H. Bombin, G. Duclos-Cianci, D. Poulin **arXiv:1103.4606**

H. Bombin **arXiv:1107.2707**

Universal Topological Phase of 2D Stabilizer Codes

Héctor Bombín

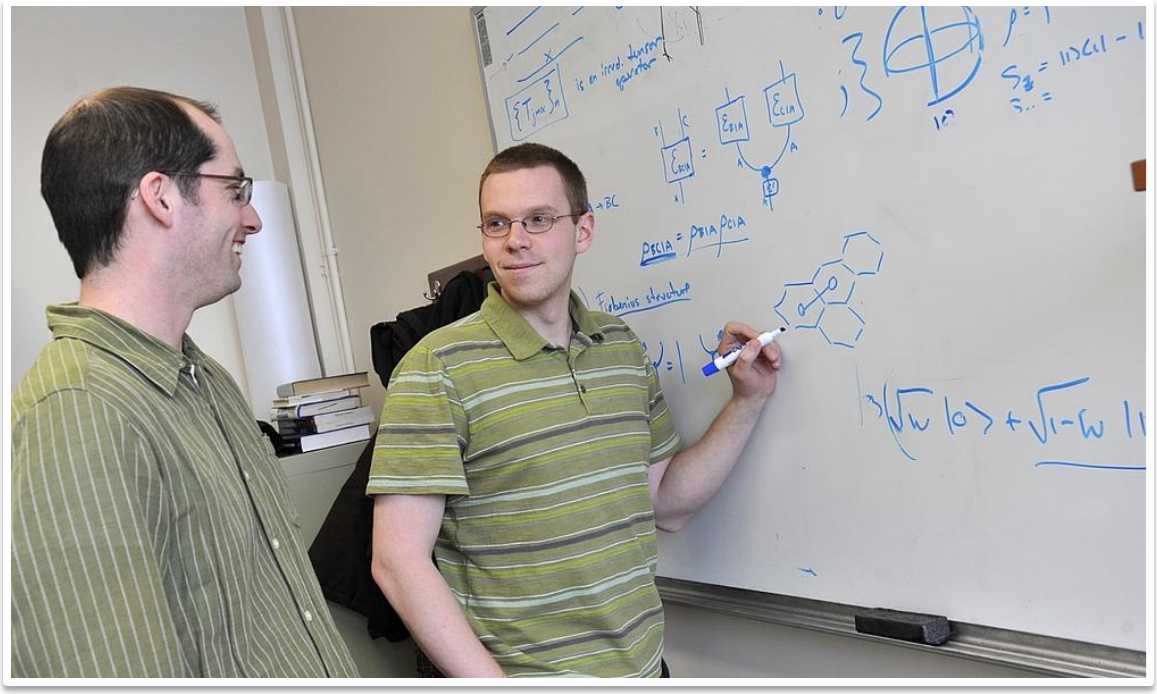
Perimeter Institute

collaborators

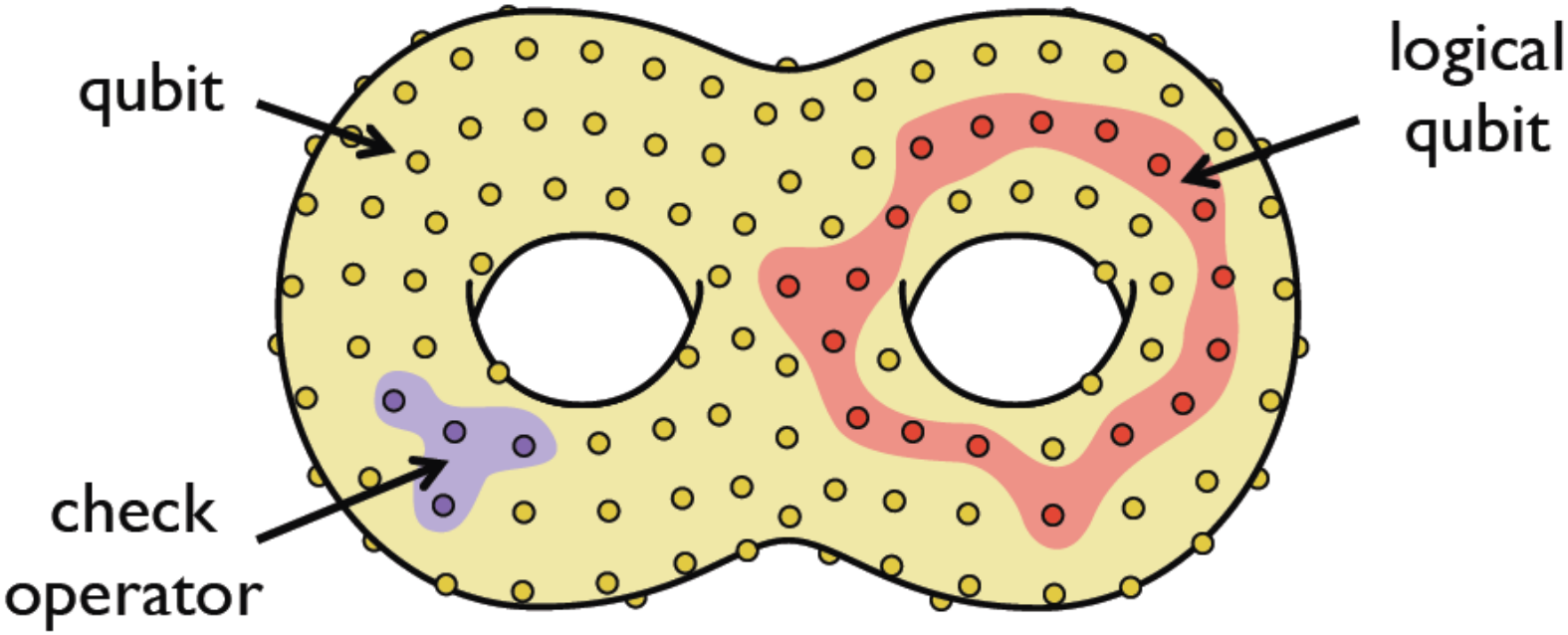
David Poulin

Guillaume Duclos-Cianci

Université de Sherbrooke



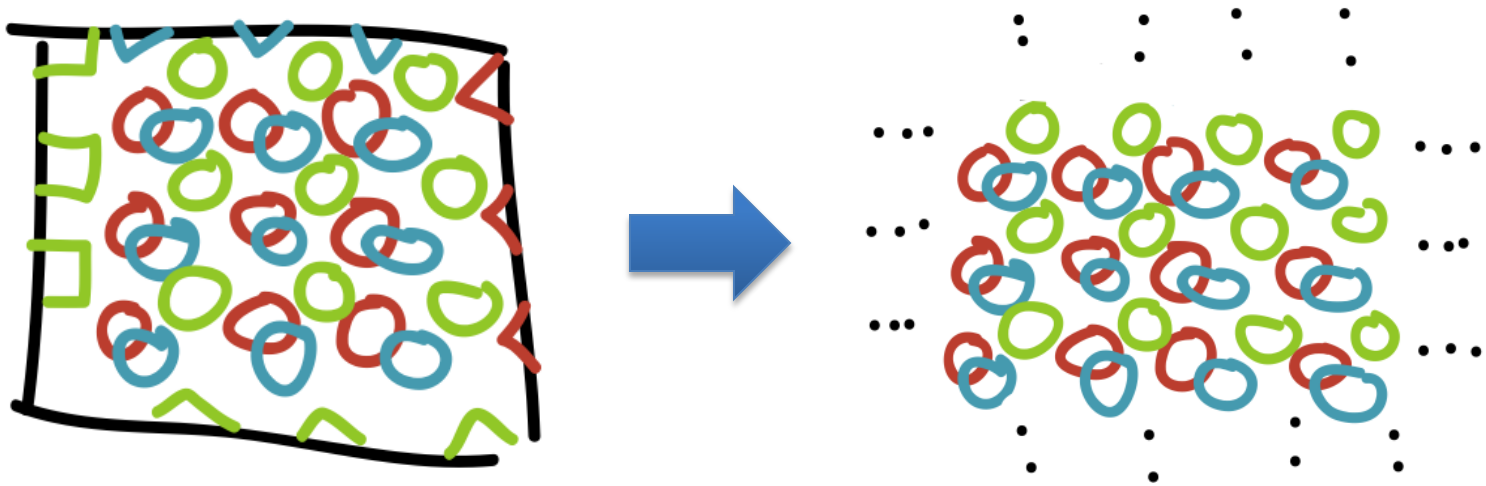
topological codes



general structure?

