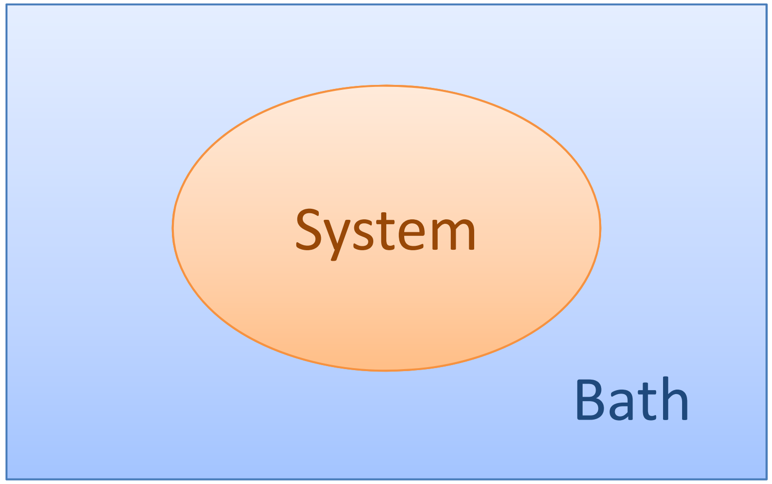
Universal Dynamical Decoupling and Quantum Walks in Functional Spaces

Liang Jiang
Caltech, IQIM
In collaboration with: Adilet Imambekov (Rice Univ.)
2011.12.7.

arXiv:1104.5021
(also see related work by Kuo & Lidar aXi:1106.2151)

System-Bath Interaction



Universal Dynamical Decoupling (DD): How to efficiently decouple system from bath for *arbitrary* system-bath interaction?

- The system-bath interaction (e.g., a qubit system)

$$H_{SB} = \hat{I} \otimes \hat{B}_0 + \hat{S}_x \otimes \hat{B}_x + \hat{S}_y \otimes \hat{B}_y + \hat{S}_z \otimes \hat{B}_z$$

Unitary evolution $U = \mathcal{T} e^{-i \int_0^T H_{SB} d\tau}$ entangles system & bath and cause decoherence of the system.

- **Dynamical Decoupling:** Introduce system evolution (e.g., using pulses X, Y, Z)

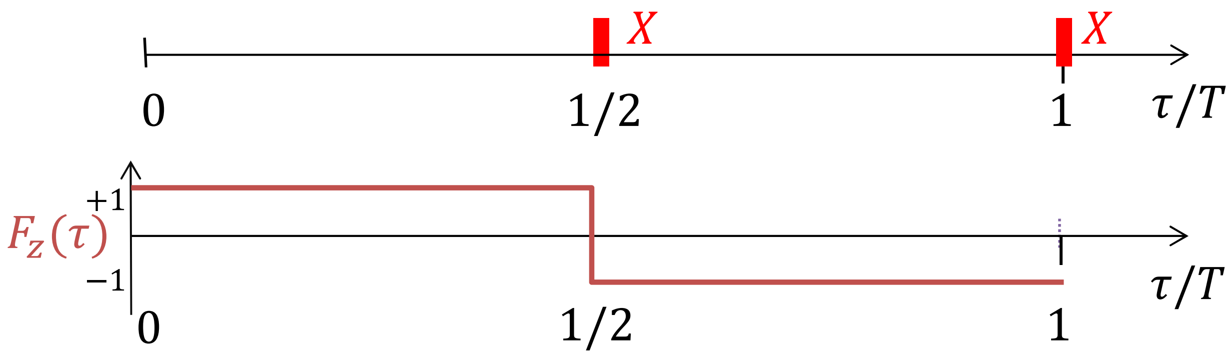
$$H_C(\tau) = \Omega_x(\tau)\hat{S}_x + \Omega_y(\tau)\hat{S}_y + \Omega_z(\tau)\hat{S}_z$$

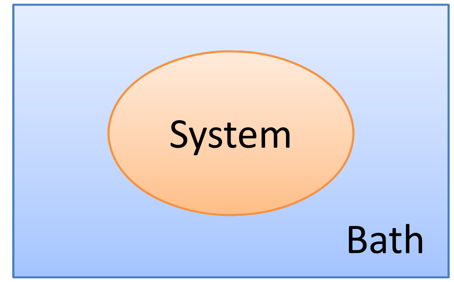
to decouple the system-bath interaction

$$U = \mathcal{T} e^{-i \int_0^T (H_{SB} + H_C) dt} \approx \hat{I} \otimes U_B$$

Hahn Echo Sequence

Hahn Echo sequence:





- The system-bath interaction (for a qubit system)

$$H_{SB} = \hat{I} \otimes \hat{B}_0 + \hat{S}_z \otimes \hat{B}_z$$

- Toggling frame Hamiltonian

$$H(\tau) = \hat{I} \otimes \hat{B}_0 + F_z(\tau)\hat{S}_z \otimes \hat{B}_z$$

- Unitary Evolution

$$U = \mathfrak{T} e^{-i\int_0^T H(\tau)d\tau}$$

$$= e^{-i(\hat{I}\otimes\hat{B}_0 - \hat{S}_z\otimes\hat{B}_z)T/2} e^{-i(\hat{I}\otimes\hat{B}_0 + \hat{S}_z\otimes\hat{B}_z)T/2}$$

$$\approx \hat{I} \otimes U_B + O(T^2)$$

Universal DD Sequences

- The system-bath interaction

$$H_{SB} = \sum_{\alpha} \hat{S}_{\alpha} \otimes \hat{B}_{\alpha}$$

- Toggling frame Hamiltonian (associated with DD sequence)

