Bounds on achievable rates of sparse quantum codes over the quantum erasure channel

Nicolas Delfosse (with Gilles Zémor)

Institute of Mathematics - Univ. of Bordeaux - France

Second Int. Conf. on Quantum Error Correction - QEC 11
USC Los Angeles - December 5, 2011

Capacity of a classical channel

- The channel introduces errors

[1] C. Shannon - A mathematical theory of communication. The Bell System Technical Journal, Vol. 27, pp. 379–423, 623–656, July, October, 1948.

Capacity of a classical channel

- The channel introduces errors
  $\longrightarrow$ We add redundancy

[1] C. Shannon - A mathematical theory of communication. The Bell System Technical Journal, Vol. 27, pp. 379–423, 623–656, July, October, 1948.

Capacity of a classical channel

- The channel introduces errors
  $\longrightarrow$ We add redundancy
- What is the highest rate $R = k/n$ with $P_{err} \to 0$?
  $\longrightarrow$ It is the capacity of the channel. [1]

[1] C. Shannon - A mathematical theory of communication. The Bell System Technical Journal, Vol. 27, pp. 379–423, 623–656, July, October, 1948.

Capacity of a classical channel

- The channel introduces errors
  $\longrightarrow$ We add redundancy
- What is the highest rate $R = k/n$ with $P_{err} \to 0$?
  $\longrightarrow$ It is the capacity of the channel. [1]
- We want fast encoding and decoding
  $\longrightarrow$ sparse codes
  $\longrightarrow$ IN COMPENSATION: a bit below the capacity.

---

[1] C. Shannon - A mathematical theory of communication. The Bell System Technical Journal, Vol. 27, pp. 379–423, 623–656, July, October, 1948.

Capacity of a classical channel

- The channel introduces errors
  $\longrightarrow$ We add redundancy
- What is the highest rate $R = k/n$ with $P_{err} \to 0$?
  $\longrightarrow$ It is the capacity of the channel. [1]
- We want fast encoding and decoding
  $\longrightarrow$ sparse codes
  $\longrightarrow$ IN COMPENSATION: a bit below the capacity.
- With stabilizers of small weight, we can use degeneracy.

---

[1] C. Shannon - A mathematical theory of communication. The Bell System Technical Journal, Vol. 27, pp. 379–423, 623–656, July, October, 1948.

Plan

1. **Capacity of the quantum erasure channel**
   1. Capacity of the quantum erasure channel
   2. Stabilizer codes
   3. A combinatorial proof

2. **With sparse quantum codes**
   1. Expected rank of a random sparse submatrix
   2. Achievable rates of sparse quantum codes over the QEC

3. **An application to percolation theory**
   1. Kitaev's toric code and percolation
   2. Hyperbolic quantum codes
   3. Bound on the critical probability

Capacity of the quantum erasure channel

What is the highest rate $R = k/n$ of quantum codes with $P_{err} \to 0$?

Theorem (Bennet, DiVicenzo, Smolin - 97)

The capacity of the quantum erasure channel is $1 - 2p$.

Proved with no-cloning [2] $\longrightarrow$ independent of quantum codes properties.

Goal: find a combinatorial proof and improve it for particular families of codes

[2] C. H. Bennett, D. P. DiVincenzo, and J. A. Smolin - Capacities of Quantum Erasure Channels. Phys. Rev. Lett. 78, 3217–3220 (1997)

Stabilizer codes

- $S = < S_1, \ldots, S_r >$ a stabilizer group of rank r.
- $C(S) = \text{Fix}(S)$ is the stabilizer code.
- $R = \frac{n-r}{n}$ is the rate of the stabilizer code.
- The syndrome of $E \in \mathcal{P}_n$ is $\sigma(E) \in \mathbb{F}_2^r$ such that:

$$\sigma_i = 0 \Leftrightarrow E \text{ and } S_i \text{ commute .}$$

$\longrightarrow$ If $E' \in S$ then E and EE' have the same effect.
$\longrightarrow$ We can measure the syndrome.

Using the syndrome, we search the most probable error.

