Local stabilizer codes in 3D without string logical operators

arXiv:1101.1962

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Quantum Error Correction 2011

Problem

Theme

- Find a noise-free subspace/subsystem from a physical system,

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- Find a noise-free subspace/subsystem from a physical system,
- on which manipulations are still possible.

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Topological QC ?

- TQC with anyons $\sim$ Excited states
- (Self-correcting) Memory $\sim$ Ground state

Why hard?

Kitaev 2D toric code

$$H = -\sum_s \begin{array}{ccc} & | & \\ & X & \\ -X- & & -X- \\ & X & \\ & | & \end{array} \quad - \sum_p \begin{array}{ccc} & -Z- & \\ Z & & Z \\ & -Z- & \end{array}$$

- **Locally indistinguishable** 4-fold degenerate ground state.

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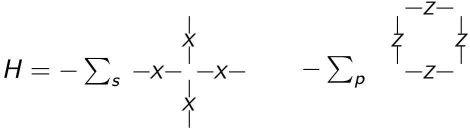

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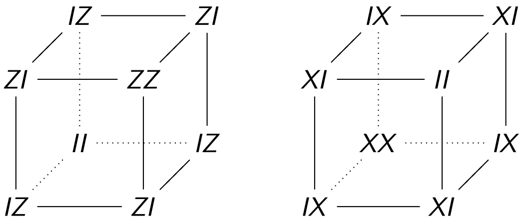
The same *had* happened for *all* low dimensional models.
Self-correction is possible in 4D or higher.

3D Cubic Code



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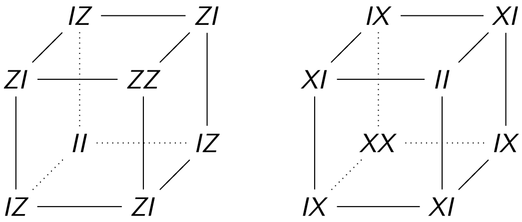

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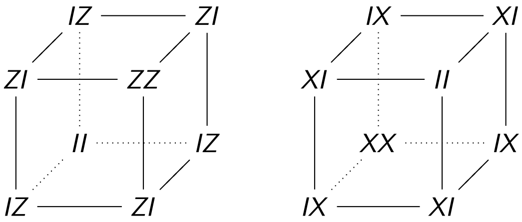

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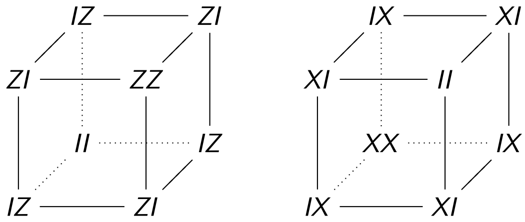

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- Local interaction. All terms commuting. Frustration-free.
- Rigorous definition for **string** in the discrete lattice.
- The model has only **short** string segments.
- Exotic degeneracy: $k(L = 2^p) = 4L - 2$, $k(L = 2^p + 1) = 2$.

Stabilizer/Additive codes

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Abelianized Pauli group

- $\sigma_X \cdot \sigma_X = I \iff (10) + (10) = (00)$
- $\sigma_X \cdot \sigma_Z = -i\sigma_Y \iff (10) + (01) = (11)$

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Codes

- Commuting set of Pauli operators = Null subspace S = stabilizer group
- Symmetry group = Hyperbolic subspace in $S^\perp$ = group of logical operators.

Cubic codes

Generally

- Simple cubic lattice with m qubits per site.
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- $t = 2$: one for X, one for Z type.
- Product of all terms in the Hamiltonian be Id.
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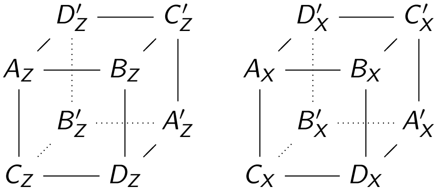
Finally

- Translate the conditions into linear algebra equations on $\mathcal{P}_m$
- $\rightarrow$ Linear equations, rank constraints.