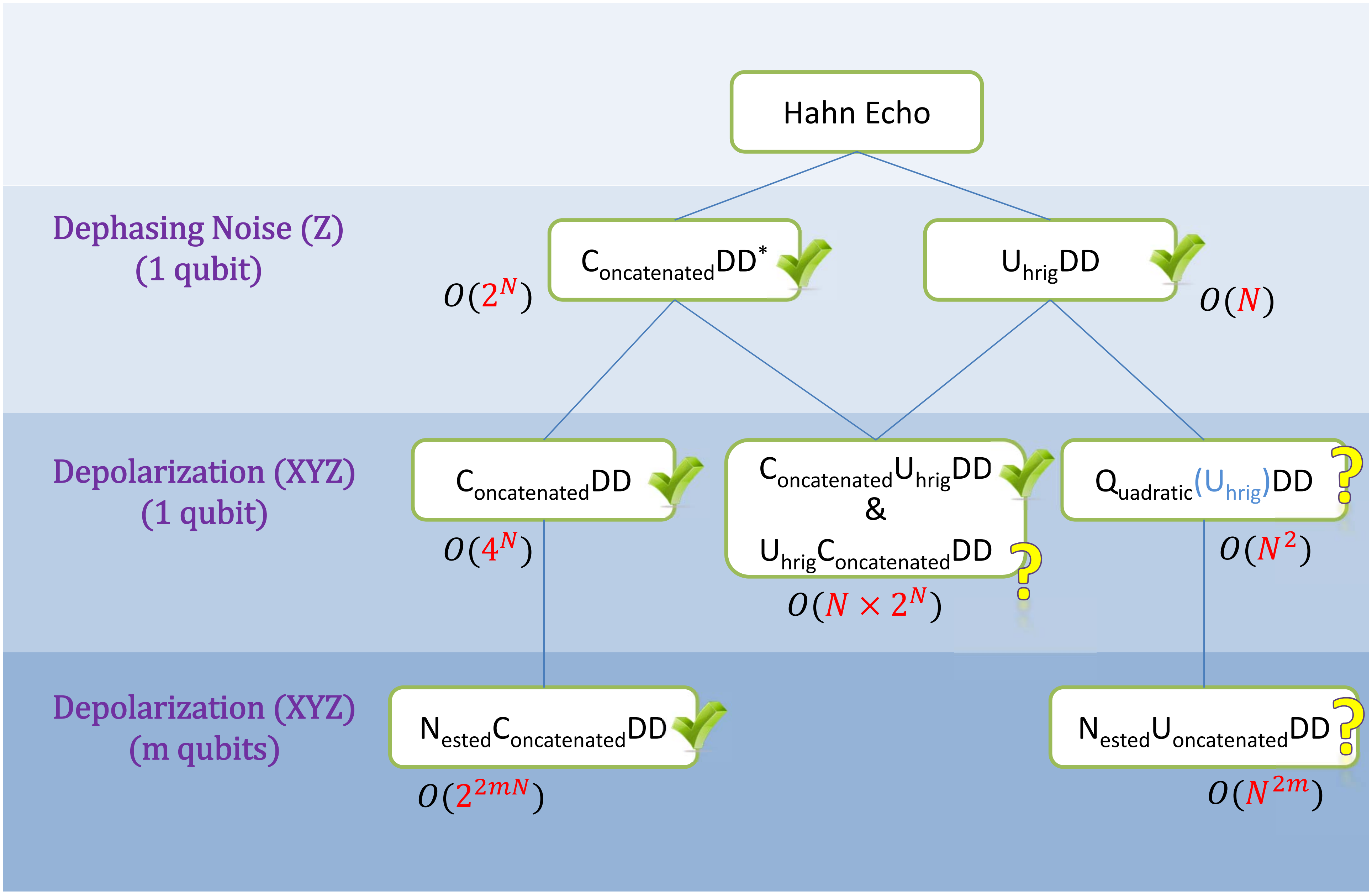
$$H(\tau) = \sum_{\alpha} F_{\alpha}(\tau) \hat{S}_{\alpha} \otimes \hat{B}_{\alpha}$$

- Unitary Evolution

$$U = \mathcal{T} e^{-i \int_0^T H(\tau) d\tau} = \hat{I} \otimes U_B + O(T^{N+1})$$

DD Sequence	Performance	# of pulses	Refs
$\mathrm{C_{oncatenated}DD}$	XYZ, 1 qubit	✓ $O(4^N)$	Khodjasteh & Lidar (05’)
$\mathrm{U_{hrig}DD}$	Z, 1 qubit	✓ $O(N)$	Uhrig (07’), Yang & Liu (08’)
$\mathrm{C_{oncatenated}U_{hrig}DD}$	XYZ, 1 qubit	✓ $O(2^N N)$	Uhrig (09’)
$\mathrm{U_{hrig}C_{oncatenated}DD}$	XYZ, 1 qubit	? $O(2^N N)$	L.J. & Imambekov (11’)
$\mathrm{Q_{uadratic}DD}$	XYZ, 1 qubit	? $O(N^2)$	West & Lidar (10’)
$\mathrm{N_{ested}U_{hrig}DD}$	XYZ, m qubits	? $O(N^{2m})$	Mukhtar et al. (10’); Wang & Liu (11’)