translational invariance

$$S_i|\phi\rangle = |\phi\rangle \qquad -1 \notin \langle S_i \rangle \subset \mathcal{P}$$



classify the bulk!

topological order

$$S_i|\phi\rangle = |\phi\rangle \qquad -1 \notin \langle S_i \rangle \subset \mathcal{P}$$

$$H := -\sum_i S_i$$

- gapped excitations
- GS degeneracy
- locally indistinguishable GS

topological order

$$S_i|\phi\rangle = |\phi\rangle \qquad -1 \notin \langle S_i \rangle \subset \mathcal{P}$$

$$H := -\sum_i S_i$$

topological order describes the equivalent classes defined by local unitary evolutions

(Chen, Gu, Wen '10)

topological order

- they argue that **two gapped GSs…**

…are in the same phase

⇕

…can be connected adiabatically without closing the gap

⇕

…are connected by a local unitary transformation

⇕

…are connected by a quantum circuit of constant depth

topological order

classify topological codes/order =
classify **long-range entanglement** patterns

goals

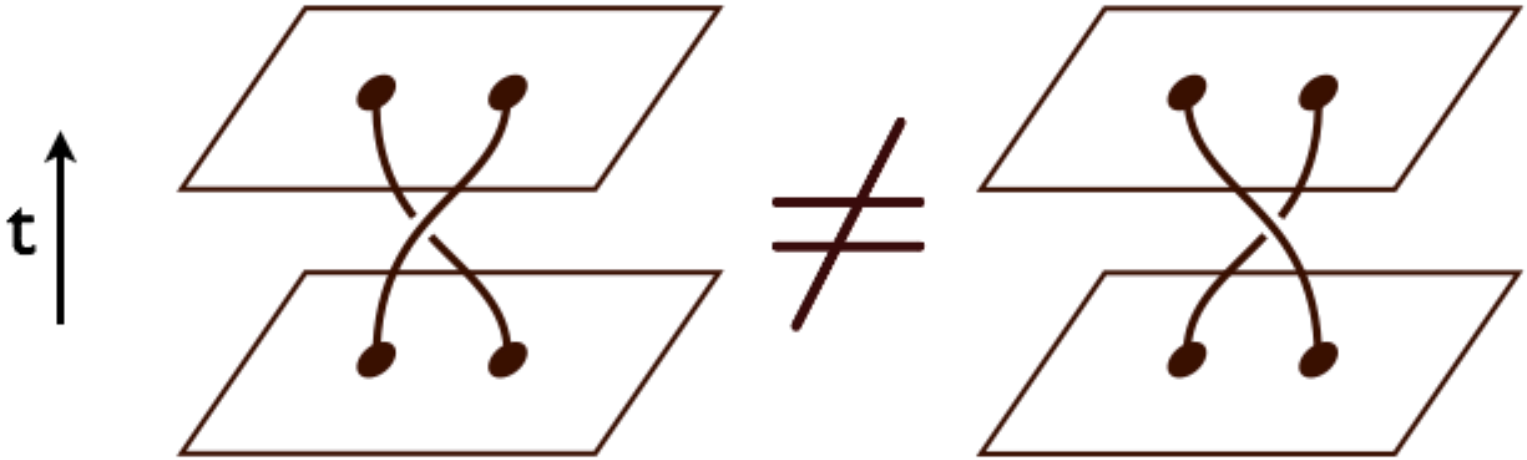
structure of 2D topological stabilizer (subsystem) codes

equivalence up to local transformations

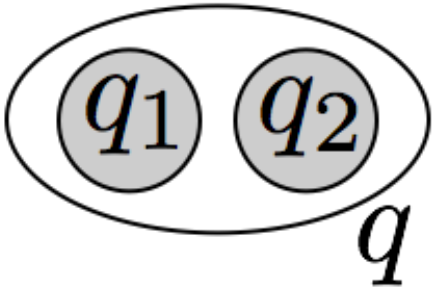
outline

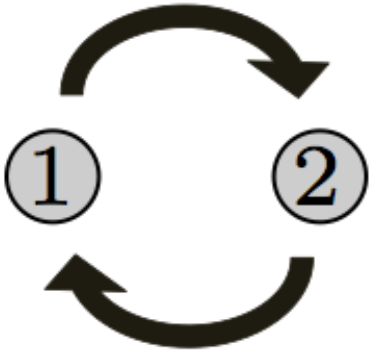
- **anyons & toric code**
- universality
- subsystem codes
- conclusions