$17 \ / \ 2^{37}$



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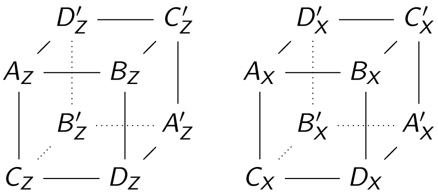

- 64 commutation relations determine everything.

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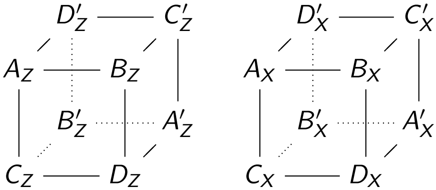

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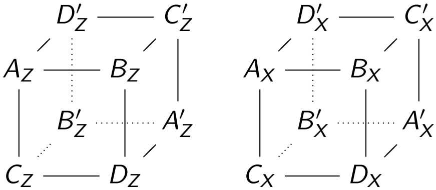

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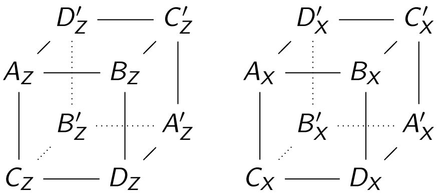

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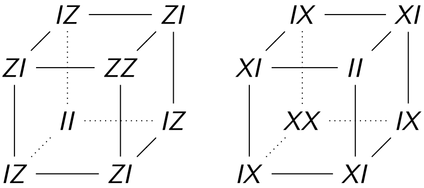
- Find by brute-force the solutions of remaining 30 equations.
- 17 different solutions up to symmetry group of cube !

Code 1 / 17



Duality $Q_Z \leftrightarrow Q_X$.

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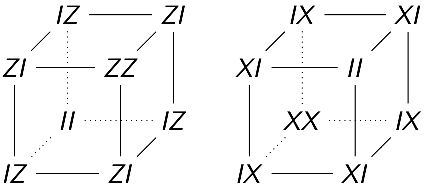

Duality $Q_Z \leftrightarrow Q_X$.

This model has no string...

- Wait, is it topologically ordered?
- What do you mean by **string**? What if there is a *thick* string?

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- Claim: Any Pauli operator of bounded support commuting with all stabilizers is a stabilizer.
- Any operator is a linear combination of Pauli operators.
- Because $Q_X \leftrightarrow Q_Z$, it suffices to consider X-type Pauli operator.

Tool – Eraser

Exercise

If ? is one of II, XI, IX, XX;

- $[ZZ, ?] = 0$

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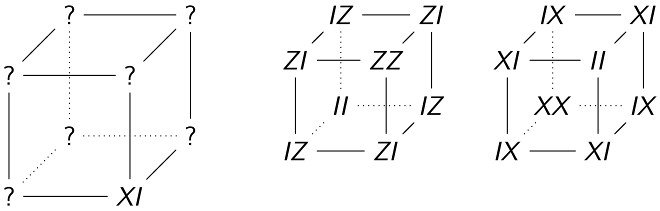

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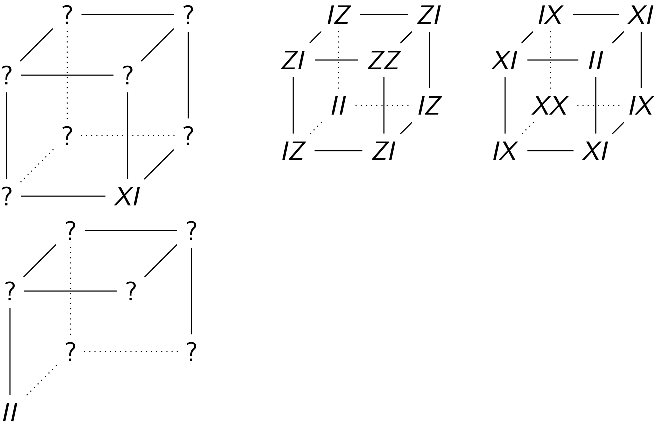