Family Tree of Universal DD Sequences



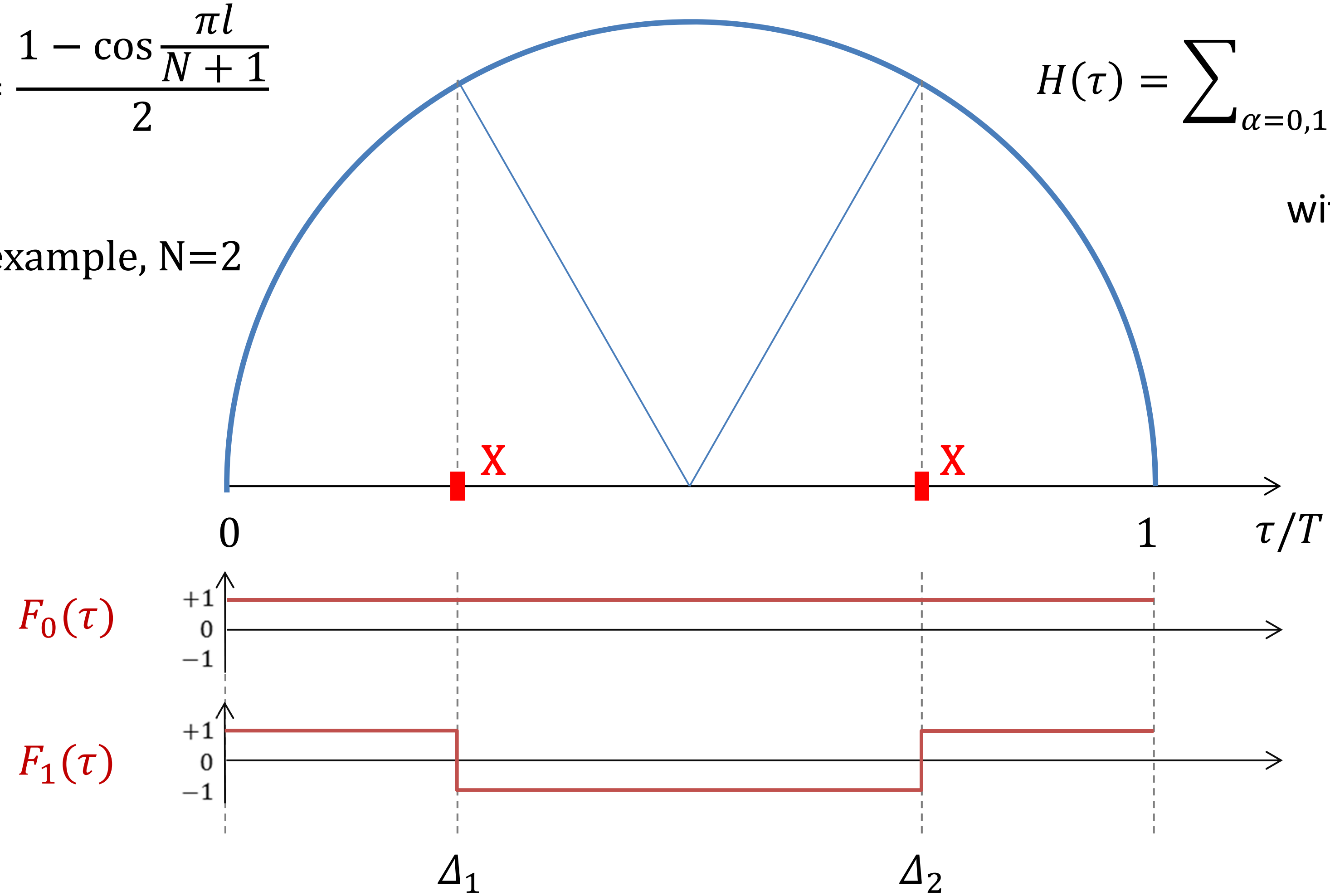
Uhrig DD Sequence

$$\Delta_l = \frac{1 - \cos\frac{\pi l}{N+1}}{2}$$

For example, N=2

$$H(\tau) = \sum_{\alpha=0,1} F_\alpha(\tau)\hat{S}_\alpha \otimes \hat{B}_\alpha$$