anyons in 2D systems



abelian anyons





$q \in \{a, b, \dots\}$

topological
charges

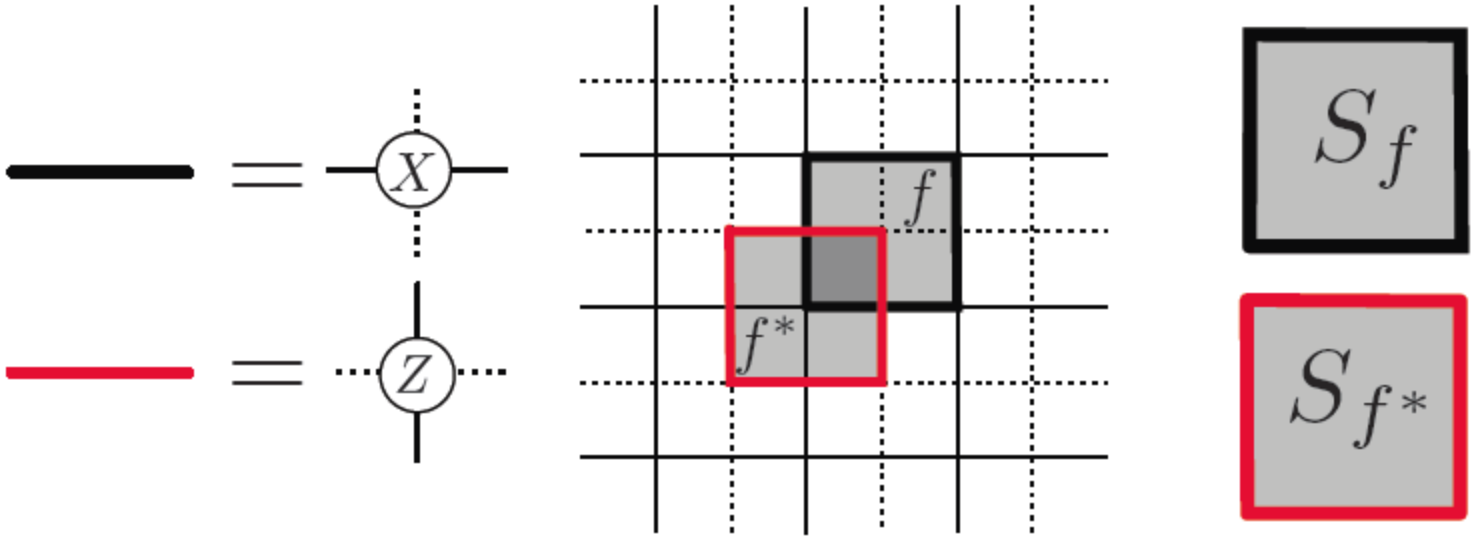
$q_1 \times q_2 = q$

fusion
rules

$|\psi\rangle \to e^{i\alpha_{12}}|\psi'\rangle$

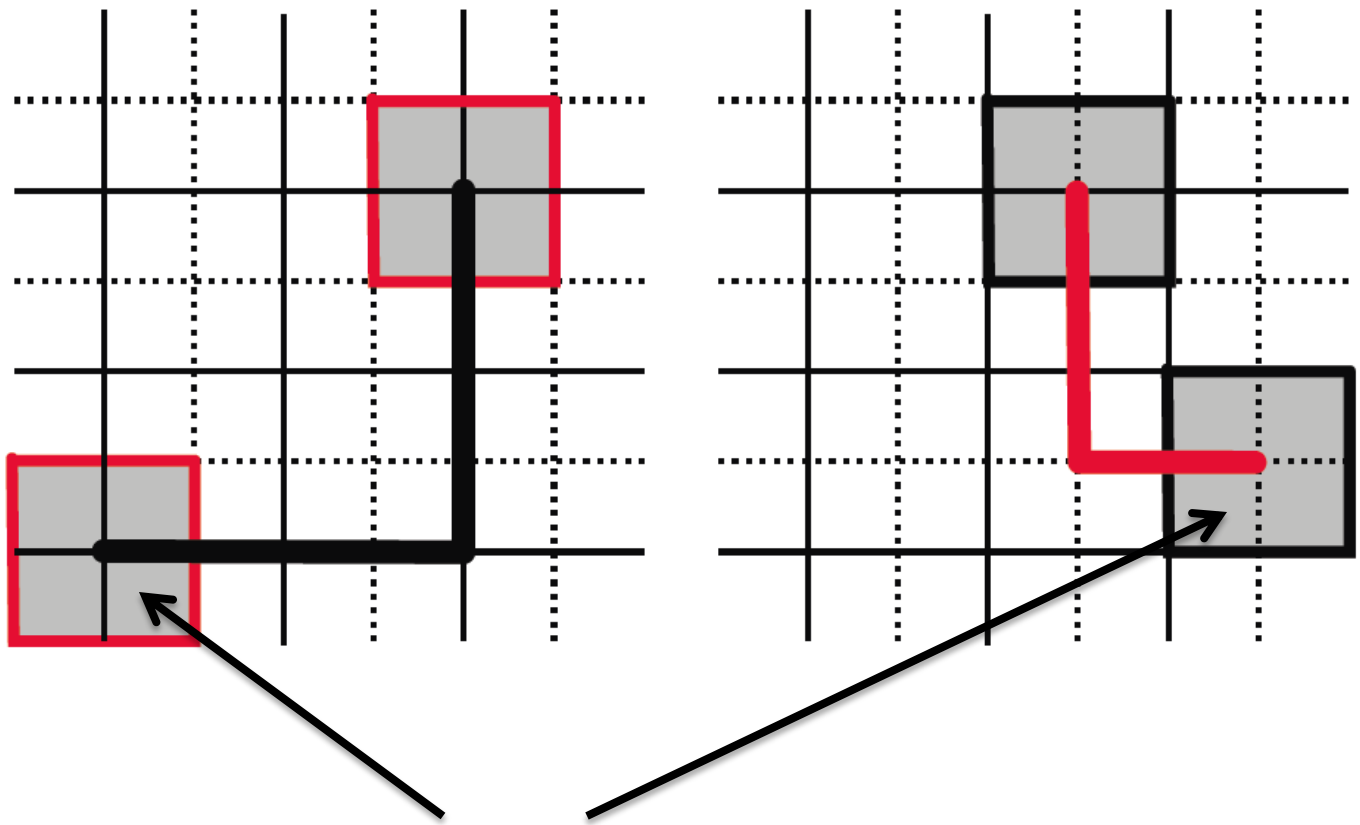
braiding
rules

toric code



$$H := -\sum_{f} S_f - \sum_{f^*} S_{f^*}$$

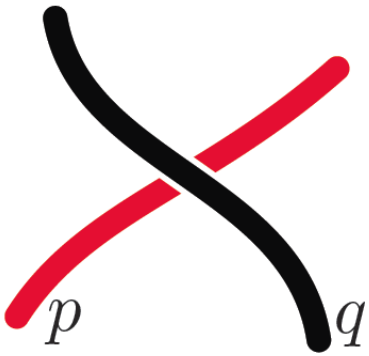
strings & charges



$$1,\ e,\ m,\ \epsilon = e \times m$$

mutual statistics

- semionic interactions:

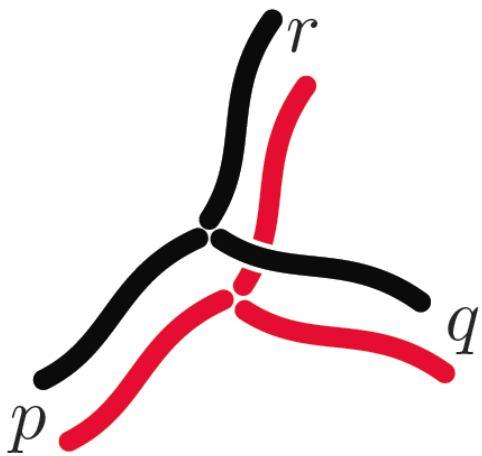


$$pq = -qp$$

$$p^{-1}q^{-1}pq = -p^{-1}pq^{-1}q = -1$$

self-statistics

- e, m are bosons, their composite is fermionic:



$$q^{-1}rp^{-1}qr^{-1}p = -q^{-1}qp^{-1}rr^{-1}p = -1$$

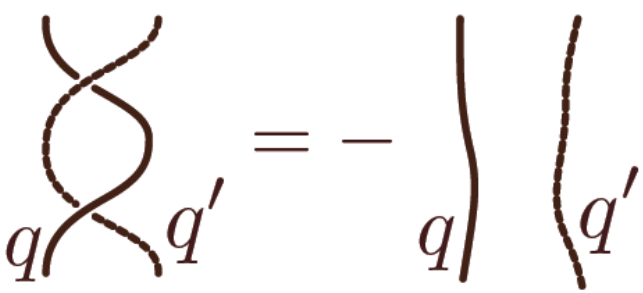
anyon model

- charges: $\{\mathbf{1}, e, m, \epsilon\}$
- fusion:

$$e \times m = \epsilon \qquad e \times \epsilon = m \qquad m \times \epsilon = e$$

$$e \times e = m \times m = \epsilon \times \epsilon = \mathbf{1}$$

- braiding: $e, m \to \text{bosons} \qquad \epsilon \to \text{fermion}$



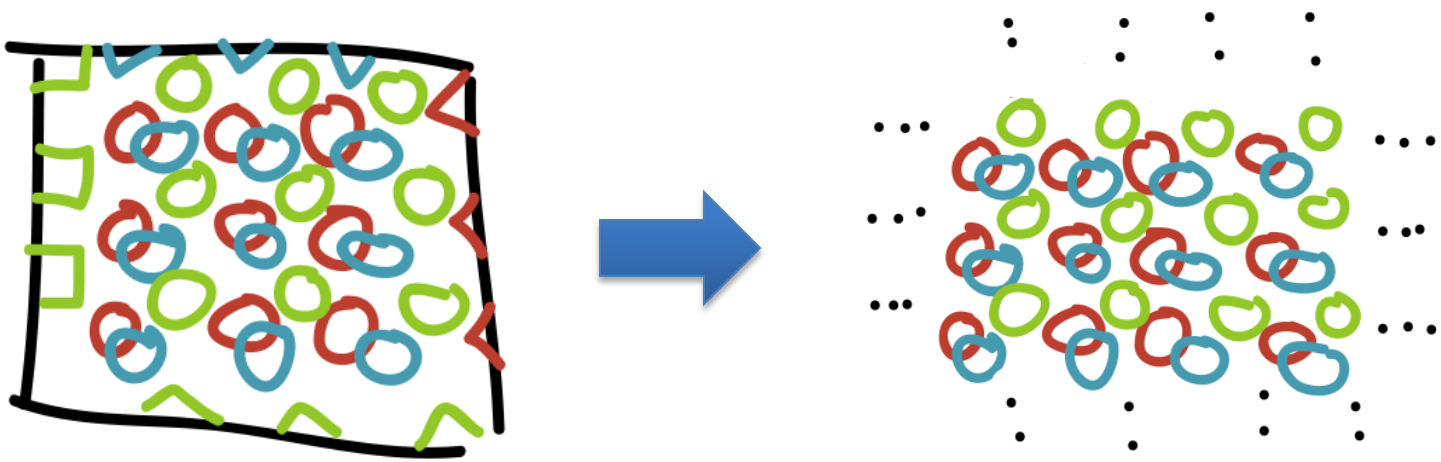
$$\mathbf{1} \neq q \neq q' \neq \mathbf{1}$$

