Thank you!

Entanglement-Assisted Quantum LDPC Codes from Combinatorial Designs

Yuichiro Fujiwara and Vladimir D. Tonchev

Department of Mathematical Sciences
Michigan Technological University

Entanglement-assisted stabiliser formalism
Brun, Devetak, and Hsieh (2006)

Our goal:
Quantum error correcting codes with
Good error-correcting performance,
Flexibility,
Low docoding complexity,
Systematic constructions.

Combinatorial design theory
- Pairwise balanced designs
- Finite geometry

Classical coding theory
- Low-density parity-check codes

Gallager (1960)

References
- F et al., Phys. Rev. A 82 042338 (2010)
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- F, Hsieh, Proc. ISIT 279 - 283 (2011)

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A pairwise balanced design, PBD(v, K, 1), is an ordered pair (V, B), where
- V: finite set (points),
- B: family of subsets of V (blocks),
- each unordered pair of distinct points is contained in exactly one block,
- the sizes of blocks consist of the elements of K.

A PBD is said to be odd-replicate if each point appears in an odd number of blocks.

This kind of mathematical object has been studied since the 19th century (i.e., we've got a bunch of mathematical tools).

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Low-density parity-check (LDPC) codes

LDPC codes are simply linear codes which are decodable by certain sub-optimal decoders.

The point is that LDPC codes can

- almost achieve the Shannon limit
- be decoded fast (in linear time)

To obtain better perfomrance, it is desirable for the Tanner graphs of LDPC codes to be of "girth" 6 (or larger).

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Stabilizer formalism

- requirs a sever condition (symplectic orthogonality),
- can emply only a limited range of classical codes (e.g., self-containing codes).

Entanglement-assisted stabilizer formalism

- removes the orthogonality condition with the help of preshared entanglement,
- allows the code designer to emply ANY binary or quaternary linear code.

EA-LDPC codes are quantum analogues of LDPC codes,
so they are defined by quantum versions of parity-check matrices.

Calderbank-Shor-Steane (CSS) construction

A quantum check matrix (of a homogenous quantum LDPC code) looks like:

$$\begin{bmatrix} H & 0 \\ 0 & H \end{bmatrix}$$

where H is a parity-check matrix of a binary linear code (which forms a regular LDPC code).

$H : [n, k, d]\text{code} \Rightarrow [[n, 2k - n + c, d; c]]$ EA-LDPC code, where $c = \mathrm{rank}(HH^T)$

c is the amount of preshared entanglement, i.e., the required number of ebits.

Desirable EA-LDPC codes

- have large girth,
- consume a small number of ebits.

We consider EA-LDPC codes consuming only one ebit with the largest possible girth.

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Homogeneous EA-LDPC codes consuming only one ebit with girth 6 (which is the largest possible) are equivalent to odd-replicate PBDs. If the LDPC code used as an ingredient is regular, its a Steiner 2-design. (i.e., $S(2, k, v)$ with $\frac{v-1}{k-1}$ odd)

H is an incidence matrix of a PBD with index 1 and odd replication number.

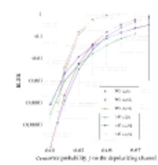
Combinatorial design theory characterizes EA-LDPC codes

We can
- derive bounds on the minimum distance, dimensions, girth, etc.
- give necessary and sufficient conditions for the existence,
- explicit constructions,
- and more (e.g., EA-LDPC codes for channels with biased noise).

Theorem 2.4 *A necessary condition for the existence of a regular entanglement-assisted quantum LDPC code which requires only one ebit and is of length n, girth six, and column weight μ is that the number $\frac{-1+\sqrt{1+4n\mu(\mu-1)}}{2(\mu-1)}$ is an odd integer. Conversely, there exists a constant n_μ such that for every pair of positive integers $n > n_\mu$ and μ the necessary condition is sufficient.*

Theorem 3.4 *Let n be an integer greater than seven. Then, there exists a regular entanglement-assisted quantum LDPC code of length n, dimension k, girth six, and column weight three which require only one ebit if and only if $\sqrt{24n+1} \equiv 5 \pmod 8$ and $n - \sqrt{24n+1} \leq k \leq n - \sqrt{24n+1} + 2t - 2$, where t is the integer satisfying $\sqrt{24n+1} = 2^{t+1}u - 3$ with u odd.*

Theorem 3.7 *The number of 6-cycles in the classical ingredient of a regular entanglement-assisted quantum code which requires only one ebit and is of length n, girth six, and column weight μ is $\frac{n\mu(\mu-1)(1-2\mu+\sqrt{1+4n\mu(\mu-1)})}{12}$.*



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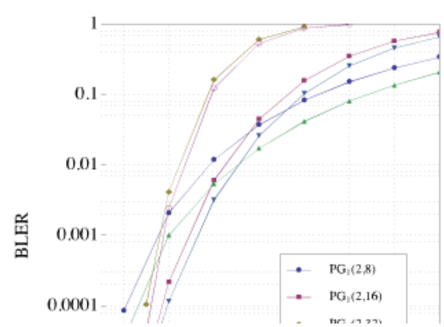