with $\alpha \in \{0,1\}$



$S_0 = I$

$S_1 = S_z$

$$F_\alpha(\tau)F_{\alpha'}(\tau) = F_{\alpha\oplus\alpha'}(\tau)$$

$$S_\alpha S_{\alpha'} = S_{\alpha\oplus\alpha'}$$

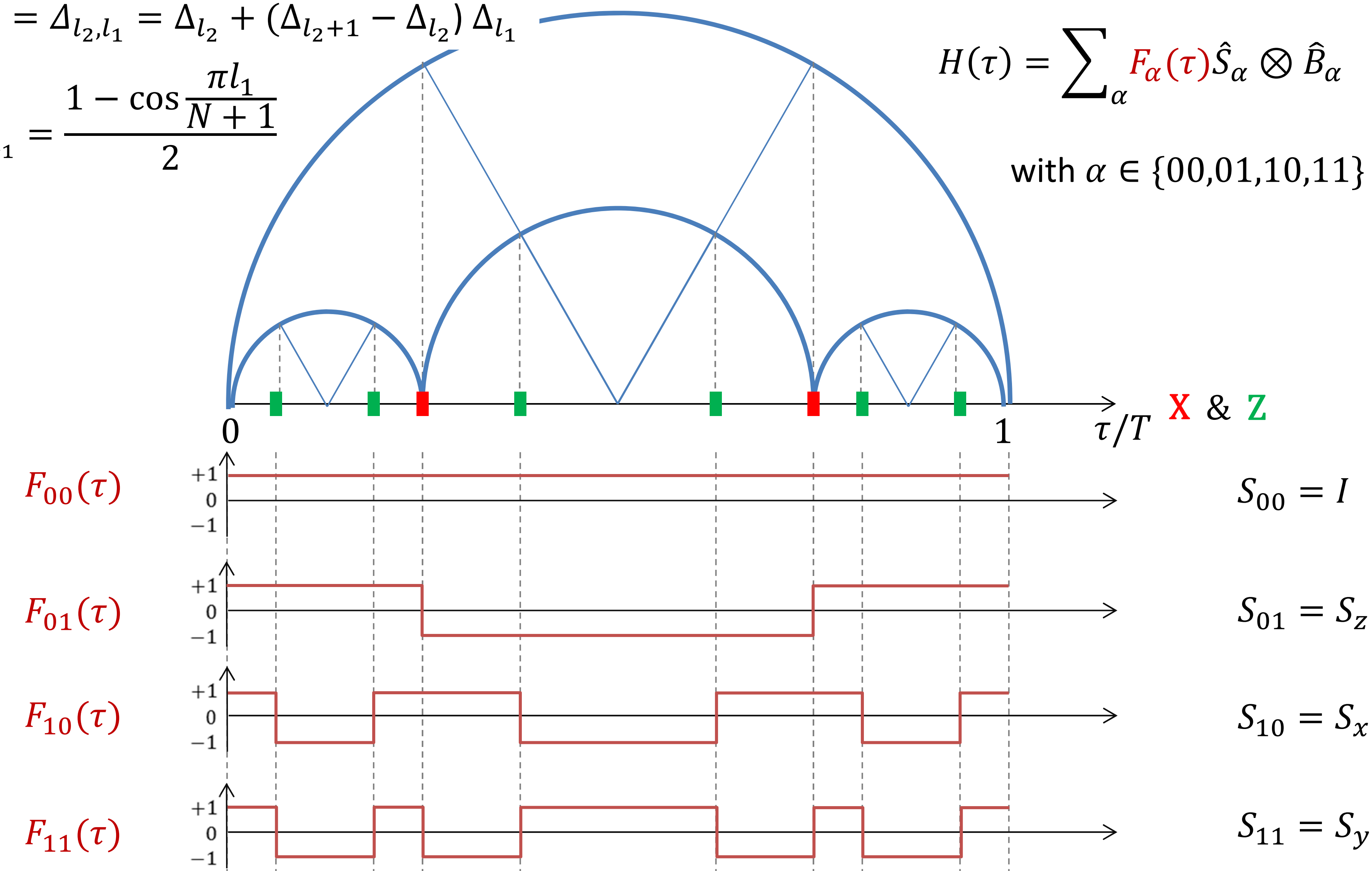
Quadratic (Uhrig) DD Sequence

$$\Delta_\lambda = \Delta_{l_2,l_1} = \Delta_{l_2} + (\Delta_{l_2+1} - \Delta_{l_2})\,\Delta_{l_1}$$

$$\Delta_{l_1} = \frac{1 - \cos\frac{\pi l_1}{N+1}}{2}$$

$$H(\tau) = \sum_\alpha F_\alpha(\tau)\hat{S}_\alpha \otimes \hat{B}_\alpha$$

with $\alpha \in \{00,01,10,11\}$



$$F_\alpha(\tau)F_{\alpha'}(\tau) = F_{\alpha\oplus\alpha'}(\tau)$$

$$S_\alpha S_{\alpha'} = \pm S_{\alpha\oplus\alpha'}$$

Prove Universal DD Sequences

- The system-bath interaction

$$H_{SB} = \sum_{\alpha} \hat{S}_{\alpha} \otimes \hat{B}_{\alpha}$$

- Toggling frame Hamiltonian (associated with DD sequence)

$$H(\tau) = \sum_{\alpha} F_{\alpha}(\tau) \hat{S}_{\alpha} \otimes \hat{B}_{\alpha}$$

- Unitary Evolution

$$U = \mathfrak{T} e^{-i \int_0^T H(\tau) d\tau} = \hat{I} \otimes U_B + O(T^{N+1})$$

Dyson Expansion

- $U = \mathcal{T} e^{-i \int_0^T H(\tau) d\tau} = 1 + \sum_{n=1}^{\infty} (-i)^n \int_0^T d\tau_n \cdots \int_0^{\tau_2} d\tau_1 H(\tau_n) \cdots H(\tau_1)$

$$= 1 + \sum_{n=1}^{\infty} T^n (-i)^n \sum_{\{\alpha_j\}} F_{\alpha_n,\cdots,\alpha_1} \hat{S}_{\alpha_n,\cdots,\alpha_1} \otimes \hat{B}_{\alpha_n,\cdots,\alpha_1} \overset{?}{=} \hat{I} \otimes U_B + O(T^{N+1})$$

where
$$F_{\alpha_n,\cdots,\alpha_1} = \int_0^1 dt_n \cdots \int_0^{t_2} dt_1 F_{\alpha_n}(t_n) \cdots F_{\alpha_1}(t_1)$$
$$\hat{S}_{\alpha_n,\cdots,\alpha_1} = \hat{S}_{\alpha_n} \cdots \hat{S}_{\alpha_1} = \pm \hat{S}_{\alpha_n \oplus \alpha_{n-1} \cdots \oplus \alpha_1}$$
$$\hat{B}_{\alpha_n,\cdots,\alpha_1} = \hat{B}_{\alpha_n} \cdots \hat{B}_{\alpha_1}$$

- It is sufficient to satisfy exp(N) number of equations

$$F_{\alpha_n,\cdots,\alpha_1} = \int_0^1 dt_n \cdots \int_0^{t_2} dt_1 F_{\alpha_n}(t_n) \cdots F_{\alpha_1}(t_1) = 0$$

for n $\leq N$ and $\hat{S}_{\alpha_n,\cdots,\alpha_1} \neq I$ (i.e., $\bigoplus_{j'=1}^{n} \alpha_{j'} \neq 0$).

- $F_\alpha(t)$ is determined by poly(N) pulses, with timing $\{\Delta_\lambda\}_{\lambda=1,\cdots,Q}$ (e.g., $Q = O(N)$ for UDD, $Q = O(N^2)$ for QDD, $Q = O(N^{2m})$ for NUDD sequence).
- Surprisingly, there exists such poly(N) parameters $\{\Delta_\lambda\}_{\lambda=1,\cdots,Q}$ to fulfill these exp(N) number of equations.

