-Shor

We can infer error syndromes using two-body measurements

Example (Bacon-Shor gauge operators)



X gauge operator

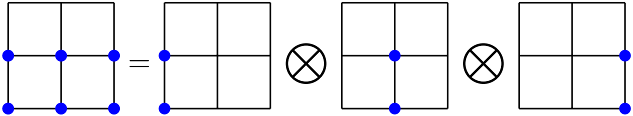


Z gauge operator

These are known as gauge operators.

Bacon-Shor gauge decomposition

By taking products of gauge operators, we can obtain the syndrome associated with a stabilizer



(1)

This is good because

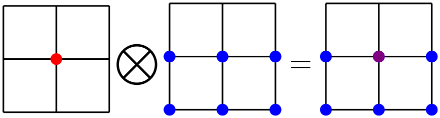
- Lower-weight operators $\Rightarrow$ lower probability of errors
- Can parallelise measurement $\Rightarrow$ advantage in speed means fewer errors
- Simple fault-tolerant gate set to measure these operators

Error correction in Bacon-Shor

Error correction takes an especially simple form.

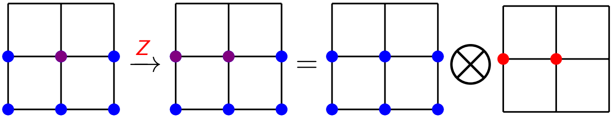
Example

Say we have an Z error on a qubit. This will anticommute with this stabilizer



(2)

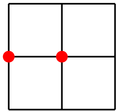
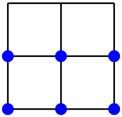
But if we apply a second Z on the qubit shown, we have a gauge operator



(3)

Error correction in Bacon-Shor

Example

The operator  overlaps with  in two places.

Since $-1 \times -1 = 1$ gauge operators commute with stabilizers

$\Rightarrow$ we have corrected the error!

- We call them gauge operators because they do not affect logical information
- $\mathcal{H}_{\mathcal{T}} = \mathcal{H}_{\mathcal{G}} \bigotimes \mathcal{H}_{\mathcal{L}}$
  - They do affect gauge qubits, but we don't store useful information in these

Intermezzo

Research question

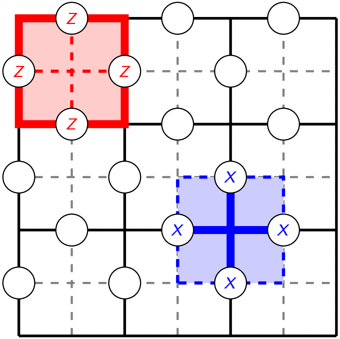
Motivation
- Can we generalise the Bacon-Shor code?

Requirements
- We want our generalisations keep all the nice properties of the Bacon-Shor code
  1. Two-body gauge decomposition
  2. Error correction procedure

What's next?
- We want to consider generalisations using the quantum double models

Toric code

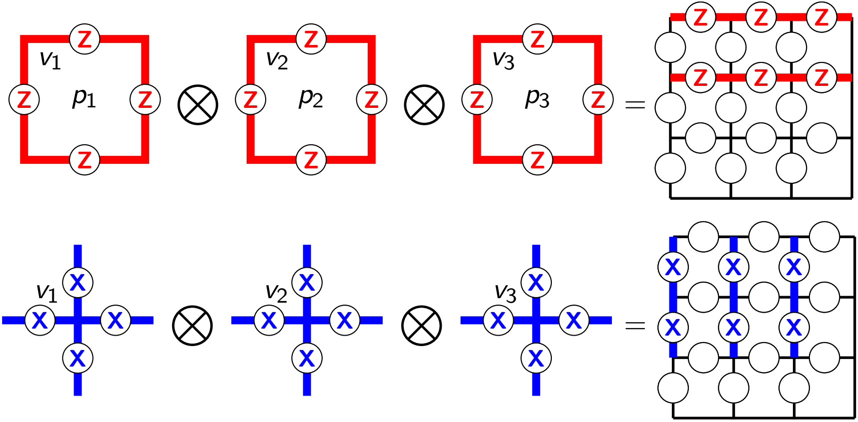
- Defined on a lattice embedded in a torus
- Associate a qubit with each edge
- Encodes two logical qubits



- Z stabilizers are plaquettes; X stabilizers are vertices
- Plaquettes and vertices commute
- Products of stabilizers are also stabilizers
  - This gives us a link with the Bacon-Shor code
  - Toric code stabilizers are Bacon-Shor stabilizers

Bacon-Shor decomposition of the toric code

We can multiply adjacent plaquettes and vertices



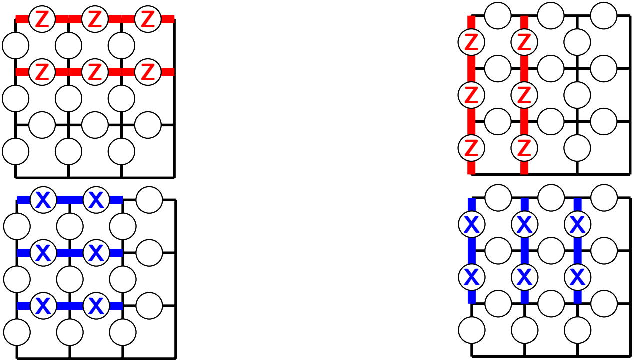
The overlapping edges cancel, since $ZZ = XX = I$, producing Bacon strips. We do the same for the vertical direction as well.

