Universal Topological Phase of 2D Stabilizer Codes

Héctor Bombín
Perimeter Institute
collaborators

David Poulin

Guillaume Duclos-Cianci

Université de Sherbrooke
topological codes

qubit

check operator

logical qubit

general structure?
translational invariance

\[ S_i |\phi\rangle = |\phi\rangle \quad -1 \notin \langle S_i \rangle \subset \mathcal{P} \]

classify the bulk!
topological order

\[ S_i |\phi\rangle = |\phi\rangle \quad -1 \not\in \langle S_i \rangle \subset \mathcal{P} \]

\[ H := - \sum_i S_i \]

- gapped excitations
- GS degeneracy
- locally indistinguishable GS
topological order

\[ S_i |\phi\rangle = |\phi\rangle \quad -1 \notin \langle S_i \rangle \subset P \]

\[ H := - \sum_i S_i \]

topological order describes the equivalent classes defined by local unitary evolutions

(Chen, Gu, Wen ’10)
topological order

- They argue that two gapped GSs...
  - ...are in the same phase
  - ...can be connected adiabatically without closing the gap
  - ...are connected by a local unitary transformation
  - ...are connected by a quantum circuit of constant depth
topological order

classify topological codes/order = classify long-range entanglement patterns
goals

structure of 2D topological stabilizer (subsystem) codes

equivalence up to local transformations
outline

• anyons & toric code
• universality
• subsystem codes
• conclusions
anyons in 2D systems
abelian anyons

$q \in \{a, b, \ldots\}$

_topological charges_

$q_1 \times q_2 = q$

 fusion rules

$|\psi\rangle \rightarrow e^{i\alpha_{12}} |\psi'\rangle$

_braiding rules_
toric code

\[ H := - \sum_f S_f - \sum_{f^*} S_{f^*} \]
strings & charges

1, e, m, $\epsilon = e \times m$
mutual statistics

- semionic interactions:

\[ pq = -qp \]

\[ p^{-1}q^{-1}pq = -p^{-1}pq^{-1} = -1 \]

\[ t \]
self-statistics

- $e, m$ are bosons, their composite is fermionic:

$$q^{-1}rp^{-1}qr^{-1}p = -q^{-1}qp^{-1}rr^{-1}p = -1$$
anyon model

- charges: \( \{1, e, m, \epsilon\} \)

- fusion:
  \[
  e \times m = \epsilon \quad e \times \epsilon = m \quad m \times \epsilon = e \\
  e \times e = m \times m = \epsilon \times \epsilon = 1
  \]

- braiding: \( e, m \rightarrow \text{bosons} \quad \epsilon \rightarrow \text{fermion} \)
• anyons & toric code
• universality
• subsystem codes
• conclusions
topological stabilizer groups

- local and translationally invariant (LTI) generators

\[ S^l = \langle S_i^l \rangle \quad \text{and} \quad S = \langle S_i \rangle \]
topological stabilizer groups

• local and translationally invariant (LTI) generators

local undetectable errors do not affect encoded states
topological stabilizer groups

$p \in \mathcal{Z}_\mathcal{P}(S^l) \quad\Rightarrow\quad p \in \langle i1 \rangle S^l$

$p \in \mathcal{Z}_\mathcal{P}(S) \quad\Rightarrow\quad p \in \langle i1 \rangle S$
topological stabilizer groups

\[ \mathbb{Z}_p(S) \times S \]
local equivalence

- coarse graining
- LTI Clifford mapping
- add/remove disentangled qubits
universality result

- same anyon model $\leftrightarrow$ equivalent!
- ruling out chiral anyons (Kitaev ‘06):

*every 2D TSG is locally equivalent to a finite number of copies of the toric code*
proof outline

1. 2D TSGs admit LTI independent generators
2. \# charges \(< \infty\)
3. anyon model from string operators
4. string segment framework, plaquette stabilizers
5. other stabilizers have no charge
6. map: string segments \(<--\) string segments, uncharged stabilizers \(\langle--\rangle\) single qubit stabilizers
string operators