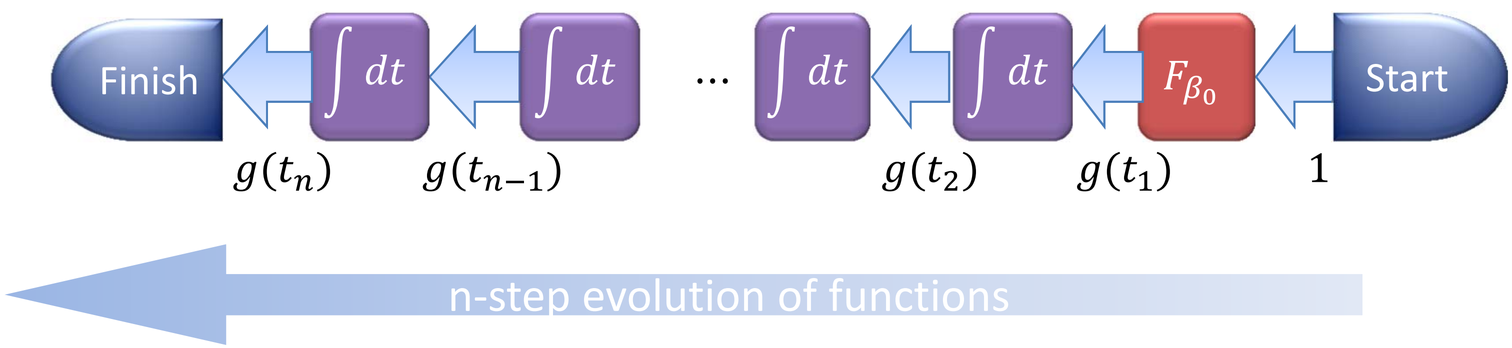
Step 1 – Operations on functions

Using the property: $F_{\alpha}(\tau)F_{\alpha'}(\tau) = F_{\alpha\oplus\alpha'}(\tau)$

$$F_{\alpha_n,\cdots,\alpha_1} = \int_0^1 dt_n\, F_{\alpha_n}(t_n) \int_0^{t_n} dt_{n-1} F_{\alpha_{n-1}}(t_{n-1}) \cdots \int_0^{t_3} dt_2 F_{\alpha_2}(t_2) \int_0^{t_2} dt_1 F_{\alpha_1}(t_1) \times 1$$

Introduce $\beta_j = \oplus_{j'=j+1}^{n} \alpha_{j'}$ and $\int_{0,[\beta]}^{t} dt' = F_{\beta}(t) \int_0^t dt' F_{\beta}(t')$,

$$F_{\alpha_n,\cdots,\alpha_1} = \int_0^1 dt_n \int_{0,[\beta_{n-1}]}^{t_n} dt_{n-1} \cdots \int_{0,[\beta_2]}^{t_3} dt_2 \int_{0,[\beta_1]}^{t_2} dt_1 \times F_{\beta_0}(t_1) \times 1$$



Look for <u>generic features</u> in evolution of functions

Step 2 – Choose functional basis

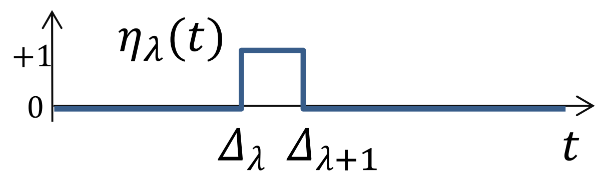
$$F_{\alpha_n,\cdots,\alpha_1} = \int_0^1 dt_n \int_{0,[\beta_{n-1}]}^{t_n} dt_{n-1} \cdots \int_{0,[\beta_2]}^{t_3} dt_2 \int_{0,[\beta_1]}^{t_2} dt_1 \times F_{\beta_0}(t_1) \times 1$$

- Choose functional basis:

$$\eta_{q,\lambda}(t) = t^q \eta_\lambda(t)$$

for $q = 0, \cdots, N$ and $\lambda = 0, \cdots, Q$.

Here $\eta_\lambda(t) = \begin{cases} 1 & \text{for } t \in (\Delta_\lambda, \Delta_{\lambda+1}] \\ 0 & \text{otherwise} \end{cases}$



- Based on the consideration:

1. $F_\alpha(t)$ is piecewise-continuous spanned by $\{\eta_\lambda(t)\}$
2. $\int dt$ maps from t^q to t^{q+1}.

Step 3 – Matrix representation

$$F_{\alpha_n,\cdots,\alpha_1} = \int_0^1 dt_n \int_{0,[\beta_{n-1}]}^{t_n} dt_{n-1} \cdots \int_{0,[\beta_2]}^{t_3} dt_2 \int_{0,[\beta_1]}^{t_2} dt_1 \times F_{\beta_0}(t_1) \times 1$$

$$= \vec{v}_L \cdot \mathbf{G}_{\beta_{n-1}} \cdot \mathbf{G}_{\beta_{n-2}} \cdots \mathbf{G}_{\beta_1} \cdot \mathbf{F}_{\beta_0} \cdot \vec{v}_R$$

- Matrix/vector representation in $(N+1) \times Q$ dimensional function space

Operation	**Matrix/Vector Form**
$t \cdot \eta_{q,\lambda} = \eta_{q+1,\lambda}$	$\mathbf{M}_{q,\lambda}^{q',\lambda'} = \delta_{q+1}^{q'} \delta_{\lambda}^{\lambda'}$
$F_\beta(t) \cdot \eta_{q,\lambda} = (-1)^{\beta \cdot \lambda} \eta_{q,\lambda}$	$(\mathbf{F}_\beta)_{q,\lambda}^{q',\lambda'} = \delta_q^{q'} (B_\beta)_\lambda^{\lambda'}$
$\int_{0,[\beta]}^{t} dt' \cdot \eta_{q,\lambda} = (\mathbf{G}_\beta)_{q,\lambda}^{q',\lambda'} \eta_{q',\lambda'}$	$(\mathbf{G}_\beta)_{q,\lambda}^{q',\lambda'} = \frac{\delta_{q+1}^{q'} \delta_\lambda^{\lambda'} - \delta_0^{q'} \left(D_\beta^{q+1}\right)_\lambda^{\lambda'}}{q+1}$
$\int_0^1 dt \cdot \eta_{q,\lambda} = (\vec{v}_L)_{q,\lambda}$	$(\vec{v}_L)_{q,\lambda} = \frac{\Delta_\lambda^{q+1} - \Delta_{\lambda+1}^{q+1}}{q+1}$
$1 = \left(\vec{v}_R^T\right)_{q,\lambda} \eta_{q,\lambda}$	$\left(\vec{v}_R^T\right)_{q,\lambda} = \delta_0^q$