The quantum erasure channel

Each qubit is erased independently with proba p.

$$\text{erased qubit} \longleftrightarrow \begin{cases} \text{random Pauli error } I, X, Y, Z \\ \text{erased position known} \end{cases}$$

On n qubits: $e \in \mathbb{F}_2^n$ denotes the erased positions
$|\psi\rangle \to E|\psi\rangle$ with $Supp(E) \subset e$ (we write $E \subset e$)

- the erased positions are known: $e \in \mathbb{F}_2^n$,
- the syndrome is known: $\sigma(E) \in \mathbb{F}_2^r$,
- the error $E \subset e$ is unknown.

To correct the state, we search an error $E \subset e$ with syndrome σ.

A combinatorial bound

$$\mathbf{H} = \begin{pmatrix} I & X & Z & Y & Z \\ Z & Z & X & I & Z \\ I & Y & Y & Y & Z \end{pmatrix}$$

$$e = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

- There are 4^2 errors $E \subset e$

A combinatorial bound

$$\mathbf{H}_e = \begin{pmatrix} I & X & Z & Y & Z \\ Z & Z & X & I & Z \\ I & Y & Y & Y & Z \end{pmatrix}$$

$$e = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

- There are 4^2 errors $E \subset e$
- There are 2^2 syndromes $\sigma(E)$ with $E \subset e$

A combinatorial bound

$$\mathbf{H}_{\bar{e}} = \begin{pmatrix} I & X & Z & Y & Z \\ Z & Z & X & I & Z \\ I & Y & Y & Y & Z \end{pmatrix}$$

$$e = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

- There are 4^2 errors $E \subset e$
- There are 2^2 syndromes $\sigma(E)$ with $E \subset e$
- There are 2 errors in each degeneracy class

A combinatorial bound

$$\mathbf{H}_{\bar{e}} = \begin{pmatrix} I & X & Z & Y & Z \\ Z & Z & X & I & Z \\ I & Y & Y & Y & Z \end{pmatrix}$$

$$e = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

- There are 4^2 errors $E \subset e$
- There are 2^2 syndromes $\sigma(E)$ with $E \subset e$
- There are 2 errors in each degeneracy class

$\longrightarrow$ e can not be corrected

A combinatorial bound

$$\mathbf{H}_{\bar{e}} = \begin{pmatrix} I & X & Z & Y & Z \\ Z & Z & X & I & Z \\ I & Y & Y & Y & Z \end{pmatrix}$$

$$e = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

- There are 4^2 errors $E \subset e$
- There are 2^2 syndromes $\sigma(E)$ with $E \subset e$
- There are 2 errors in each degeneracy class

$\longrightarrow$ e can not be corrected

- $2^{\operatorname{rank} \mathbf{H}_e}$ syndromes
- $2^{\operatorname{rank} \mathbf{H} - \operatorname{rank} \mathbf{H}_{\bar{e}}}$ errors, in each class

Lemma

We can correct

$$2^{\operatorname{rank} \mathbf{H} - (\operatorname{rank} \mathbf{H}_{\bar{e}} - \operatorname{rank} \mathbf{H}_e)}$$

errors $E \subset e$.

A combinatorial bound

Let $(\mathbf{H}_t)$ be a sequence of stabilizer matrices of rate R.

Theorem (D., Zémor - 2011)

If $P_{err} \to 0$ *then*

$$R \leq 1 - 2p - g(p) \leq 1 - 2p,$$

where

$$g(p) = \limsup E_p \left(\frac{\operatorname{rank} \mathbf{H}_{\bar{e}} - \operatorname{rank} \mathbf{H}_e}{n} \right)$$

A combinatorial bound

Let $(\mathbf{H}_t)$ be a sequence of stabilizer matrices of rate R.