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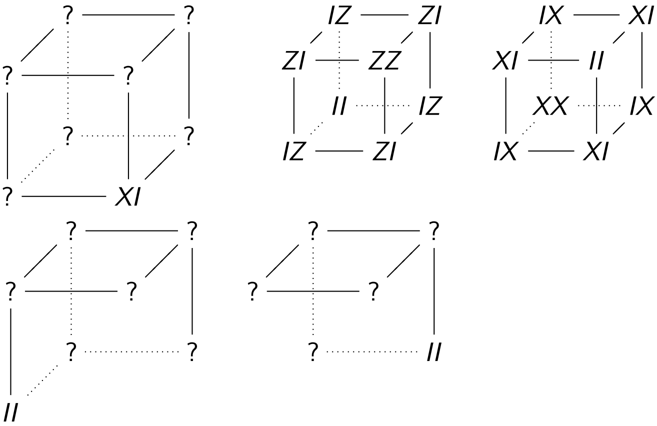

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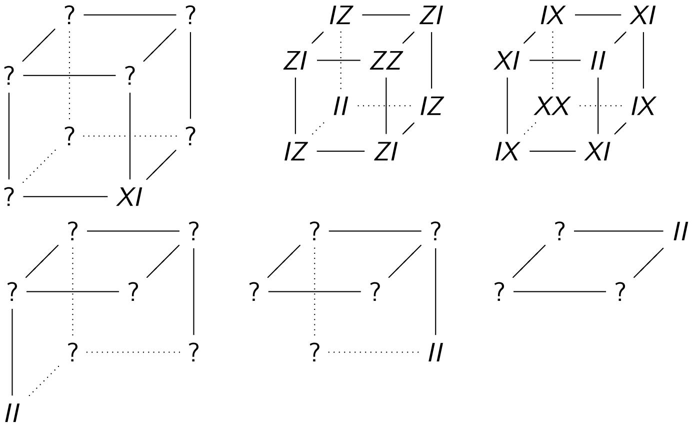

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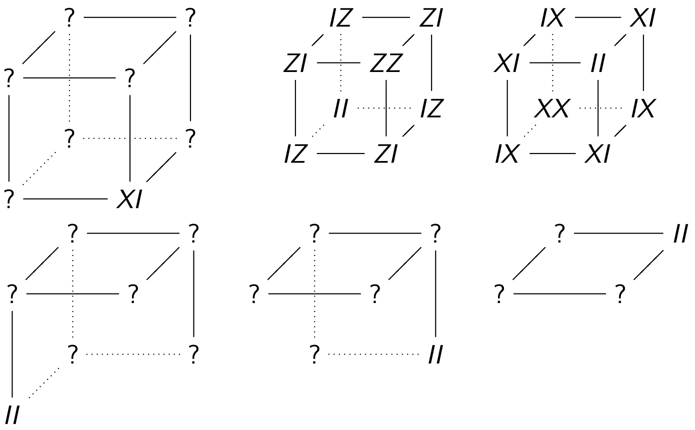
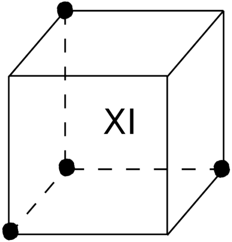


Q.E.D.

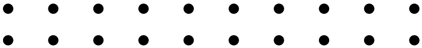
Elementary Excitations (Syndromes)



Can you make a **pair** of defects out of these ?

String?

String?