- coarse grain:
  - till each site holds any charge
  - till excitation pairs of same charge can be removed with \textit{string-like operators}
anyon model

\[ \kappa(\text{bullet}, \text{bullet}) = (p; q) := pqp^{-1}q^{-1} \]

\[ \theta(\text{bullet}) = (p; q)(p; r)(q; r) = (p; q)(p; r) \]
anyon model

• well defined:

\[(p; q) \equiv (p'; q')\]
anyon model

- well defined:

\[(p; q) = (p'; q') \quad , \quad \in S\]
anyon model

- charge generators:

\[ \langle 1, 1, 2, 2, \ldots, k, k \rangle \]

\[
\begin{align*}
\kappa(i, j) &= \kappa(i, j) = 1 \\
\kappa(i, j) &= 1 - 2\delta_{ij}
\end{align*}
\]
anyon model

• charge generators:

\[ \langle 1, 1, 2, 2, \ldots, k, k \rangle \]

- toric code
- chiral case
string framework

- **model independent** commutation relations
mapping
• anyons & toric code
• universality
• **subsystem codes**
• conclusions
subsystem codes

Hilbert space

Logical qubits

Gauge qubits

Absorbed errors
subsystem codes

stabilizer group

\[-1 \notin \mathcal{S} \subseteq \mathcal{P}\]

\[S_i \rho = \rho S_i = \rho\]

\[\mathbb{Z}_g(G) \propto S\]

gauge group

\[G \subseteq \mathcal{P}\]

\[\rho \sim G_i \rho G_i^\dagger\]
topological subsystem codes

\[ S^l, G^l \rightarrow S, G \]
topological subsystem codes

local undetectable errors do not affect encoded states
topological subsystem codes

\[ p \in \mathcal{Z}_P(S^l) \quad \Rightarrow \quad p \in \mathcal{G}^l \]

\[ p \in \mathcal{Z}_P(S) \quad \Rightarrow \quad p \in \mathcal{G} \]
topological subsystem codes

\[ \mathbb{Z}_p(S) = \mathcal{G} \]
topological subsystem codes

\[ G := \mathbb{Z}_P(S) \]

\[ \mathbb{Z}_P(\mathbb{Z}_P(S)) \propto S \]
generalized charge

• Stabilizer charge

\[ C_S = \frac{\text{morphisms} \quad \phi : S \rightarrow \pm 1 \quad \text{commutators} \quad \phi = [p, \cdot] \quad p \in P} \]
generalized charge

• Gauge charge

\[ C_g = \frac{\text{morphisms } \phi : G \rightarrow \pm 1}{\text{commutators } \phi = [p, \cdot], \quad p \in \mathcal{P}} \]
topological ‘interactions’

\[ \kappa : C_G \times C_G \longrightarrow \pm 1 \]
\[ \theta : C_G \longrightarrow \pm 1 \]
\[ \kappa : C_G \times C_S \longrightarrow \pm 1 \]
charge morphism

\[ \phi : G \rightarrow \pm 1 \quad \text{and} \quad \phi|_S : S \rightarrow \pm 1 \]

\[ C_G \rightarrow C_S \]

\[ \kappa(\text{blue, green}) = \kappa(\text{blue, square}) \]
canonical generators

• gauge charge generators:

\[ \langle 1, 1, \ldots, k, k, 1, 2, \ldots, l \rangle \]

\[ b, b, b/f, b/f, b, b, b/f \]

• **dual** stabilizer charge generators:

\[ \langle 1, 1, \ldots, k, k, 1, 2, \ldots, l \rangle \]
canonical generators

• all possible anyon models are **combinations** of

  toric code: \( \bullet b \bullet b \)

  topological subsystem color codes: \( \bullet f \bullet f \)

  subsystem toric code: \( \bullet b \bullet f \)

  honeycomb subsystem code:
string framework
string framework

- gauge generators (non-trivial charge)
- stabilizer generators (non-trivial charge)
every 2D TSC has a structure based on an anyon model
conclusions & questions

• the long-range entanglement pattern of toric codes is universal for 2D topological stabilizer models
• all 2D TSCs are anyon based
• the same approach could be used for boundaries or point defects...
• more general 2D models?
• what about 3D?