Theorem (D., Zémor - 2011)

If $P_{err} \to 0$ then

$$R \leq 1 - 2p - g(p) \leq 1 - 2p,$$

where

$$g(p) = \limsup E_p \left(\frac{\operatorname{rank} \mathbf{H}_{\bar{e}} - \operatorname{rank} \mathbf{H}_e}{n} \right)$$

- For general stabilizer codes $g(p)$ can be small (≈ 0)
- BUT for sparse matrices, this bound is below the capacity

Goal: estimate $g(p)$ for sparse matrices

Rank of a random sparse matrix

$$\left(\begin{array}{c|c} \underbrace{\quad \mathbf{H}_e \quad}_{pn \text{ columns}} & \qquad\qquad\qquad\qquad\qquad\qquad \end{array} \right)$$

- Typically: $\mathbf{H}_e$ is a $r \times np$ matrix

Rank of a random sparse matrix

$$\left(\begin{array}{c|c} \underbrace{\mathbf{H}_e}_{pn \text{ columns}} & \qquad\qquad\qquad\qquad \end{array}\right)$$

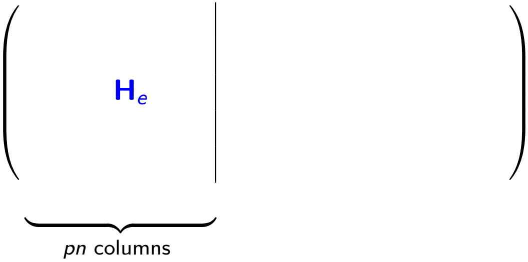
- Typically: $\mathbf{H}_e$ is a $r \times np$ matrix

Rank of a random sparse matrix

$$\left(\begin{array}{c|c} \underbrace{\quad \mathbf{H}_e \quad}_{pn \text{ columns}} & \qquad\qquad\qquad\qquad \end{array}\right)$$

- Typically: $\mathbf{H}_e$ is a $r \times np$ matrix

Rank of a random sparse matrix



- Typically: $\mathbf{H}_e$ is a $r \times np$ matrix

Rank of a random sparse matrix

$$\left(\underbrace{\qquad \mathbf{H}_e \qquad}_{pn \text{ columns}} \;\Bigg|\; \qquad\qquad\qquad \right)$$

- Typically: $\mathbf{H}_e$ is a $r \times np$ matrix
- When $np = r$, the square matrix $\mathbf{H}_e$ has almost full rank
  $\longrightarrow g(p)$ is close 0

Rank of a random sparse matrix

$$\left(\begin{array}{c|ccc} & Z & X & Z \\ \mathbf{H}_e & & & \\ & & & \end{array}\right)$$

$\underbrace{\hspace{4cm}}_{pn \text{ columns}}$

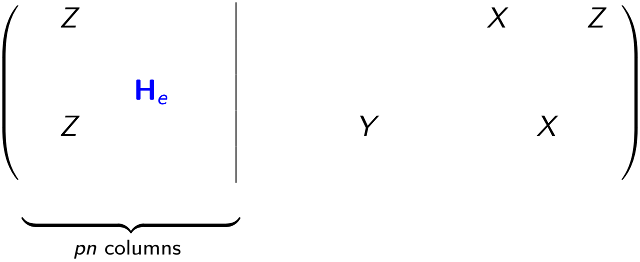
- Typically: $\mathbf{H}_e$ is a $r \times np$ matrix
- When $np = r$, the square matrix $\mathbf{H}_e$ has almost full rank $\longrightarrow g(p)$ is close 0

Rank of a random sparse matrix

$$\left(\begin{array}{c|ccc} & Z & X & Z \\ \underbrace{\quad \mathbf{H}_e \quad}_{pn \text{ columns}} & & & \end{array}\right)$$

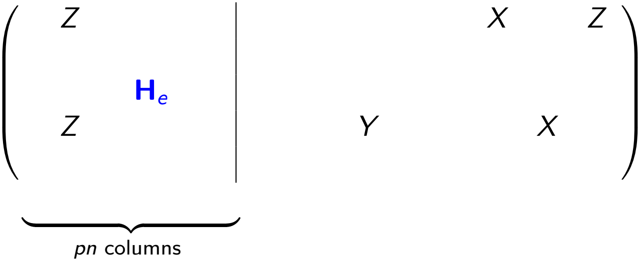
- Typically: $\mathbf{H}_e$ is a $r \times np$ matrix
- When $np = r$, the square matrix $\mathbf{H}_e$ has almost full rank
  $\longrightarrow g(p)$ is close 0
- BUT for a sparse matrix $\mathbf{H}$, there are αn null rows in $\mathbf{H}_e$
  $\longrightarrow g(p) > \lambda$
  $\longrightarrow$ Bound on achievable rates