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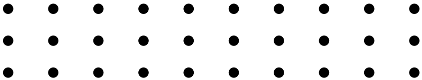

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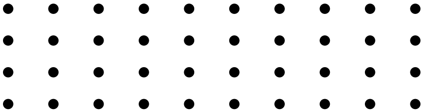

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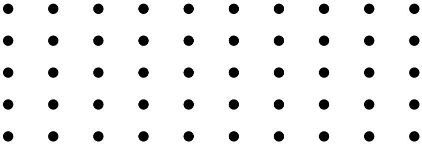

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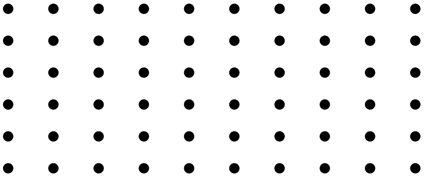

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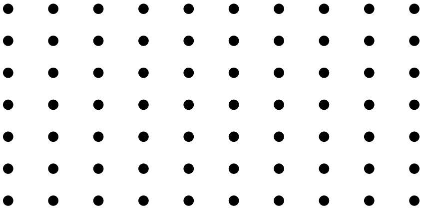

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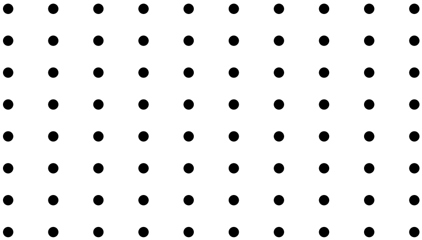

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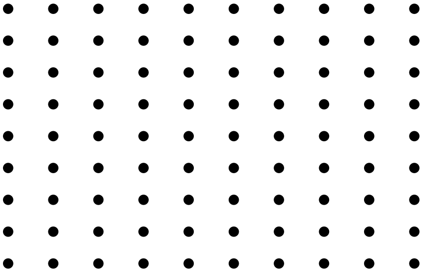

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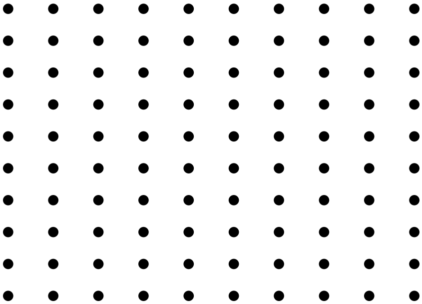

Surface?

String segment

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- **Anchors** envelop excitations.
- **width** $=$ the size of the anchors.
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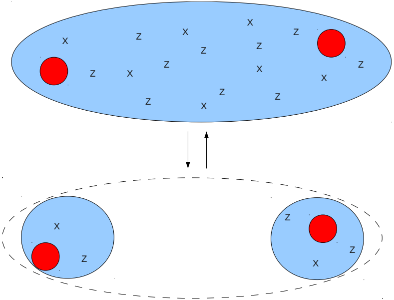
Definition

A finite Pauli operator that creates excitations at most two locations.

- **Anchors** envelop excitations.
- **width** = the size of the anchors.
- **length** = the distance between the anchors.
- No geometric restriction.

Trivial string segments



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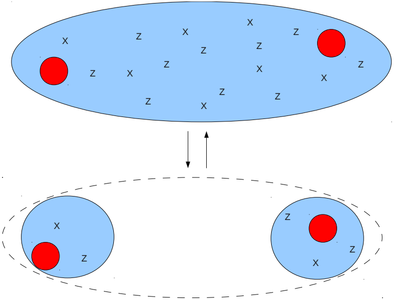

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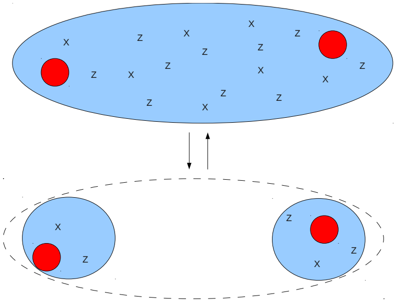
Trivial string segments



- **Trivial**, if there is a equivalent P s.t. $\mathrm{supp}(P) = C_1 \cup C_2$ and $\mathrm{dist}(C_1, C_2) > 1$.
- A trivial string segment only creates trivial charges.

examples...