outline

- anyons & toric code
- **universality**
- subsystem codes
- conclusions

topological stabilizer groups

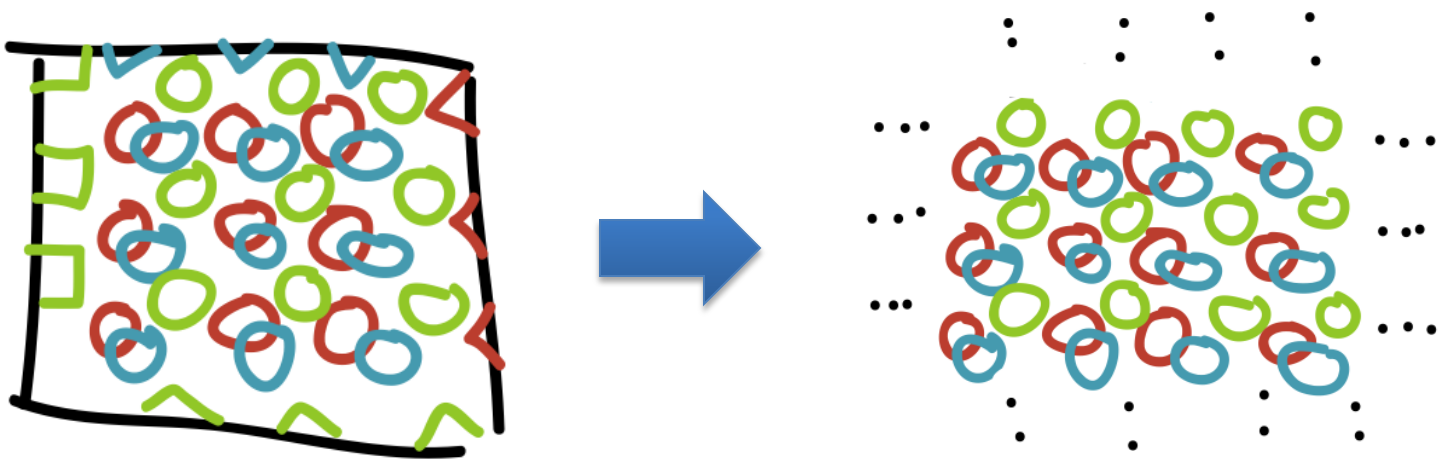
- local and translationally invariant (LTI) generators



$$\mathcal{S}^l = \langle S_i^l \rangle \qquad \mathcal{S} = \langle S_i \rangle$$

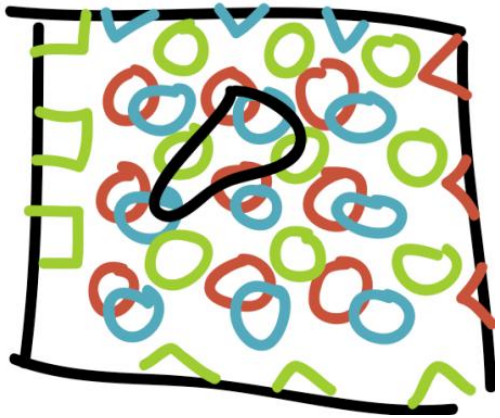
topological stabilizer groups

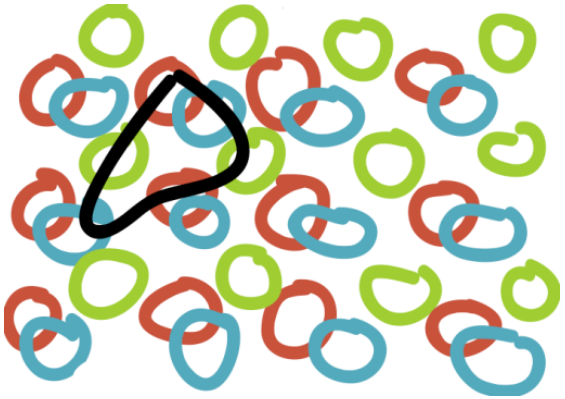
- local and translationally invariant (LTI) generators



local undetectable errors do not affect encoded states

topological stabilizer groups



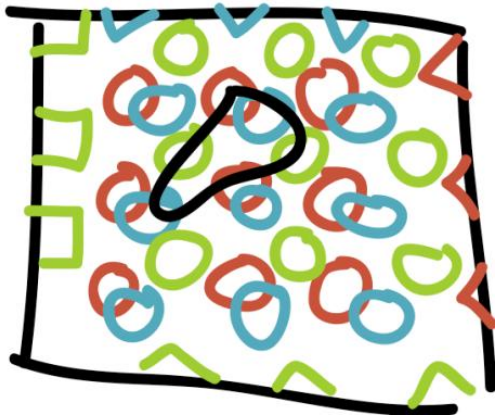


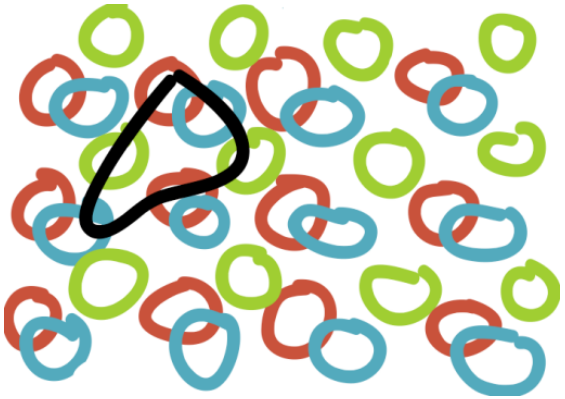
$$p \in \mathcal{Z}_{\mathcal{P}}(\mathcal{S}^l) \quad \Leftarrow \quad p \in \mathcal{Z}_{\mathcal{P}}(\mathcal{S})$$

$$\Downarrow$$

$$p \in \langle i\mathbf{1} \rangle \mathcal{S}^l \quad \Rightarrow \quad p \in \langle i\mathbf{1} \rangle \mathcal{S}$$

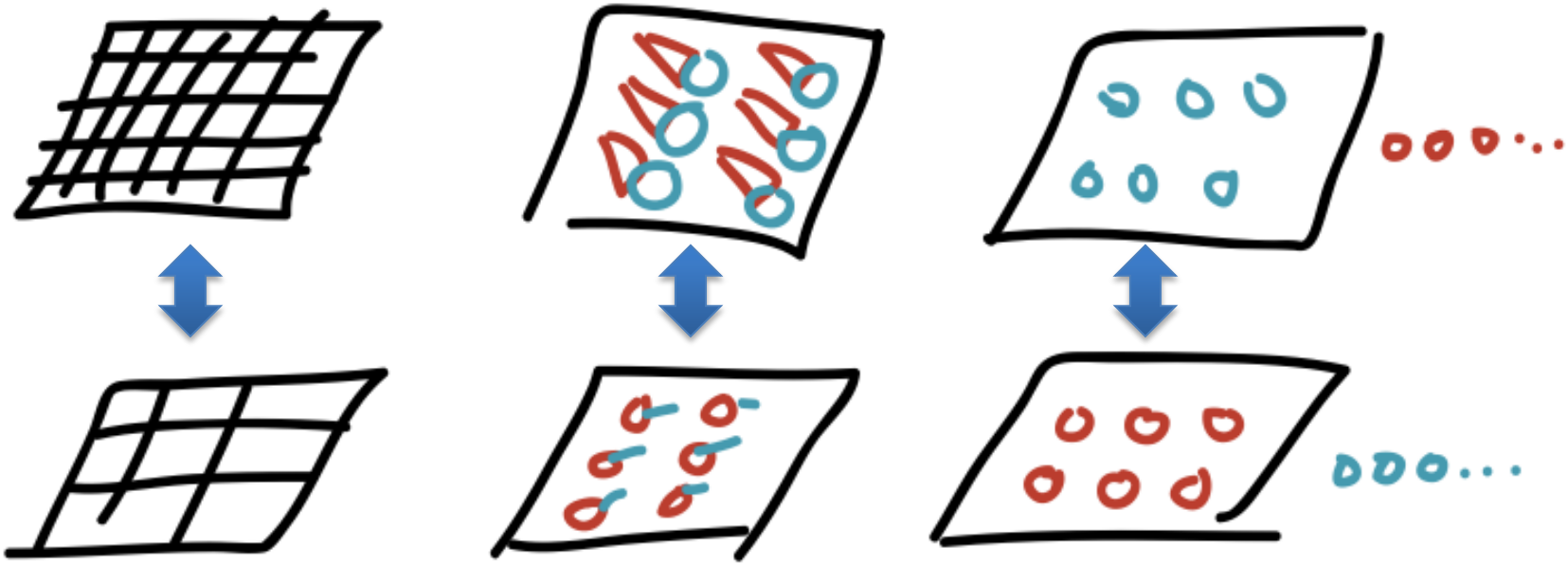
topological stabilizer groups





$$\mathcal{Z}_{\mathcal{P}}(\mathcal{S}) \propto \mathcal{S}$$

local equivalence



coarse graining

LTI Clifford mapping

add/remove disentangled qubits

universality result

- same anyon model $\leftrightarrow$ equivalent!
- ruling out chiral anyons (Kitaev '06):