Rank of a random sparse matrix



- Typically: $\mathbf{H}_e$ is a $r \times np$ matrix
- When $np = r$, the square matrix $\mathbf{H}_e$ has almost full rank
  $\longrightarrow g(p)$ is close 0
- BUT for a sparse matrix $\mathbf{H}$, there are αn null rows in $\mathbf{H}_e$
  $\longrightarrow g(p) > \lambda$
  $\longrightarrow$ Bound on achievable rates

Rank of a random sparse matrix



- Typically: $\mathbf{H}_e$ is a $r \times np$ matrix
- When $np = r$, the square matrix $\mathbf{H}_e$ has almost full rank
  $\longrightarrow g(p)$ is close 0
- BUT for a sparse matrix $\mathbf{H}$, there are αn null rows in $\mathbf{H}_e$
  $\longrightarrow g(p) > \lambda$
  $\longrightarrow$ Bound on achievable rates
- Similarly, there are βn identical rows of weight 1 ...
  $\longrightarrow$ more accurate bound

Achievable rates of sparse CSS codes

Theorem (D., Zémor - 2011)

Achievable rates of CSS(2, m) codes with $d_X, d_Z \geq 2\delta + 1$*, over the quantum erasure channel of probability* p *satisfy:*

$$\begin{aligned} R &\leq 1 - 2p - g(p) \\ &\leq (1-2p)\Big(\frac{4}{mp}\big(1 - (1-p)^m S_\delta(p(1-p)^{m-2})\big) - 1\Big) \end{aligned}$$

S_δ depends on the generating function for rooted subtrees in the m-regular tree

Achievable rates of sparse CSS codes

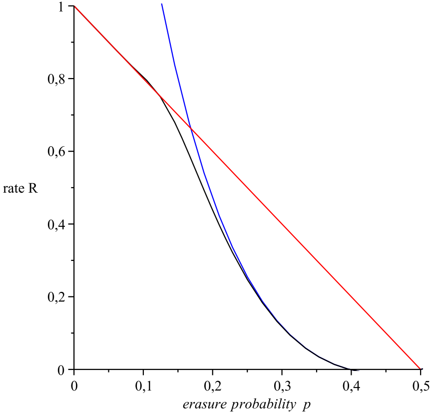


Figure: Bounds on achievable rates of CSS(2,8) codes with $\delta = 0$ (blue) and $\delta = 30$ (black)

Kitaev's toric code and percolation

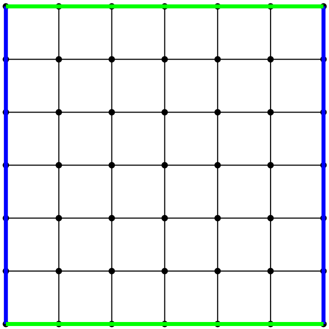


Figure: The toric code

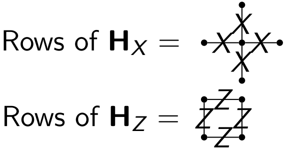


Figure: The stabilizers

- It is a $\mathrm{CSS}(2,4)$ code

Kitaev's toric code and percolation

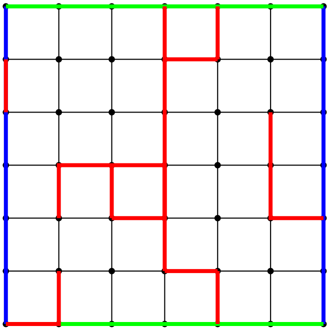
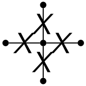


Figure: The toric code

Rows of $\mathbf{H}_X =$



Rows of $\mathbf{H}_Z =$

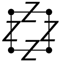


Figure: The stabilizers

- It is a CSS$(2, 4)$ code
- An erasure is problematic iff it covers an homological cycle

Kitaev's toric code and percolation

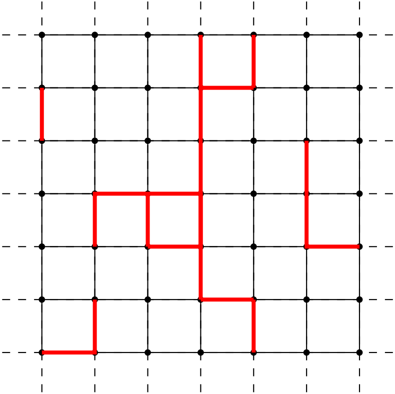
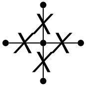