Bacon-Shor decomposition of the toric code

We end up with two one-qubit codes

- Each on only horizontal or vertical edges

Example (Toric Bacon strips)

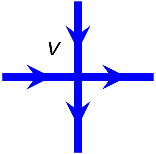


We can decompose these Bacon strips into gauge operators as with the regular Bacon-Shor code.

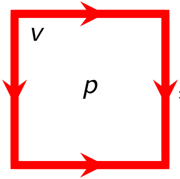
Non-Abelian quantum double

This is a more complex, non-stabilizer, generalisation

- We place qudits on edges, but these are labeled by elements of a non-Abelian finite group
- We can think of these codes using an electromagnetic analogy

‘Electric’ operators 

$= A^g(v)$ (4)

‘Magnetic’ operators 

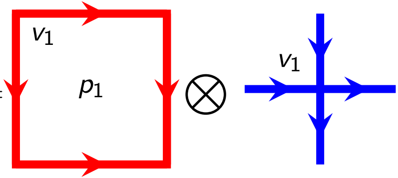
$= B^h(v, p)$ (5)

- These operators are labeled by an element of the group
- Note that these operators are not analogous to stabilizers as they don’t directly detect errors

Non-Abelian quantum double

- Errors can be thought of as excitations called anyons
  - Anyons are quasiparticles which occur in 2D systems
- The error syndrome corresponds to the ‘charge’ carried by the anyon
- Anyons with electric *and* magnetic charge are called dyons
- In order to fully describe the model we need operators on a pair of adjacent vertices and plaquettes; a site

$D^{h,g} =$



$= B^h A^g \qquad (6)$

- These operators generate the quantum double algebra
  - Consists of linear combinations of $D^{h,g}$ operators

Non-Abelian quantum double

- In order to detect errors, we need projectors onto anyon charge states

A bit of group theory

A representation
: of a group is a set of matrices which obey the multiplication rules of the group. Can think of a set of all possible representations of $\mathcal{G}$, with elements labeled by Γ

A conjugacy class
: is the set of all elements $h \in C$ such that $C(h) = g_i h \overline{g}_i$ for all $g_i \in \mathcal{G}$. Can think of a set of all conjugacy classes of $\mathcal{G}$ labeled by C

Non-Abelian quantum double

- In order to detect errors, we need projectors onto anyon charge states
- Electric charge is labelled by Γ
- Magnetic charge is labelled by C

$$A^{\Gamma} = \sum_{i} \frac{d_{\Gamma}}{|\mathcal{G}|} \sum_{g \in \mathcal{G}} [\Gamma(g)]_{ii} A^{g}(v) \tag{7}$$

$$B_{C} = \frac{1}{|\mathcal{C}|} \sum_{h \in C} B^{h}(v, p) \tag{8}$$

- We denote projectors onto electric and magnetic charge (dyons) by $D^{\Gamma, C}$
- Projectors are akin to stabilizers (insofar as they detect errors)

Non-Abelian Bacon-Shor quantum double

We have a problem here

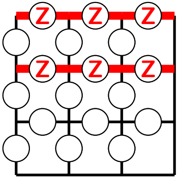
We need Bacon strip projectors to detect anyonic charge (errors)

- We cannot construct Bacon strips out of products of plaquette and vertex operators as before
- Non-Abelian models have much more structure so our construction for the qudit toric code fails

The solution?

A Bacon strip, such as this, is a closed, non-contractible loop.

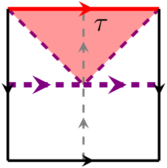
- We need operators like this

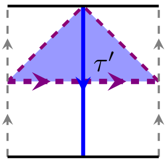


Non-Abelian Bacon-Shor quantum double

- We need *ribbon operators*
- These control the transport of anyons around the lattice
- Dual & Direct

$$L_+^g = \sum_{h \in \mathcal{G}} |gh\rangle\langle h| \qquad T_+^h = |h\rangle\langle h| \tag{9}$$

$F_\tau^{h,g} = T_\tau^{\overline{g}} =$ 

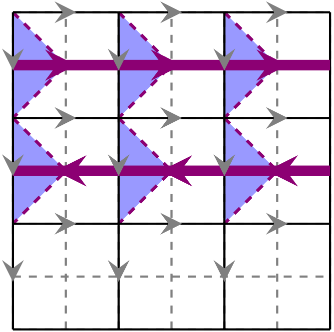
$F_{\tau'}^{h,g} = \delta_{1,g} L_{\tau'}^h =$ 

- Dual ribbons transport magnetic charge
- Direct ribbons transport electric charge

