3D local qupit quantum code without string logical operator

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Energy barriers of local quantum error correcting codes

- 2D : $O(1)$ (particle-like excitations)
- 4D : $O(L)$ (closed string-like excitations)
- 3D
  - 3D toric code family and variants : $O(1)$ (particle & string)
  - Haah's code : $O(\log L)$ (Bravyi, Haah 2011) (need to create extra particles to move particles)

Recap

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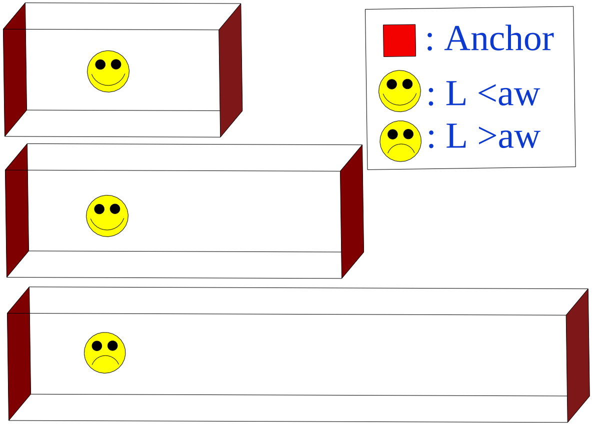
  - Answer : Existence of constant aspect ratio a



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Properties	Haah's code	Our code
Particle dimension	2	Prime
Particles/site	2	1
Generators/cube	2	1

Instead of Paulis...

Generalized Shift Operator X

$$X = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & 0 \end{bmatrix}$$

Generalized Phase Operator Z

$$Z = \begin{bmatrix} \omega & 0 & 0 & \dots & 0 \\ 0 & \omega^2 & 0 & \dots & 0 \\ 0 & 0 & \omega^3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \omega^d \end{bmatrix}$$

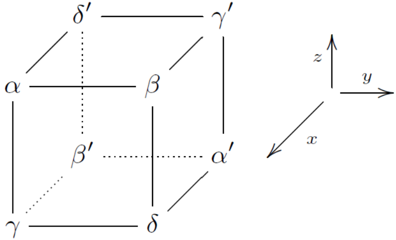
$$\omega = e^{\frac{2\pi i}{d}}$$

$$(X^{\alpha_1} Z^{\alpha_2})(X^{\beta_1} Z^{\beta_2}) = (X^{\beta_1} Z^{\beta_2})(X^{\alpha_1} Z^{\alpha_2})\omega^{\langle \alpha, \beta \rangle}$$

$$\text{Symplectic Product} : \langle \alpha, \beta \rangle = \alpha_1 \beta_2 - \beta_1 \alpha_2$$

Stabilizer generator

U=



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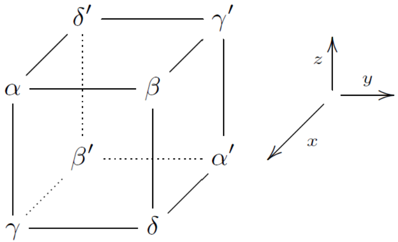

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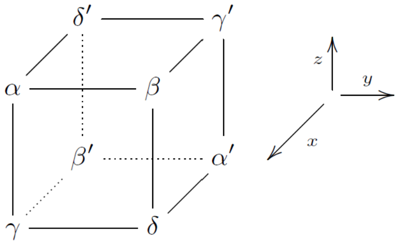

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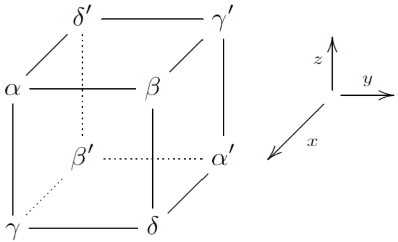

- $\alpha = (\alpha_1, \alpha_2)$ represents $X^{\alpha_1} Z^{\alpha_2}$
- Translation of U in 3 directions
- Periodic Boundary Condition
- Unitary, but not hermitian
- $H = -\sum(U + U^\dagger)$

Constraints

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  - Stabilizer generators should commute with each other.
- Absence of string logical operator
  - Deformability : sharp boundaries of logical operator can be deformed smoothly
  - Constant aspect ratio : finite segments of logical string operator cannot get too long.