Step 4 – Reduced matrix representation

$$F_{\alpha_n,\cdots,\alpha_1} = \int_0^1 dt_n \int_{0,[\beta_{n-1}]}^{t_n} dt_{n-1} \cdots \int_{0,[\beta_2]}^{t_3} dt_2 \int_{0,[\beta_1]}^{t_2} dt_1 \times F_{\beta_0}(t_1) \times 1$$

$$= \vec{v}_L \cdot \mathbf{G}_{\beta_{n-1}} \cdot \mathbf{G}_{\beta_{n-2}} \cdots \mathbf{G}_{\beta_1} \cdot \mathbf{F}_{\beta_0} \cdot \vec{v}_R$$

$$= \sum_{\sum_{j=1}^{n} i_j \le N} c_{\{\beta_j\},\{i_j\}} \left\langle u_L \middle| D_{\beta_n}^{i_n} D_{\beta_{n-1}}^{i_{n-1}} \cdots D_{\beta_1}^{i_1} B_{\beta_0} \middle| u_R \right\rangle$$

- Use block-matrix properties and reduce from $(N+1) \times Q$ to Q dimensional function space, spanned by $\{\eta_\lambda(t)\}$

$$\left| u_L \right\rangle_\lambda = \Delta_{\lambda+1} - \Delta_\lambda$$

$$\left| u_R \right\rangle_\lambda = 1$$

$$(B_\beta)_\lambda^{\lambda'} = (-1)^{\beta \cdot \lambda}\, \delta_\lambda^{\lambda'},$$

$$(D_\beta)_\lambda^{\lambda'} = \Delta_\lambda \delta_\lambda^{\lambda'} - (\Delta_{\lambda+1} - \Delta_\lambda) \sum_{\lambda''=\lambda+1}^{Q} (-1)^{\beta \cdot (\lambda' - \lambda)}\, \delta_{\lambda''}^{\lambda'}.$$

Step 5 – Quantum walk in functional space

- To shown $F_{\alpha_n,\cdots,\alpha_1} = 0$ for $n \le N$ and $\beta_0 \ne 0$, it is sufficient to show:

$$\langle u_L | D_{\beta_n}^{i_n} D_{\beta_{n-1}}^{i_{n-1}} \cdots D_{\beta_1}^{i_1} | B_{\beta_0} u_R \rangle = 0$$

for $\sum_{j=1}^{n} i_j \le N - 1$ *and* $\beta_0 \ne 0$.

Uhrig DD -- 1D quantum walk

k_1: 0 1 2 3 4 5 (S . . . #) $\xrightarrow{D}$ 0 1 2 3 4 5 (S X . . #) $\xrightarrow{D}$ 0 1 2 3 4 5 (S X X . #) $\xrightarrow{D}$ 0 1 2 3 4 5 (S X X X #)

N=4

S starting state
X explored states
unexplored target state

- Use a convenient functional basis:

$$|\kappa\rangle = c_\kappa^\lambda |\lambda\rangle \qquad \chi_\kappa(t) = c_\kappa^\lambda \eta_\lambda(t)$$

$$c_\kappa^\lambda = \sin\left(\kappa \frac{2\lambda+1}{N+1} \frac{\pi}{2}\right)$$

This is Fourier Transform!

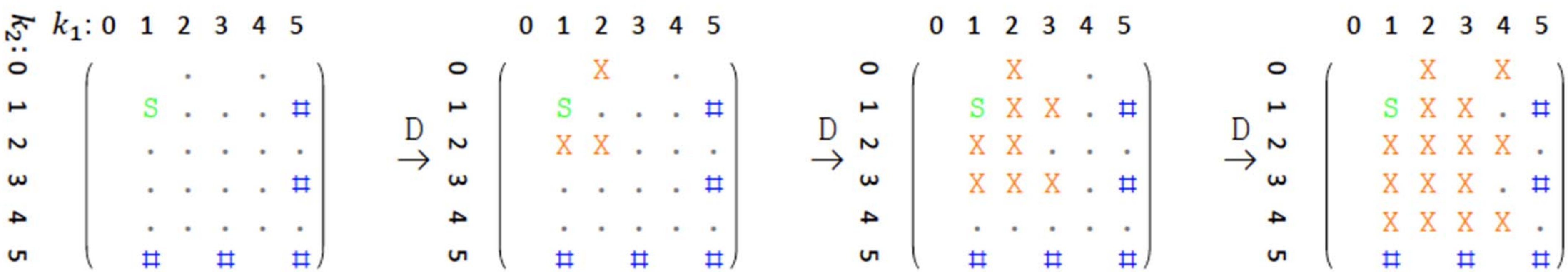
Step 5 – Quantum walk in functional space

- To shown $F_{\alpha_n,\cdots,\alpha_1} = 0$ for $n \le N$ and $\beta_0 \ne 0$, it is sufficient to show:

$$\langle u_L | D_{\beta_n}^{i_n} D_{\beta_{n-1}}^{i_{n-1}} \cdots D_{\beta_1}^{i_1} | B_{\beta_0} u_R \rangle = 0$$

for $\sum_{j=1}^{n} i_j \le N-1$ *and* $\beta_0 \ne 0$.

