

Quantum Double Subsystem Codes

Prashant Kumar

University of Sydney

`p.kumar@physics.usyd.edu.au`

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Overview

In a Nutshell

Introduce a new quantum error correcting codes which combine the useful properties of two interesting classes of codes: the Bacon-Shor codes and the quantum double models.

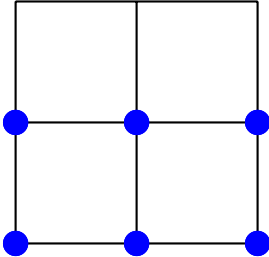
Background

- ▶ Bacon-Shor code
- ▶ Quantum double models
 - ▶ Toric code
 - ▶ Non-Abelian quantum double

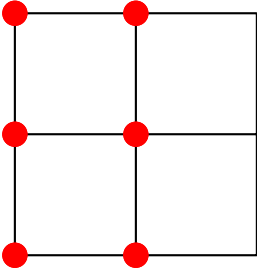
Bacon-Shor code

One particular code which has been shown to have a high error threshold is the Bacon-Shor code, which encodes one logical qubit.

Bacon-Shor stabilizers



X stabilizer



Z stabilizer

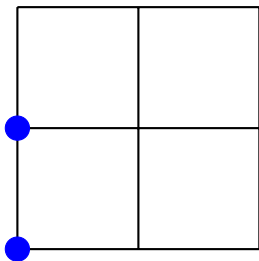
Bacon strips

This is because we can infer syndromes indirectly

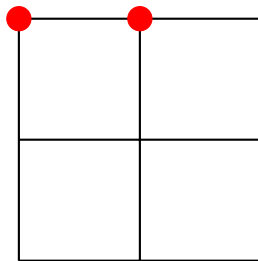
Syndrome measurement in Bacon-Shor

We can infer error syndromes using two-body measurements

Example (Bacon-Shor gauge operators)



X gauge operator



Z gauge operator

These are known as **gauge operators**.

Bacon-Shor gauge decomposition

By taking products of gauge operators, we can obtain the syndrome associated with a stabilizer

The diagram illustrates the decomposition of a Bacon-Shor stabilizer operator into a product of three weight-2 operators. On the left, a 2x3 grid of qubits has blue dots at all six positions. This is equal to the product of three operators, each represented by a 2x2 grid with a circled 'X' in the center. The first operator has blue dots at the left and right positions of the top row. The second operator has blue dots at the top and bottom positions of the right column. The third operator has blue dots at the right and left positions of the bottom row.

$$\text{Stabilizer} = \text{Operator 1} \otimes \text{Operator 2} \otimes \text{Operator 3} \quad (1)$$

This is good because

- ▶ Lower-weight operators \Rightarrow lower probability of errors
- ▶ Can parallelise measurement \Rightarrow advantage in speed means fewer errors
- ▶ Simple fault-tolerant gate set to measure these operators

Error correction in Bacon-Shor

Error correction takes an especially simple form.

Example

Say we have an Z error on a qubit. This will anticommute with this stabilizer

Equation (2) illustrates the commutation of a Z error with a stabilizer. On the left, a 2x2 grid has a red dot in the center. This is multiplied by a 3x3 grid of blue dots. The result is a 3x3 grid where the center dot is purple and all other dots are blue.

$$\begin{array}{|c|c|} \hline & \\ \hline \cdot & \\ \hline & \\ \hline \end{array} \otimes \begin{array}{|c|c|c|} \hline \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot \\ \hline \end{array} \quad (2)$$

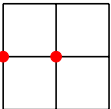
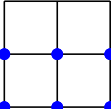
But if we apply a second Z on the qubit shown, we have a gauge operator

Equation (3) shows the application of a second Z error to a stabilizer. A 3x3 grid of blue dots with a purple center dot is transformed by a Z error (indicated by a red arrow) into a 3x3 grid of blue dots with a purple center dot. This is then multiplied by another 3x3 grid of blue dots, resulting in a 3x3 grid of blue dots with a purple center dot, which is then multiplied by a 2x2 grid with red dots in the center.