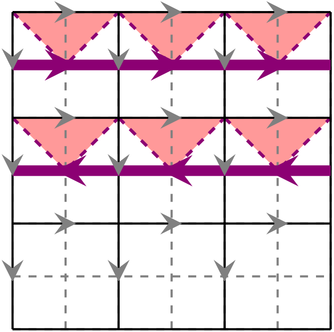
Non-Abelian Bacon-Shor quantum double

Topological Bacon strips

- We can construct analogues of the Bacon strips using ribbon operators



Magnetic ribbon



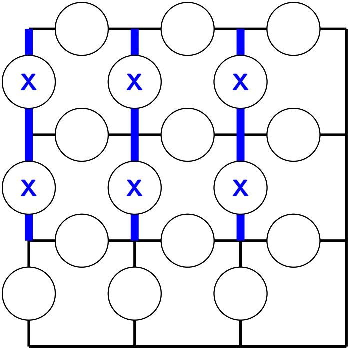
Electric ribbon

- Note that these operators are not simply products of vertices and plaquettes

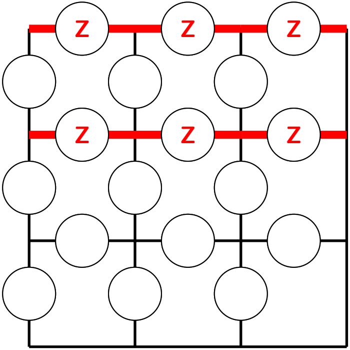
Non-Abelian Bacon-Shor quantum double

Topological Bacon strips

- Note the similarity here to the qudit Toric Bacon strips



Vertical X strip



Horizontal Z strip

Non-Abelian Bacon-Shor quantum double

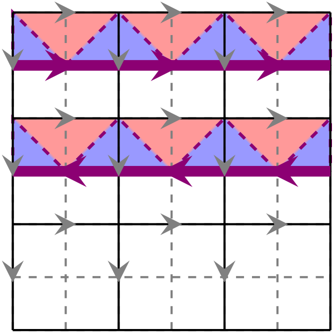
Projective Bacon strips

- We want to construct operators to detect charge (errors) within the code
  - We can take linear combinations of the Bacon strip ribbons to create electric and magnetic projectors
- We are really interested in projectors onto dyons
  - Need to project onto electric *and* magnetic charge
  - But Bacon strips consist of only one kind of charge (colour)
  - This means that they transport or create charge along their length

Non-Abelian Bacon-Shor quantum double

Closed ribbons

- We need to use *combined* direct-dual ribbons



- This operator does not affect charge along the strip
- This is like a combination of two Bacon strips from the horizontal and vertical codes in the qudit Toric code
  - This is one *two* qudit code
- These are similar to site $D^{h,g}$ operators

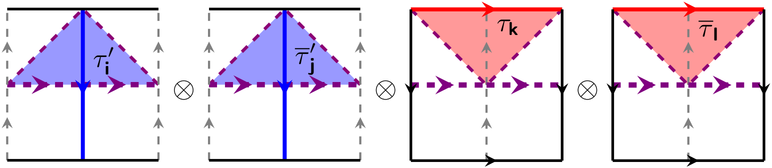
Non-Abelian Bacon-Shor quantum double

Next, we want to consider projective versions of these 'combined' ribbon operators

Unfortunately...

The equation for these projectors is a horrendous mess

$$\mathbb{F}^{\Gamma C}_{\rho_{BS}} = \frac{d_\Gamma}{|Z_k|} \sum_{D\in(Z_k)_{cj}} \overline{\chi}_\Gamma(D) \sum_{\substack{\{\prod_k g_k \prod_j g_j = g;\\ q g_k \overline{q} \in \mathcal{G};\\ d\in D;\\ q\in Q_k\}}} \bigotimes_{i,j,k,l}$$



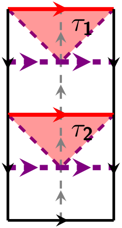
- There is no (straightforward) way to decompose this into gauge terms

Non-Abelian Bacon-Shor quantum double

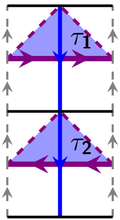
Gauge operators

- 'Bacon strip' ribbon operators define a new code
- The Bacon strip ribbon operators are equivalent to the vertex and plaquette operators
- This code has an excitation space which is a subset of that of the original code
  - This is a *gauge subspace*
- *Gauge operators* satisfy $[F^{h,g}_{\rho_G}, \mathbb{F}^{\Gamma C}_{\rho_{BS}}] = 0$
- Can get two-body gauge operators

$$F^{h,g}_{\rho_G} = \sum_{g_1 g_2 = g} T^{\overline{g}_1}_{\tau_1} T^{\overline{g}_2}_{\tau_2} =$$



$$F^{h,g}_{\rho_G} = \delta_{e,g}\, L^{h}_{\tau_1'}\, L^{\overline{h}}_{\tau_2'} =$$



Summary

- Generalised Bacon-Shor using the qudit toric code.
- Outlined a non-Abelian Bacon-Shor quantum double.