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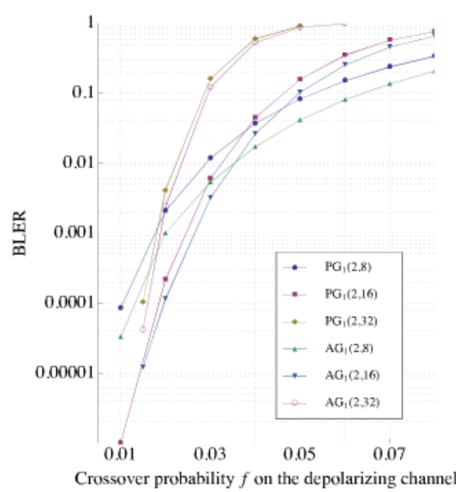

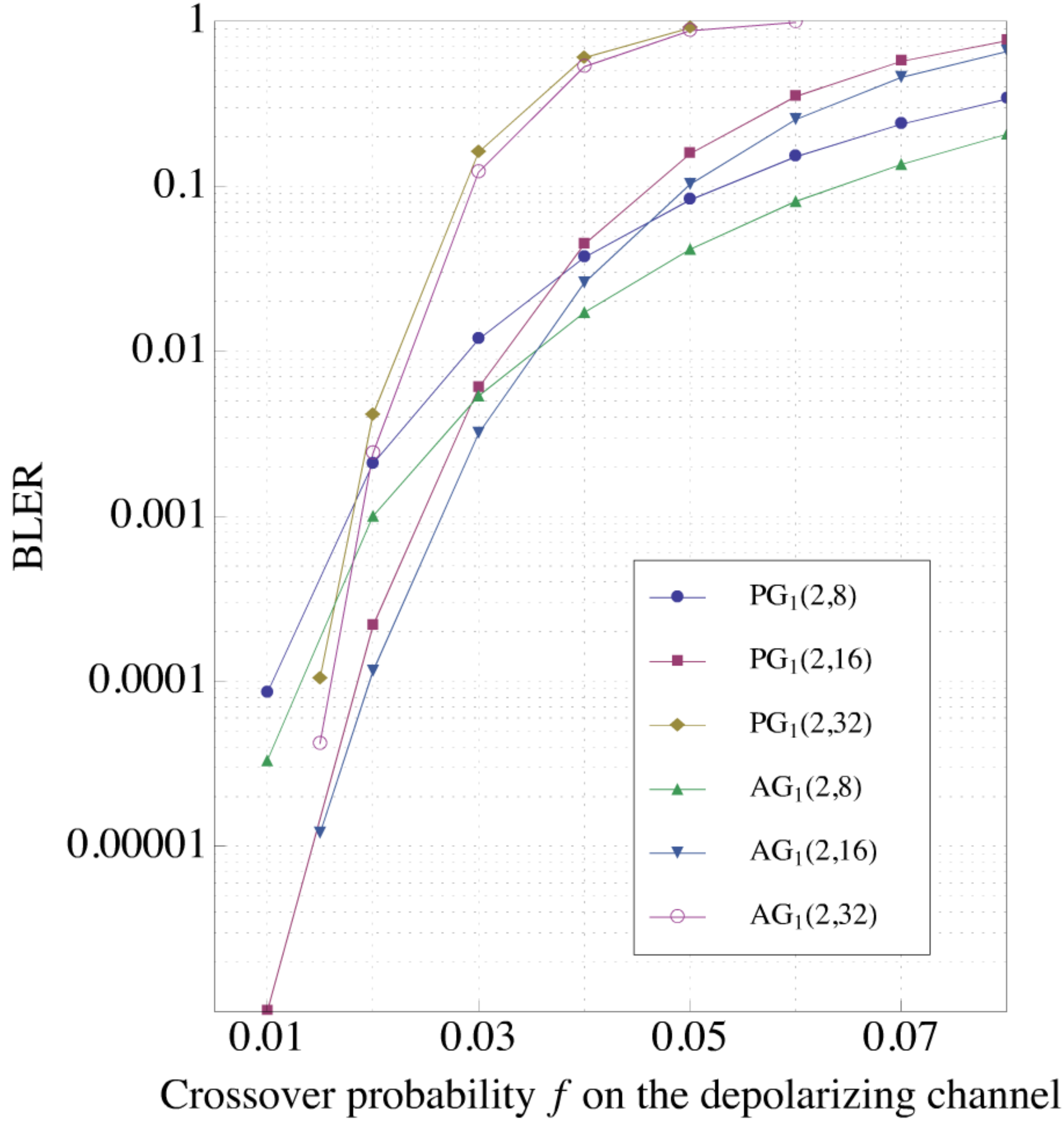

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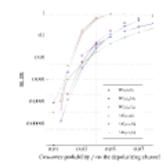

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- Can we characterize heterogeneous EA-LDPC codes?
- What if we allow multiple ebits?

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