Quadratic DD -- 2D quantum walk



S starting state
X explored states
unexplored target state

- Use a convenient functional basis:

$$|\kappa\rangle = c_\kappa^\lambda |\lambda\rangle \qquad \chi_\kappa(t) = c_\kappa^\lambda \eta_\lambda(t)$$

$$c_{\kappa,\lambda} = c_{(k_2,k_1),(l_2,l_1)} = \begin{cases} (-1)^{(k_1-1)/2} \sin\left[k_2 \frac{2l_2+1}{N+1}\frac{\pi}{2}\right] \sin\left[k_1 \frac{2l_1+1}{N+1}\frac{\pi}{2}\right] & \text{for odd } k_1 \\ (-1)^{k_1/2} \cos\left[k_2 \frac{2l_2+1}{N+1}\frac{\pi}{2}\right] \sin\left[k_1 \frac{2l_1+1}{N+1}\frac{\pi}{2}\right] & \text{for even } k_1 \end{cases}$$

This is Fourier Transform with a shift!

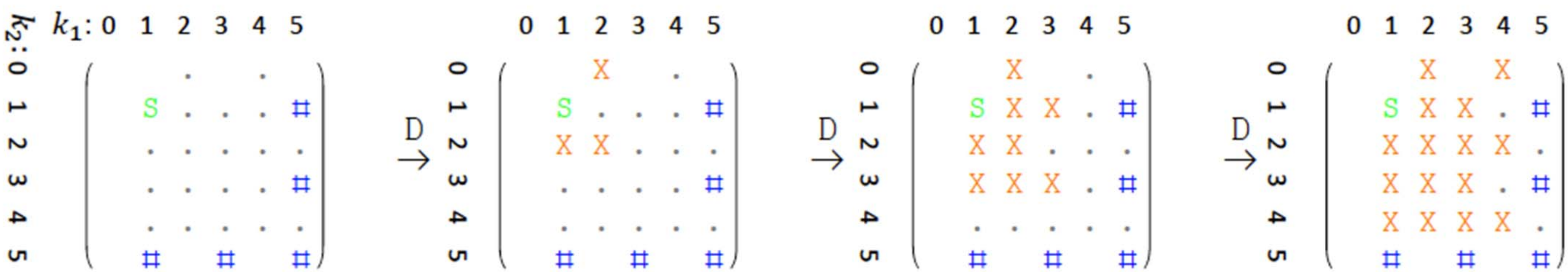
Step 5 – Quantum walk in functional space

- To shown $F_{\alpha_n,\cdots,\alpha_1} = 0$ for $n \leq N$ and $\beta_0 \neq 0$, it is sufficient to show:

$$\langle u_L | D_{\beta_n}^{i_n} D_{\beta_{n-1}}^{i_{n-1}} \cdots D_{\beta_1}^{i_1} | B_{\beta_0} u_R \rangle = 0$$

for $\sum_{j=1}^{n} i_j \leq N - 1$ *and* $\beta_0 \neq 0$.