Figure: The toric code

- It is a CSS$(2, 4)$ code
- An erasure is problematic iff it covers an homological cycle

Rows of $\mathbf{H}_X$ = 

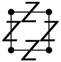
Rows of $\mathbf{H}_Z$ = 

Figure: The stabilizers

- What is the probability P_p of an inifinite red cluster in $\mathbb{Z}^2$?
  There is a critical probability p_c:

$$\begin{cases} \text{if } p < p_c, & \text{then } P_p = 0 \\ \text{if } p > p_c, & \text{then } P_p = 1 \end{cases}$$

Kitaev's toric code and percolation

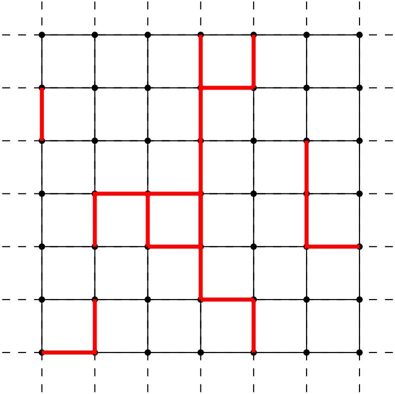


Figure: The toric code

- It is a CSS(2, 4) code
- An erasure is problematic iff it covers an homological cycle

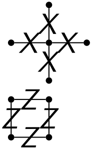
Rows of $\mathbf{H}_X =$ 

Figure: The stabilizers

- What is the probability P_p of an inifinite red cluster in $\mathbb{Z}^2$?
  There is a critical probability p_c:

$$\begin{cases} \text{if } p < p_c, \text{ then } P_p = 0 \\ \text{if } p > p_c, \text{ then } P_p = 1 \end{cases}$$

- For large graphs:
  problematic erasure ≈ infinite cluster

Hyperbolic percolation

Goal: connect percolation and quantum erasure for other graphs

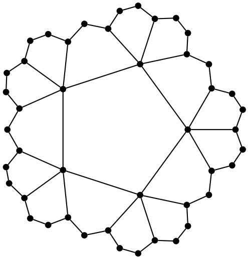


Figure: A few faces of the 5-regular graph $G(5)$

Definition

$G(m)$ is the self-dual m-regular tiling.

The determination of $p_c(G(m))$ is difficult.
(Benjamini, Schramm, and later Baek, Kim, Minnhagen)

Percolation and capacity

Using finite quotients of $G(m)$ proposed by Siran in 2001, we constructed surface codes such that:

- locally look like to $G(m)$
- constant rate R
- sparse of type $(2, m)$

Main argument: if $p < p_c$ then R is under the capacity

Percolation and capacity

The critical proba of $G(m)$ satisfy:

Using the capacity: $R \leq 1 - 2p$:

Theorem (D., Zémor - 2010)

$$\frac{1}{m-1} \leq p_c \leq \frac{2}{m}.$$

Using our bound on quantum LDPC codes:

Theorem (D., Zémor - 2011)

$p_c \leq p$ where p is solution of:

$$1 - \frac{4}{m} = (1-2p)\Big(\frac{4}{mp}(1-(1-p)^m S_{m/2}(p(1-p)^{m-2})) - 1\Big)$$

Numerical results

m	$\frac{1}{m-1} \leq p_c$	improved bound: $p_c \leq$	with the capacity: $p_c \leq \frac{2}{m}$
5	0.25	0.38	0.40
10	0.11	0.17	0.20
20	0.053	0.073	0.100
30	0.035	0.046	0.067
40	0.026	0.034	0.050
50	0.020	0.026	0.040

Numerical results

CONCLUSION:

We obtained:

- Similar for CSS (ℓ, m) codes using hypergraphs
- Similar for stabilizer (ℓ, m) codes

OPEN QUESTIONS:

- With the depolarizing channel?
- Do stabilizer codes surpass CSS codes?
- What is exactly $p_c(G(m))$?

Numerical results

Questions?

Thank you for your attention!