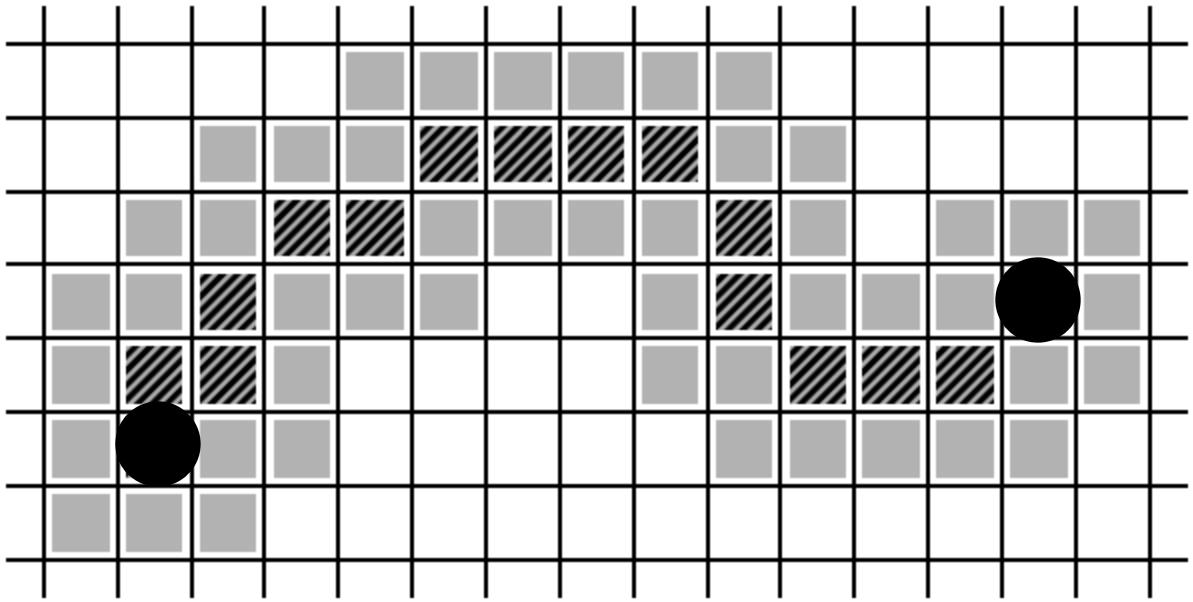
every 2D TSG is locally equivalent to a finite number of copies of the toric code

proof outline

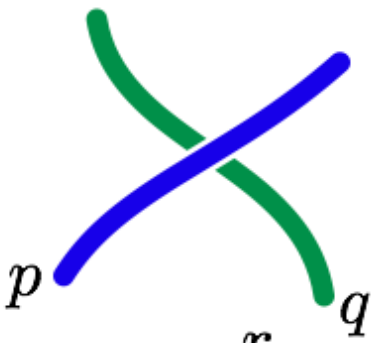
1. 2D TSGs admit LTI **independent** generators
2. # charges < ∞
3. anyon model from **string operators**
4. string segment **framework**, plaquette stabilizers
5. other stabilizers have **no charge**
6. **map:** string segments <--> string segments, uncharged stabilizers <--> single qubit stabilizers

string operators

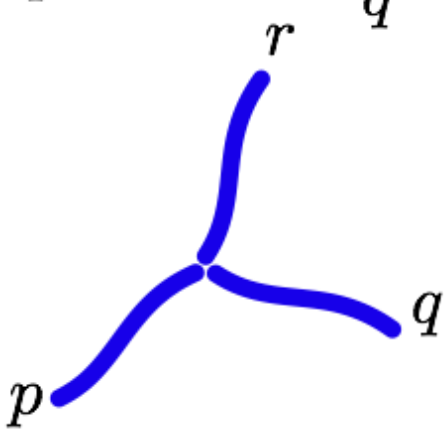
- coarse grain:
  - till each site holds any charge
  - till excitation pairs of same charge can be removed with **string-like operators**



anyon model



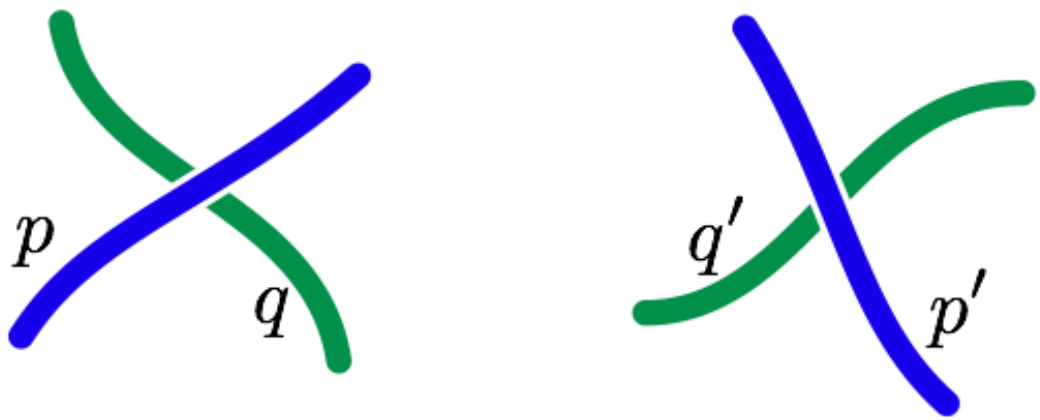
$$\kappa(\bullet\,,\,\bullet) = (p;q) := pqp^{-1}q^{-1}$$



$$\theta(\bullet) = (p;q)(p;r)(q;r) =$$
$$= (p;q)(p;r)$$

anyon model

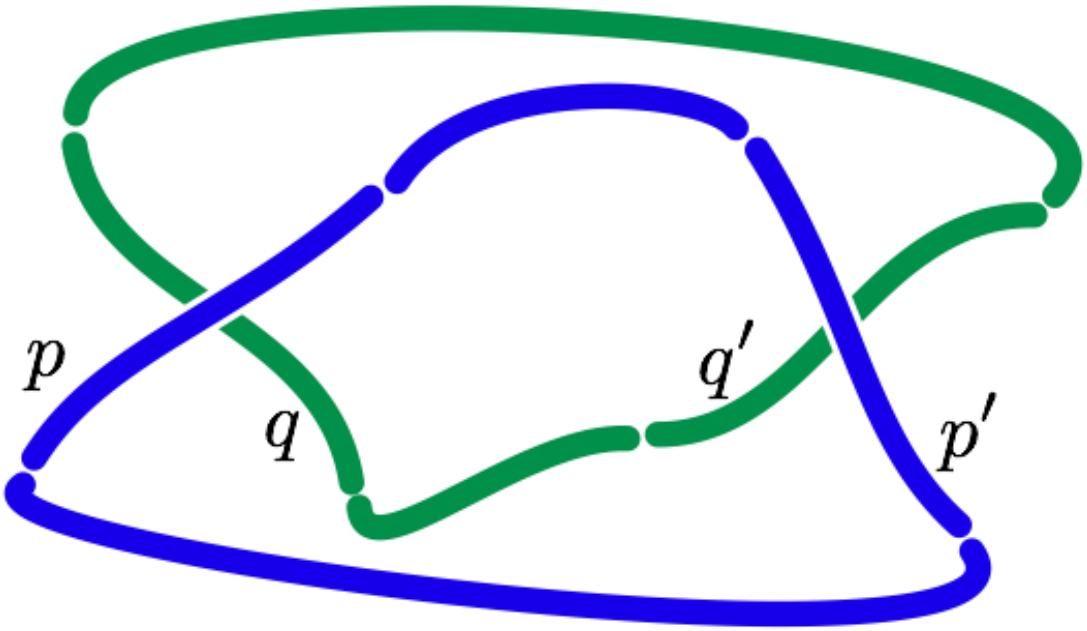
- well defined:



$$(p;q) \stackrel{?}{=} (p';q')$$

anyon model

- well defined:



$$(p;q) = (p';q')$$

$\in \mathcal{S}$

anyon model

- charge generators:

$$\langle \bullet_{1}, \circ_{1}, \bullet_{2}, \circ_{2}, \cdots, \bullet_{k}, \circ_{k} \rangle$$