$$\begin{array}{|c|c|c|} \hline \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot \\ \hline \end{array} \xrightarrow{Z} \begin{array}{|c|c|c|} \hline \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot \\ \hline \end{array} \otimes \begin{array}{|c|c|} \hline & \\ \hline \cdot & \cdot \\ \hline & \\ \hline \end{array} \quad (3)$$

Error correction in Bacon-Shor

Example

The operator  overlaps with  in two places.

Since $-1 \times -1 = 1$ gauge operators commute with stabilizers

\Rightarrow we have corrected the error!

- ▶ We call them gauge operators because they do not affect logical information
- ▶ $\mathcal{H}_{\mathcal{T}} = \mathcal{H}_{\mathcal{G}} \otimes \mathcal{H}_{\mathcal{L}}$
 - ▶ They do affect gauge qubits, but we don't store useful information in these

Intermezzo

Research question

Motivation ▶ Can we generalise the Bacon-Shor code?

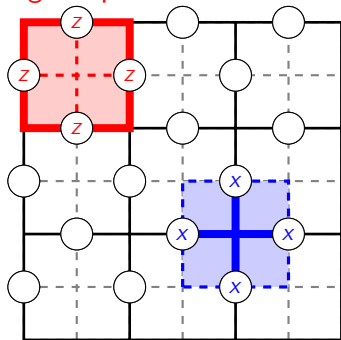
Requirements ▶ We want our generalisations keep all the nice properties of the Bacon-Shor code

1. Two-body gauge decomposition
2. Error correction procedure

What's next? ▶ We want to consider generalisations using the quantum double models

Toric code

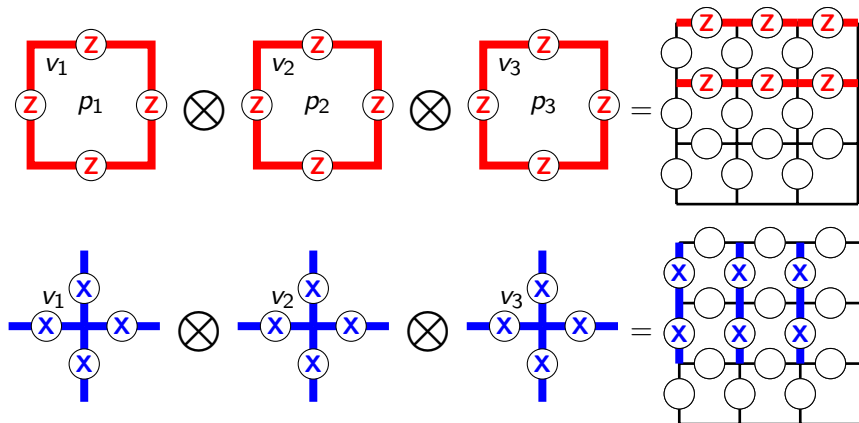
- ▶ Defined on a lattice embedded in a torus
- ▶ Associate a qubit with each edge
- ▶ Encodes **two logical qubits**



- ▶ **Z** stabilizers are **plaquettes**; **X** stabilizers are **vertices**
- ▶ **Plaquettes** and **vertices** commute
- ▶ Products of stabilizers are also stabilizers
 - ▶ This gives us a link with the Bacon-Shor code
 - ▶ Toric code stabilizers are Bacon-Shor stabilizers

Bacon-Shor decomposition of the toric code

We can multiply adjacent **plaquettes** and **vertices**



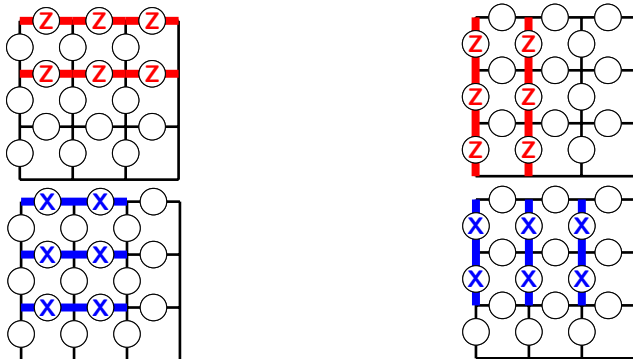
The overlapping edges cancel, since $ZZ = XX = I$, producing Bacon strips. We do the same for the vertical direction as well.