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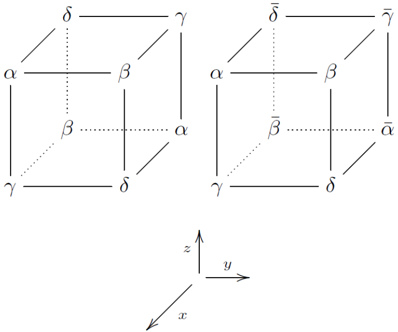

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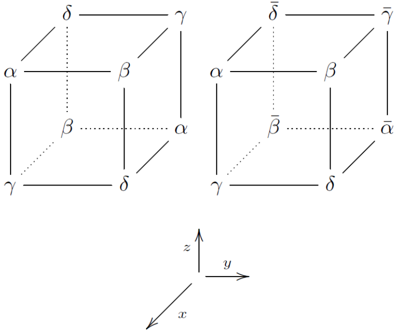

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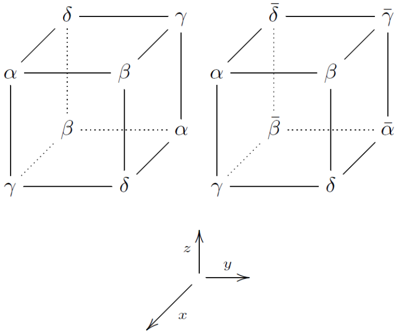

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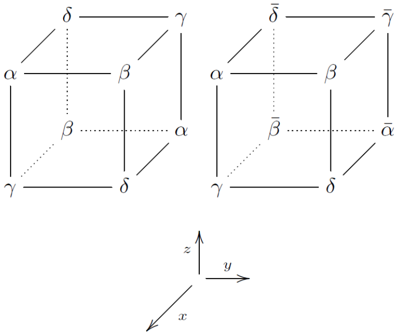
- Commutation & deformability implies inversion symmetric/antisymmetric stabilizer generators: $\langle A, B\rangle \neq 0$ for $A \neq B \in \{\alpha, \beta, \gamma, \delta\}$
- Symmetric, Antisymmetric code $= (\mathcal{C}_S^{\alpha\beta\gamma\delta}, \mathcal{C}_A^{\alpha\beta\gamma\delta})$

Equivalence Relations



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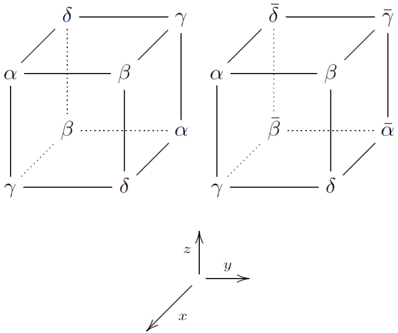

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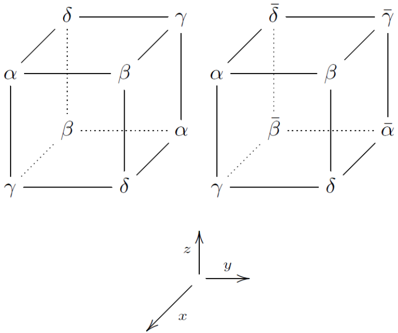

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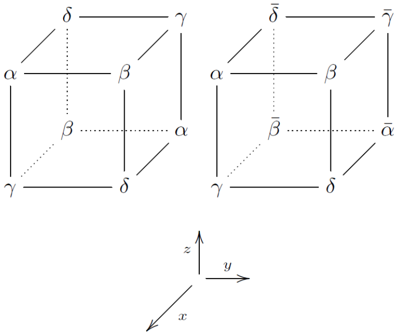

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- Local Clifford Transformation
  - $SL(2, d)$
- $\mathcal{C}_S^{\alpha\beta\gamma\delta} \cong \mathcal{C}_A^{\alpha\beta\gamma\delta}$ in the bulk
  - Not so with periodic boundary condition in general.