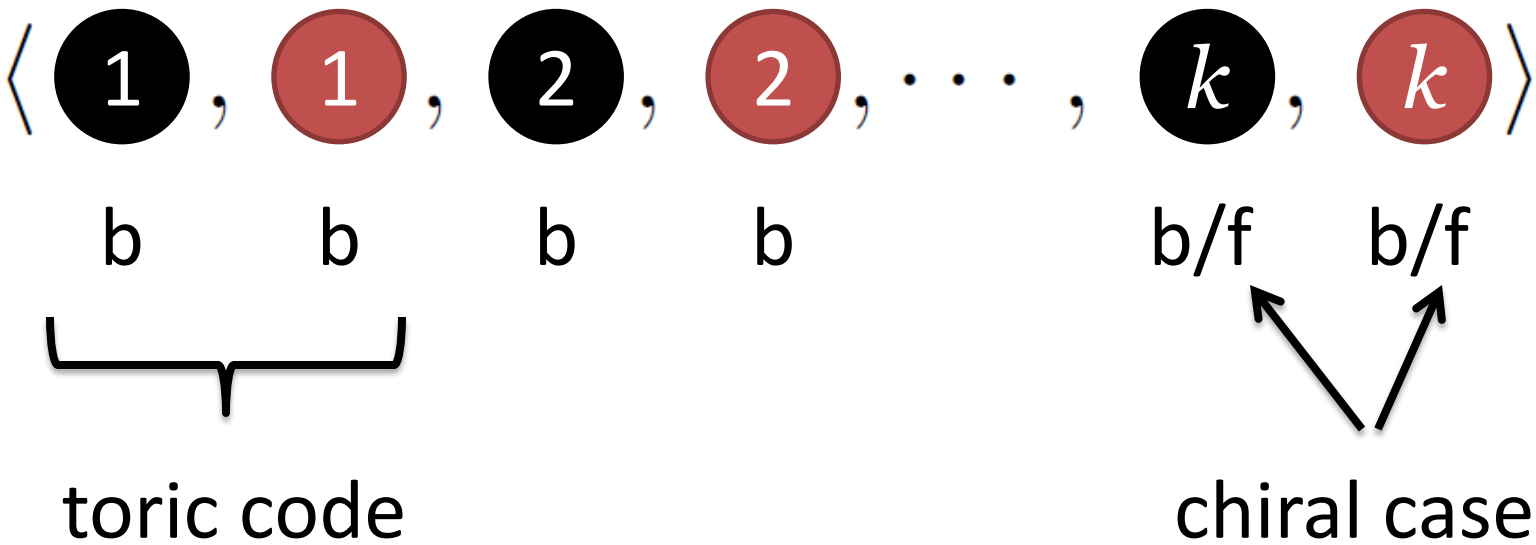
b b b b b/f b/f

$$\kappa(\circ_{i}, \circ_{j}) = \kappa(\bullet_{i}, \bullet_{j}) = 1$$

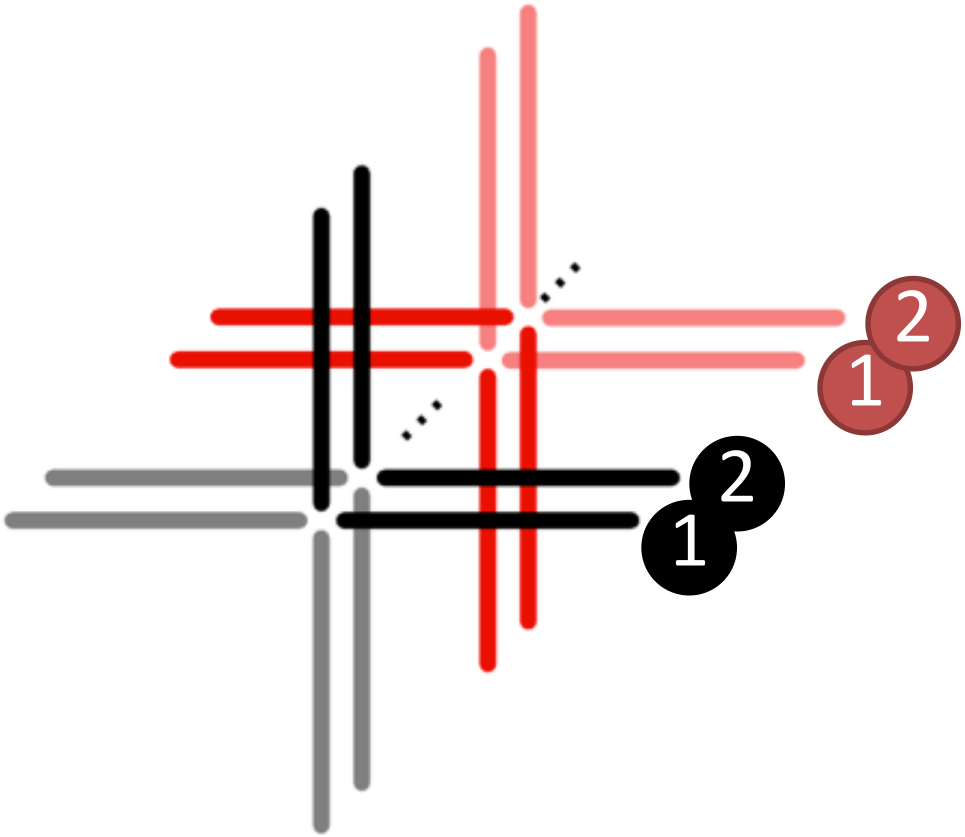
$$\kappa(\circ_{i}, \bullet_{j}) = 1 - 2\delta_{ij}$$

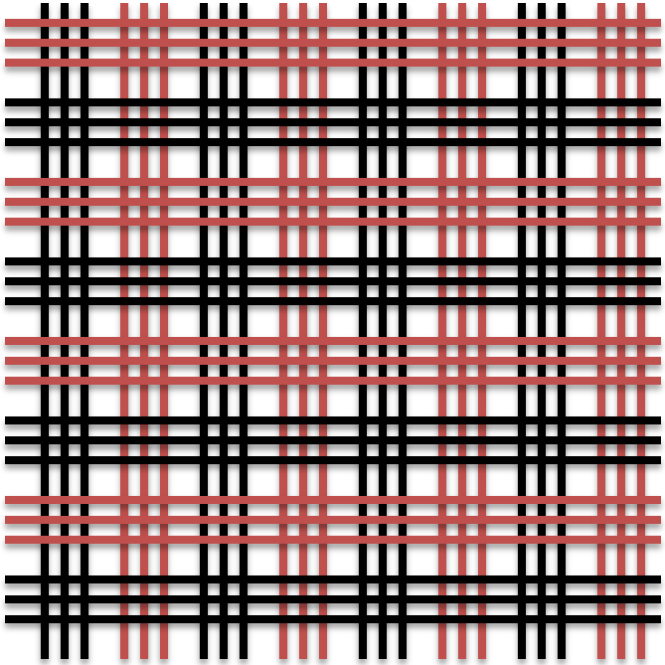
anyon model

- charge generators:



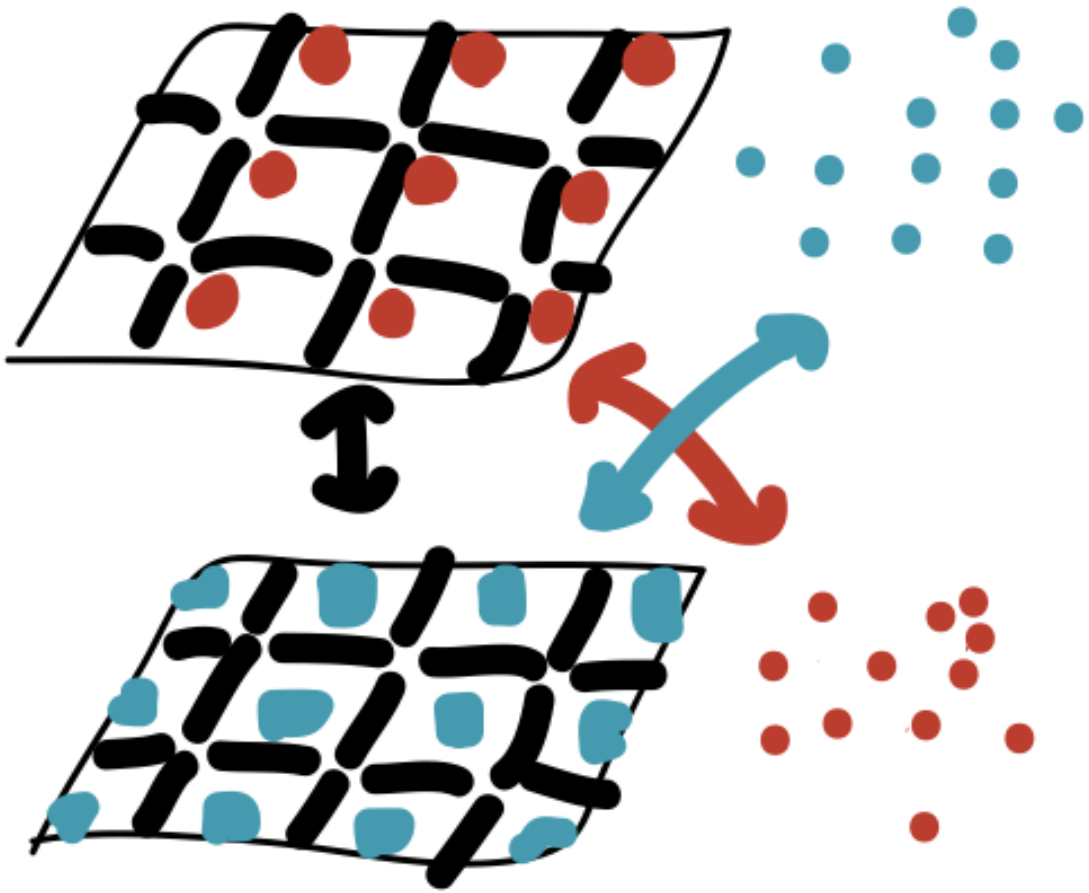
string framework





- **model independent** commutation relations

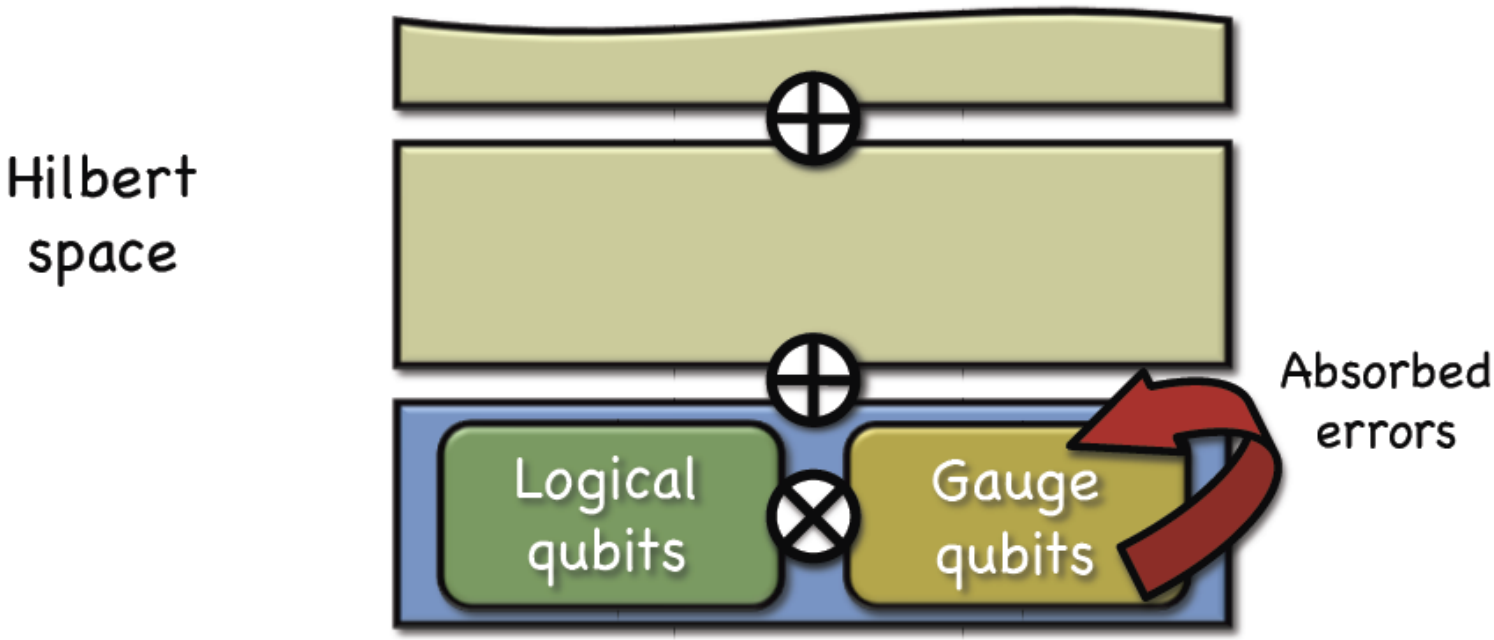
mapping



outline

- anyons & toric code
- universality
- **subsystem codes**
- conclusions

subsystem codes



subsystem codes

stabilizer group

$$-1 \notin \mathcal{S} \subset \mathcal{P}$$

$$S_i \, \rho = \rho \, S_i = \rho$$

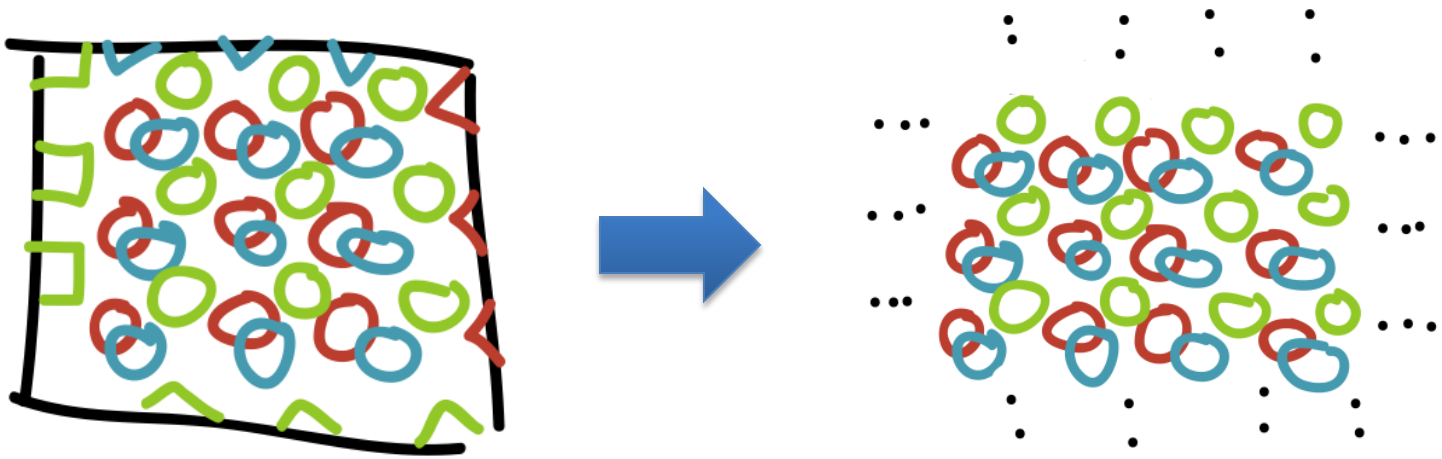
gauge group

$$\mathcal{G} \subset \mathcal{P}$$

$$\rho \sim G_i \, \rho \, G_i^\dagger$$

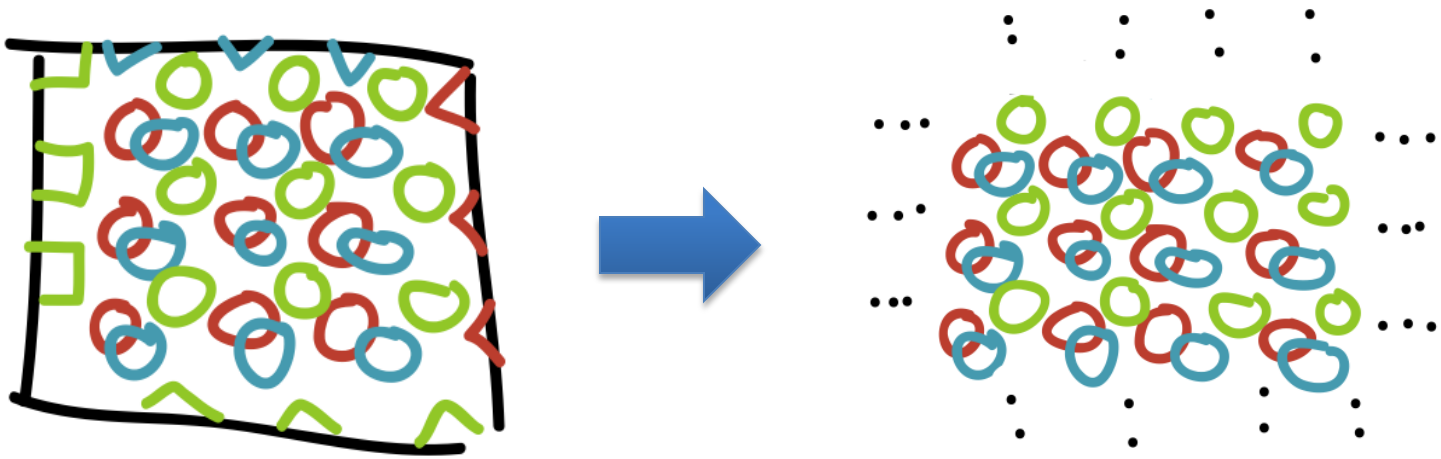
$$\mathcal{Z}_{\mathcal{G}}(\mathcal{G}) \propto \mathcal{S}$$