Bacon-Shor decomposition of the toric code

We end up with two one-qubit codes

- ▶ Each on only horizontal or vertical edges

Example (Toric Bacon strips)

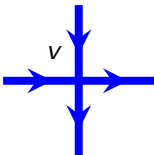


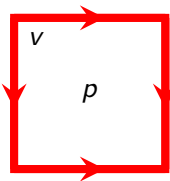
We can decompose these Bacon strips into gauge operators as with the regular Bacon-Shor code.

Non-Abelian quantum double

This is a more complex, non-stabilizer, generalisation

- ▶ We place qudits on edges, but these are labeled by elements of a non-Abelian finite group
- ▶ We can think of these codes using an electromagnetic analogy

'Electric' operators  = $A^g(v)$ (4)

'Magnetic' operators  = $B^h(v, p)$ (5)

- ▶ These operators are labeled by an element of the group
- ▶ Note that these operators are not analogous to stabilizers as they don't directly detect errors

Non-Abelian quantum double

- ▶ Errors can be thought of as excitations called **anyons**
 - ▶ Anyons are quasiparticles which occur in 2D systems
- ▶ The error syndrome corresponds to the 'charge' carried by the anyon
- ▶ Anyons with **electric** and **magnetic** charge are called dyons
- ▶ In order to fully describe the model we need operators on a pair of adjacent vertices and plaquettes; a site

$$D^{h,g} = \begin{array}{c} \text{---} \xrightarrow{\quad} \text{---} \\ \uparrow \text{ } \downarrow \\ \text{ } \text{ } \\ \downarrow \text{ } \uparrow \\ \text{---} \xleftarrow{\quad} \text{---} \end{array} \otimes \begin{array}{c} \downarrow \\ \uparrow \\ \leftarrow \\ \rightarrow \\ \downarrow \end{array} = B^h A^g \quad (6)$$

The diagram shows the operator $D^{h,g}$ as a tensor product of two operators. The first operator is a red square with arrows on all four sides pointing clockwise. The top edge is labeled v_1 and the center is labeled p_1 . The second operator is a blue cross with arrows pointing outwards from the center along all four axes. The top edge of the cross is labeled v_1 . The two operators are connected by a tensor product symbol \otimes . The result of the tensor product is labeled $B^h A^g$.

- ▶ These operators generate the quantum double algebra
 - ▶ Consists of linear combinations of $D^{h,g}$ operators

Non-Abelian quantum double

- ▶ In order to detect errors, we need **projectors** onto anyon charge states

A bit of group theory

A representation of a group is a set of matrices which obey the multiplication rules of the group.
Can think of a set of all possible representations of \mathcal{G} , with elements labeled by Γ

A conjugacy class is the set of all elements $h \in C$ such that $C(h) = g_i h g_i^{-1}$ for all $g_i \in \mathcal{G}$.
Can think of a set of all conjugacy classes of \mathcal{G} labeled by C

Non-Abelian quantum double

- ▶ In order to detect errors, we need **projectors** onto anyon charge states
- ▶ **Electric** charge is labelled by Γ
- ▶ **Magnetic** charge is labelled by C

$$A^\Gamma = \sum_i \frac{d_\Gamma}{|\mathcal{G}|} \sum_{g \in \mathcal{G}} [\Gamma(g)]_{ii} A^g(v) \quad (7)$$

$$B_C = \frac{1}{|C|} \sum_{h \in C} B^h(v, p) \quad (8)$$

- ▶ We denote projectors onto electric and magnetic charge (dyons) by $D^{\Gamma, C}$
- ▶ Projectors are akin to stabilizers (insofar as they detect errors)