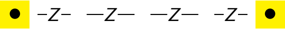
String segment: Example

In 2D toric code:

● –Z– —Z— —Z— –Z– ●

String segment: Example

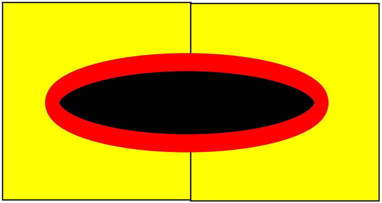
In 2D toric code:



- Dots are enclosed by two anchors.
- Pauli operator commutes with local terms in H except at the anchors.
- Non-trivial string segment.
- Width $= 1$, length $= \infty$

String segment: Example

In 2D Ising model: $H = -\sum_{\langle ij \rangle} Z_i Z_j$



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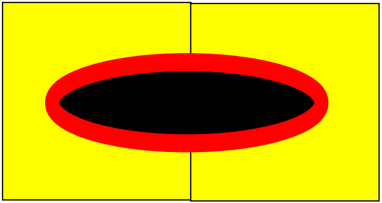
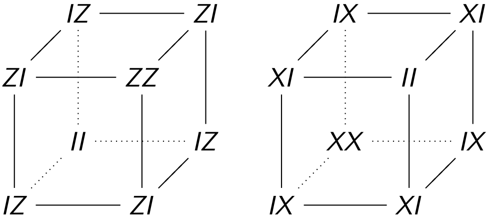


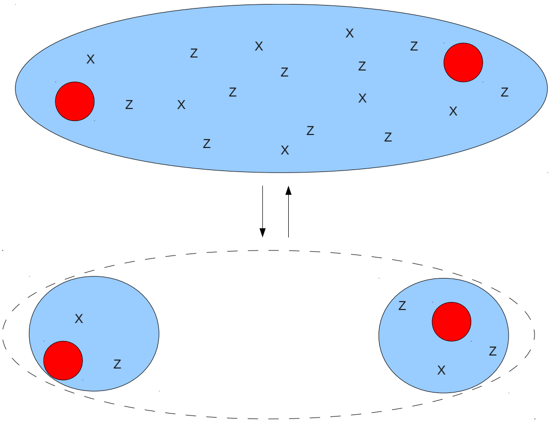
- Non-trivial X-type string segment.
- Two anchors must be adjacent.
- length $= 0$

No-strings rule



A string segment of width w and length $\geq 15w$ is trivial.

No-strings rule

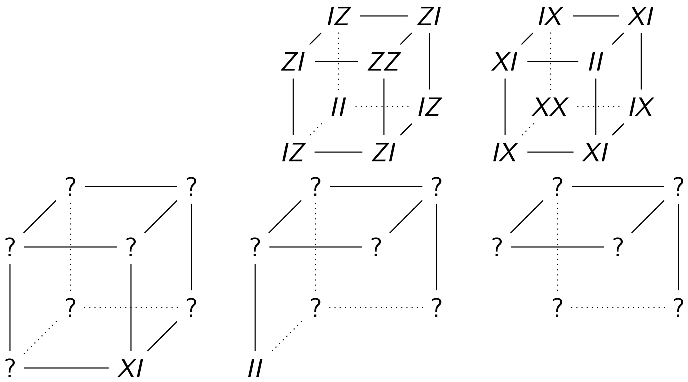


Proof

- Any long string segment is equivalent to a product of "flat" ones.
- Any long flat segment is trivial.

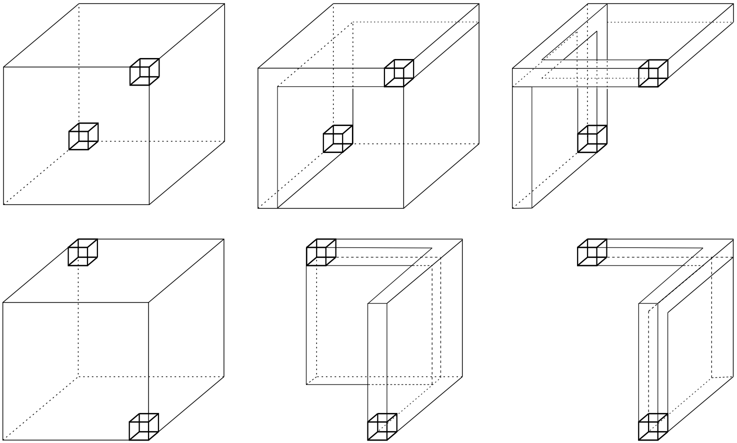
No-strings rule : Proof

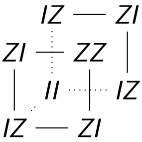
Recall the Eraser:



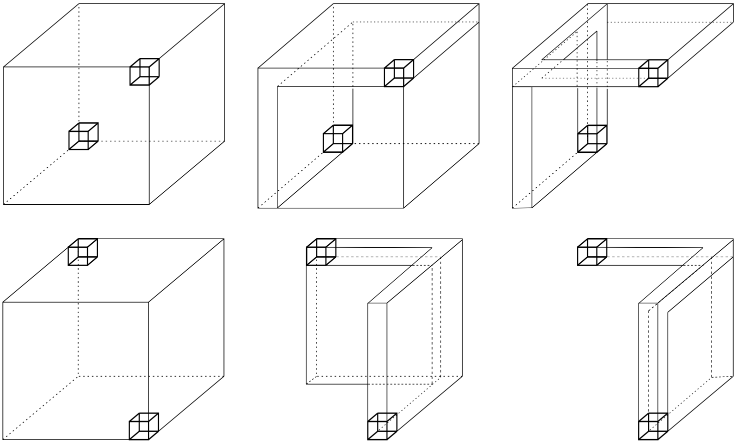
- *IZ*–*ZI* has two independent ends.

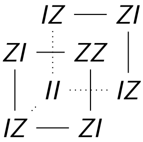
No-strings rule : Proof





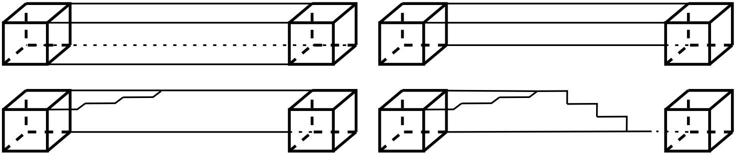
No-strings rule : Proof

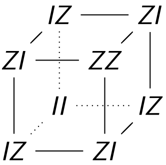




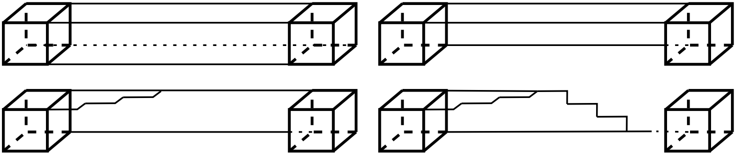
Reduced to flat segments.

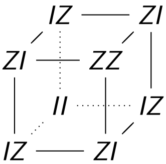
No-strings rule : Proof





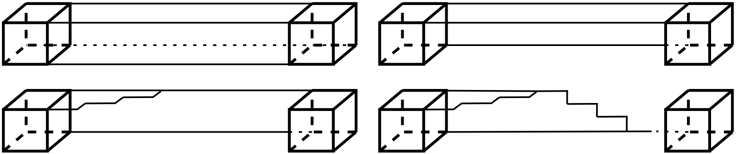
No-strings rule : Proof

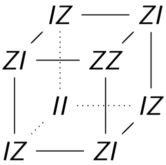




$$[O_1 - O_2, IZ - ZI] = 0$$
$$[O_1 - O_2, II - IZ] = 0$$

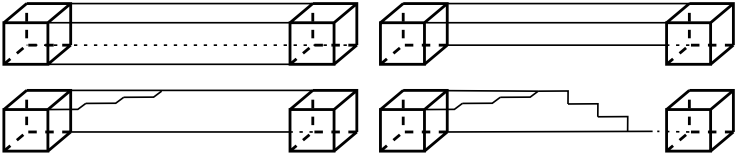
No-strings rule : Proof

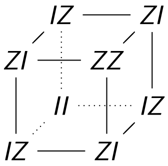




$$[O_1 - O_2, IZ - ZI] = 0$$
$$[O_1 - O_2, II - IZ] = 0$$
$$\iff O_1 - O_2 = \begin{cases} II - II \\ XI - II \\ IX - XI \\ XX - XI \end{cases}$$