Equivalence Relations

- Lattice Symmetry
  - $\mathcal{C}_{S,A}^{\alpha\beta\gamma\delta} \cong \mathcal{C}_{S,A}^{S}$, $S = \{\alpha, \beta, \gamma, \delta\}$
- Local Clifford Transformation
  - $\mathcal{C}_{S,A}^{S} \cong \mathcal{C}_{S,A}^{S'}$ for $S' = aS$, $a \in SL(2, d)$.
- Bulk equivalence of symmetric and antisymmetric code
  - It suffices to check the absence of string logical operator for only one of them.

Main Result : Sufficient condition for finite aspect ratio

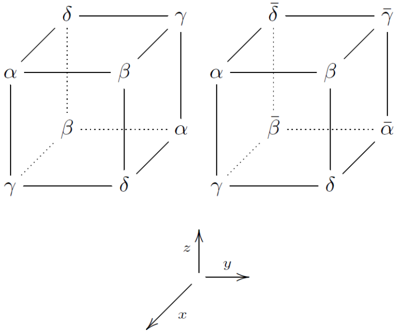
Theorem : Following three conditions on $S = \{\alpha, \beta, \gamma, \delta\}$ imply aspect ratio of 5 for $\mathcal{C}^{S}_{S,A}$.

- Deformability : $\langle A, B\rangle \neq 0 \ \forall A \neq B,\ A, B \in S$.
- Absence of width $w = 1$ string logical operator.
- $\langle A, B\rangle^2 \neq \langle C, D\rangle^2 \ \forall A, B, C, D \in S$. A,B,C,D are distinct.

Observations

- Any $d = 2, 3$ code do not satisfy the condition.
- When $d = 5$, $S = \{(1, 0), (0, 1), (1, 1), (3, -3)\}$ satisfies the condition.
- For sufficiently large d, there is always a code that satisfies the condition.
- Such codes have a logarithmic energy barrier for logical error (Bravyi, Haah 2011)

Encoded Qudits



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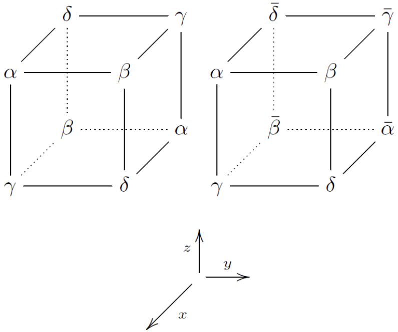

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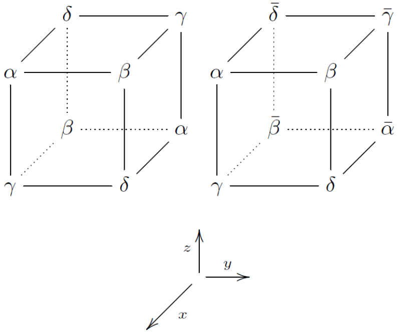

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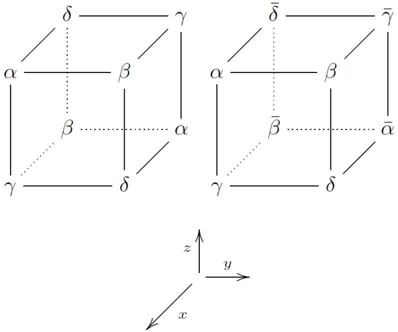

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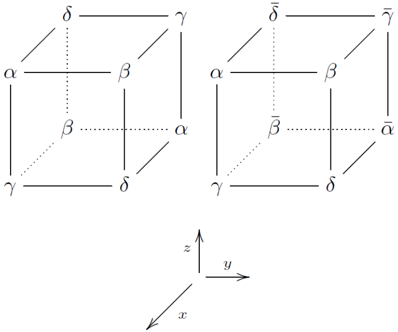

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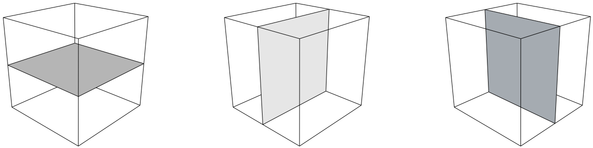

Conclusion & Open Problems

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  - Logarithmic energy barrier (from finite aspect ratio)
  - Ground state degeneracy changes with system size.
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