Non-Abelian Bacon-Shor quantum double

We have a problem here

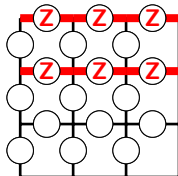
We need Bacon strip projectors to detect anyonic charge (errors)

- ▶ We cannot construct Bacon strips out of products of **plaquette** and **vertex** operators as before
- ▶ Non-Abelian models have much more structure so our construction for the qudit toric code fails

The solution?

A Bacon strip, such as this, is a closed, non-contractible loop.

- ▶ We need operators like this



Non-Abelian Bacon-Shor quantum double

- ▶ We need *ribbon operators*
- ▶ These control the transport of anyons around the lattice
- ▶ Dual & Direct

$$L_+^g = \sum_{h \in \mathcal{G}} |gh\rangle \langle h| \quad T_+^h = |h\rangle \langle h| \quad (9)$$

$$F_{\tau}^{h,g} = T_{\tau}^{\bar{g}} = \text{[Diagram: A square with a red shaded triangle at the top. A dashed red line forms the boundary of the triangle. A solid red arrow labeled \tau points down from the top vertex. A dashed purple line with arrows points left from the top-left vertex, right from the top-right vertex, and down from the bottom vertex. A dashed grey line with arrows points up from the bottom vertex and down from the top vertex.]}$$

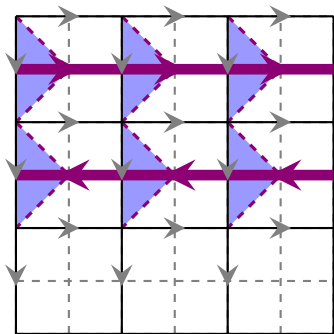
$$F_{\tau'}^{h,g} = \delta_{1,g} L_{\tau'}^h = \text{[Diagram: A square with a blue shaded triangle at the top. A dashed blue line forms the boundary of the triangle. A solid blue arrow labeled \tau' points down from the top vertex. A dashed purple line with arrows points left from the top-left vertex, right from the top-right vertex, and down from the bottom vertex. A dashed grey line with arrows points up from the bottom vertex and down from the top vertex.]}$$

- ▶ Dual ribbons transport magnetic charge
- ▶ Direct ribbons transport electric charge