Quadratic DD -- 2D quantum walk

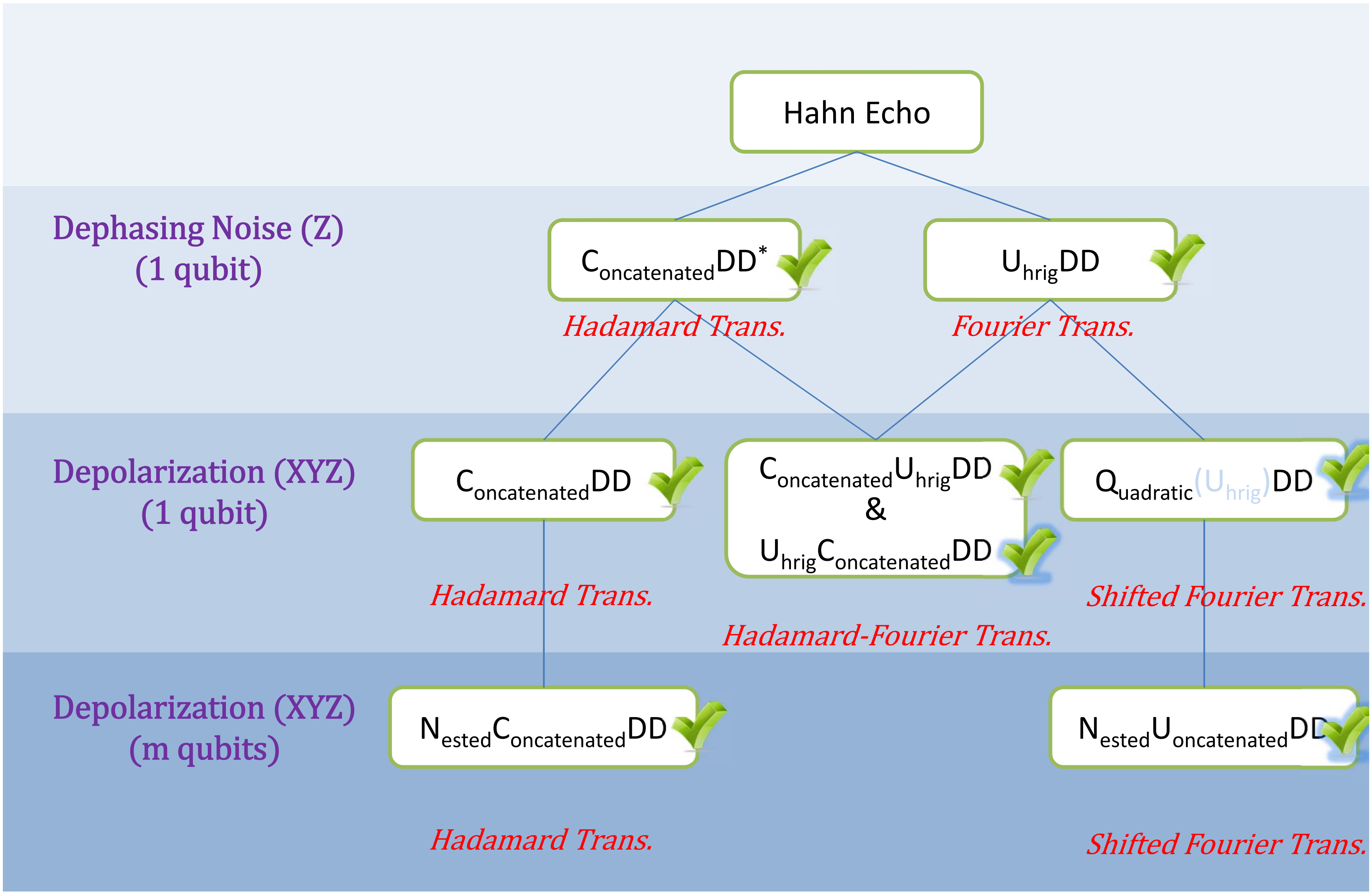


- Use a convenient functional basis:

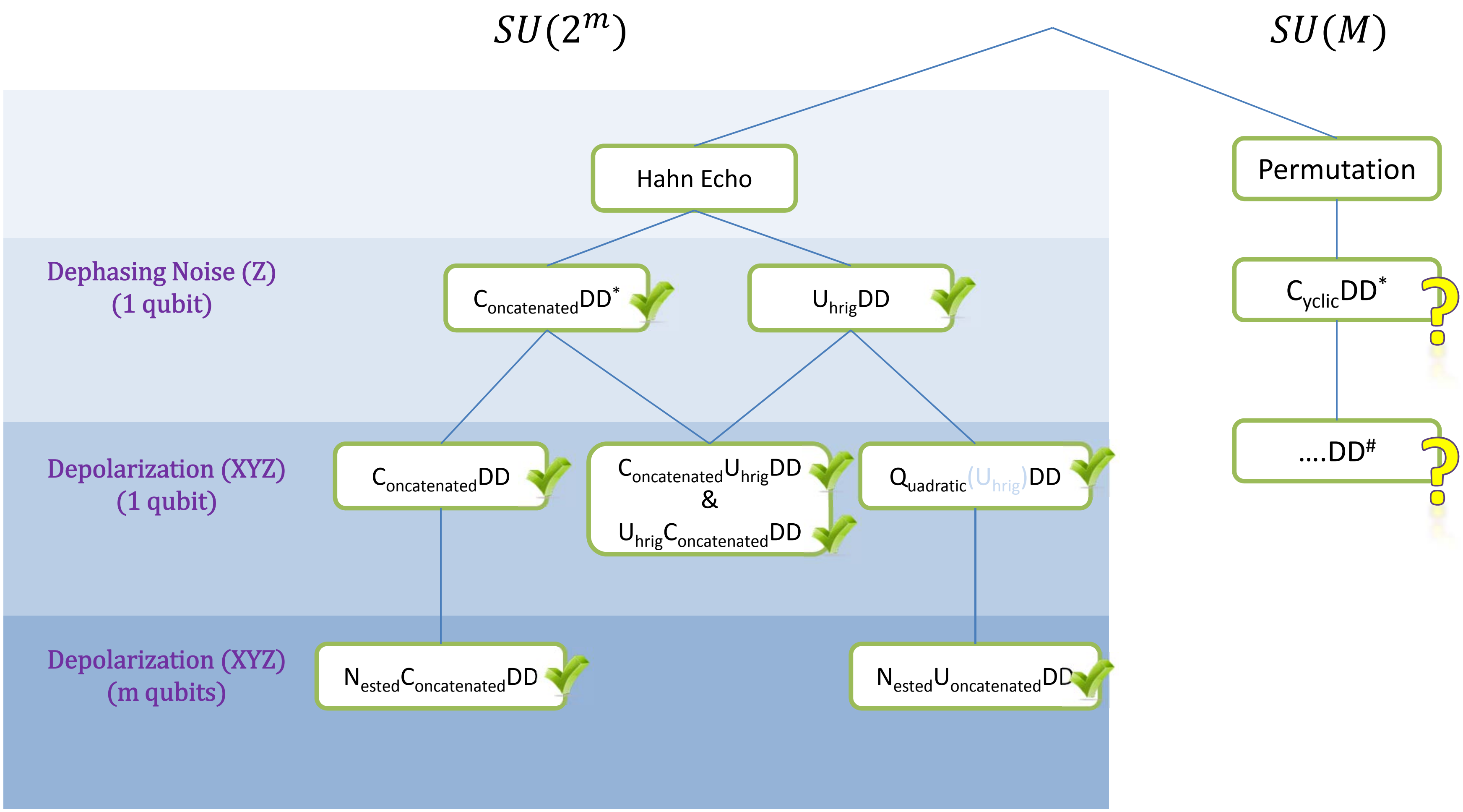
$$|\kappa\rangle = c_\kappa^\lambda |\lambda\rangle \qquad \chi_\kappa(t) = c_\kappa^\lambda \eta_\lambda(t)$$

- Proof generalizes for NUDD and all other known cases CDD, CUDD, UCDD, …, by identifying the orthogonal transformation $\boldsymbol{c_\kappa^\lambda}$.
- Proof generalizes for analytically time-dependent system bath coupling, as long as $B_\alpha(t)$ has a Taylor expansion.

Family Tree of Universal DD Sequences



Outlook: Family Tree of Universal DD Sequences



*Shukla, Imambekov, Preskill, L.J. (in preparation)

#Bryan Fong's talk yesterday (QEC11)