No-strings rule : Proof

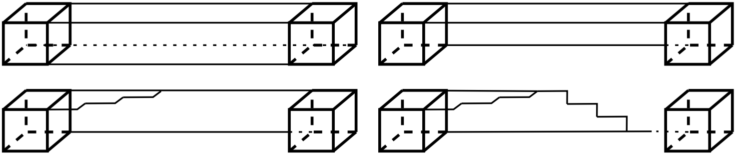


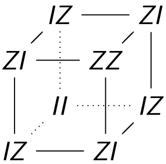


$$\begin{aligned}[O_1 - O_2, IZ - ZI] &= 0 \\ [O_1 - O_2, II - IZ] &= 0\end{aligned} \iff O_1 - O_2 = \begin{cases} II - II \\ XI - II \\ IX - XI \\ XX - XI \end{cases}$$

$$\begin{cases} II - II - II - \cdots \\ XX - XI - II - \cdots \\ IX - XI - II - \cdots \\ XI - II - II - \cdots \end{cases}$$

No-strings rule : Proof





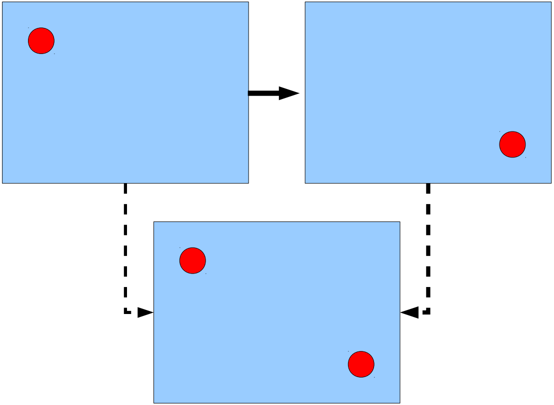
$$[O_1 - O_2, IZ - ZI] = 0$$
$$[O_1 - O_2, II - IZ] = 0$$

$$\Longleftrightarrow O_1 - O_2 = \begin{cases} II - II \\ XI - II \\ IX - XI \\ XX - XI \end{cases}$$

$$\begin{cases} II - II - II - \cdots \\ XX - XI - II - \cdots \\ IX - XI - II - \cdots \\ XI - II - II - \cdots \end{cases}$$

Q.E.D.

No-strings rule



- You can't drag the defect.
- Annihilate it, and then create it.

Code space dimension under periodic boundary conditions

L	k	L	k
2	6	3	2
4	14	5	2
6	6	7	2
8	30	9	2
10	6	11	2
12	14	13	2
14	6	15	50
16	62	17	2

Code space dimension under periodic boundary conditions

L	k	L	k
2	6	3	2
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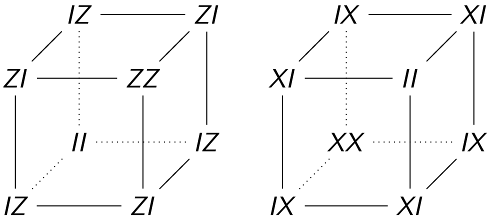
8	30	9	2
10	6	11	2
12	14	13	2
14	6	15	50
16	62	17	2

Formula(?) for odd L

$$\frac{k+2}{4} = \deg_x \gcd\left[1+(1+x)^L, 1+(1+tx)^L, 1+(1+t^2x)^L\right]_{\mathbb{F}_4}$$

where $t^2 + t + 1 = 0.$

Summary



- Strictly local commuting Hamiltonian, frustration-free.
- Defined and proved **no-strings rule**, with aspect ratio 15.
- Exotic degeneracy: $k(L=2^p)=4L-2$, $k(L=2^p+1)=2$.
- Logarithmic energy barrier leads to very long memory time $\rightarrow$ Bravyi's talk 4:40.