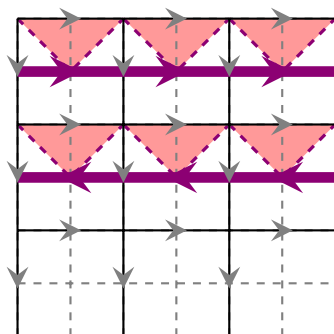
Non-Abelian Bacon-Shor quantum double

Topological Bacon strips

- ▶ We can construct analogues of the Bacon strips using ribbon operators



Magnetic ribbon



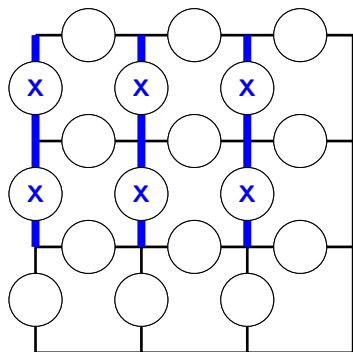
Electric ribbon

- ▶ Note that these operators are not simply products of vertices and plaquettes

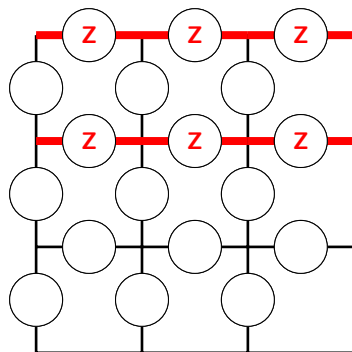
Non-Abelian Bacon-Shor quantum double

Topological Bacon strips

- ▶ Note the similarity here to the qudit Toric Bacon strips



Vertical X strip



Horizontal Z strip

Non-Abelian Bacon-Shor quantum double

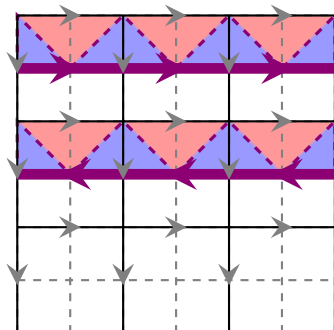
Projective Bacon strips

- ▶ We want to construct operators to detect charge (errors) within the code
 - ▶ We can take linear combinations of the Bacon strip ribbons to create electric and magnetic projectors
- ▶ We are really interested in projectors onto dyons
 - ▶ Need to project onto **electric** and **magnetic** charge
 - ▶ But Bacon strips consist of only one kind of charge (colour)
 - ▶ This means that they transport or create charge along their length

Non-Abelian Bacon-Shor quantum double

Closed ribbons

- ▶ We need to use *combined direct-dual* ribbons



- ▶ This operator does not affect charge along the strip
- ▶ This is like a combination of two Bacon strips from the horizontal and vertical codes in the qudit Toric code
 - ▶ This is one *two* qudit code
- ▶ These are similar to site $D^{h,g}$ operators

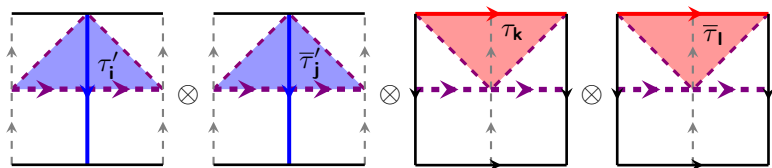
Non-Abelian Bacon-Shor quantum double

Next, we want to consider projective versions of these 'combined' ribbon operators

Unfortunately...

The equation for these projectors is a horrendous mess

$$\mathbb{F}_{\rho_{BS}}^{\Gamma C} = \frac{d_{\Gamma}}{|Z_k|} \sum_{D \in (Z_k)_{C_j}} \bar{\chi}_{\Gamma}(D) \sum_{\{\prod_k g_k \prod_j g_j = g; \ i,j,k,l; \\ qg_k \bar{q} \in \mathcal{G}; \\ d \in D; \\ q \in Q_k\}} \otimes$$

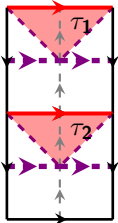
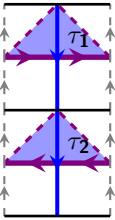


- ▶ There is no (straightforward) way to decompose this into gauge terms

Non-Abelian Bacon-Shor quantum double

Gauge operators

- ▶ 'Bacon strip' ribbon operators define a new code
- ▶ The Bacon strip ribbon operators are equivalent to the vertex and plaquette operators
- ▶ This code has an excitation space which is a subset of that of the original code
 - ▶ This is a *gauge subspace*
- ▶ Gauge operators satisfy $[F_{\rho_G}^{h,g}, \mathbb{F}_{\rho_{BS}}^{\Gamma C}] = 0$
- ▶ Can get two-body gauge operators

$$F_{\rho_G}^{h,g} = \sum_{g_1 g_2 = g} T_{\tau_1}^{\bar{g}_1} T_{\tau_2}^{\bar{g}_2} =$$

$$F_{\rho_G}^{h,g} = \delta_{e,g} L_{\tau_1}^h L_{\tau_2}^{\bar{h}} =$$


Summary

- ▶ Generalised Bacon-Shor using the qudit toric code.
- ▶ Outlined a non-Abelian Bacon-Shor quantum double.