topological subsystem codes



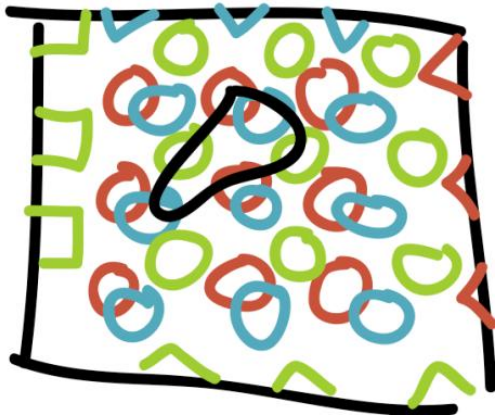
$\mathcal{S}^l, \mathcal{G}^l$ $\qquad$ $\mathcal{S}, \mathcal{G}$

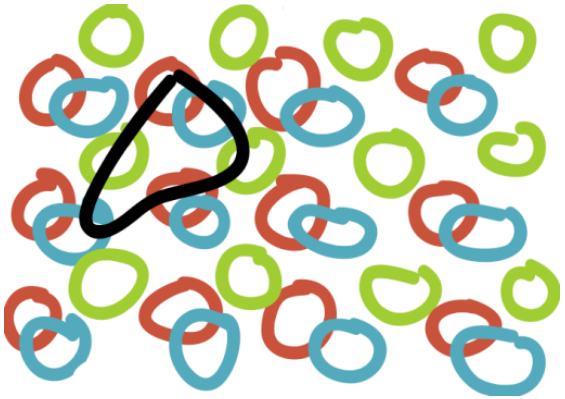
topological subsystem codes



local undetectable errors do not affect encoded states

topological subsystem codes



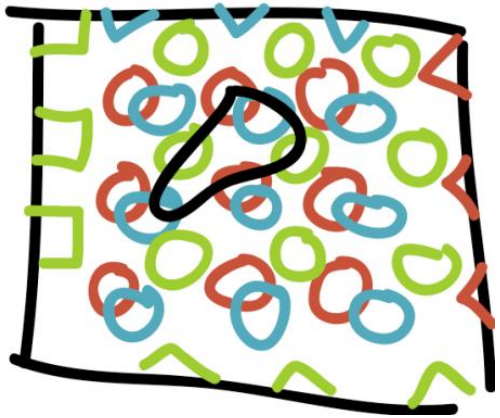


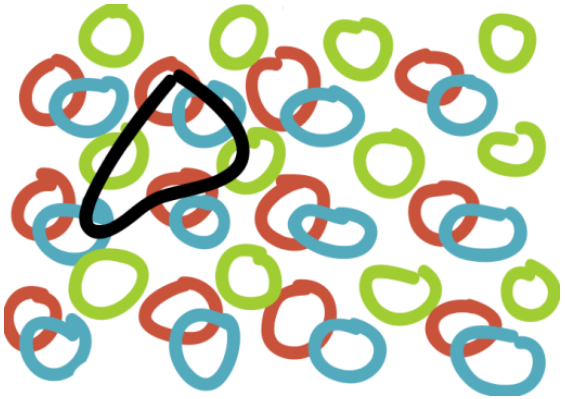
$$p \in \mathcal{Z}_{\mathcal{P}}(\mathcal{S}^l) \quad \Leftarrow \quad p \in \mathcal{Z}_{\mathcal{P}}(\mathcal{S})$$

$$\Downarrow$$

$$p \in \mathcal{G}^l \quad \Rightarrow \quad p \in \mathcal{G}$$

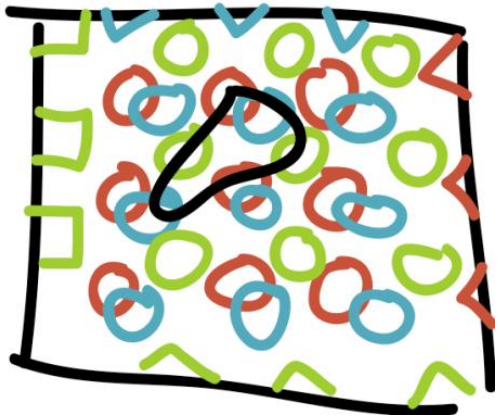
topological subsystem codes

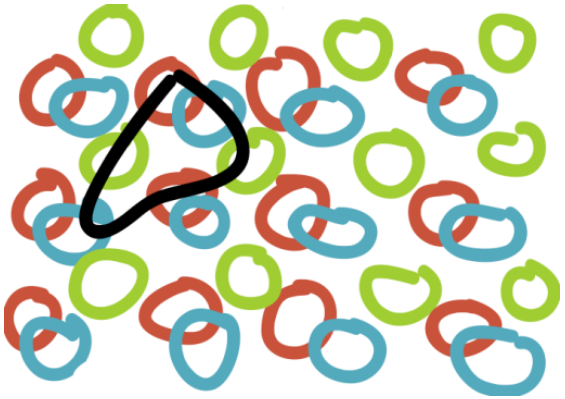




$$\mathcal{Z}_{\mathcal{P}}(\mathcal{S}) = \mathcal{G}$$

topological subsystem codes





$$\mathcal{G} := \mathcal{Z}_{\mathcal{P}}(\mathcal{S})$$

$$\mathcal{Z}_{\mathcal{P}}(\mathcal{Z}_{\mathcal{P}}(\mathcal{S})) \propto \mathcal{S}$$

generalized charge

- Stabilizer charge

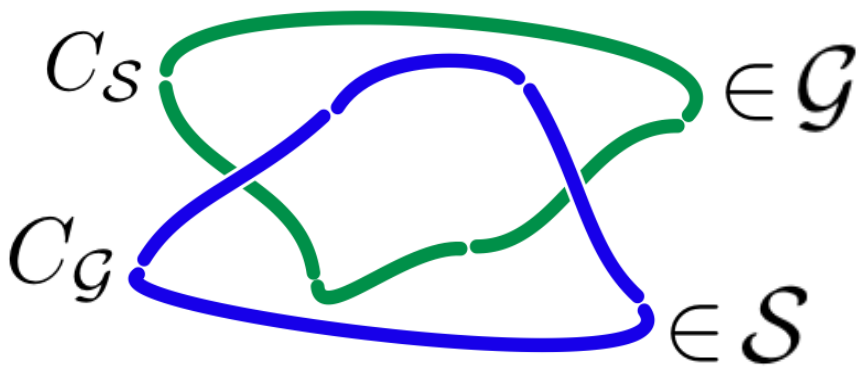
$$C_{\mathcal{S}} = \frac{\text{morphisms} \quad \phi : \mathcal{S} \longrightarrow \pm 1}{\text{commutators} \quad \phi = [p, \cdot] \quad p \in \mathcal{P}}$$

generalized charge

- Gauge charge

$$C_{\mathcal{G}} = \frac{\text{morphisms} \quad \phi : \mathcal{G} \longrightarrow \pm 1}{\text{commutators} \quad \phi = [p, \cdot] \quad p \in \mathcal{P}}$$

topological ‘interactions’



$$\kappa : C_{\mathcal{G}} \times C_{\mathcal{G}} \longrightarrow \pm 1$$

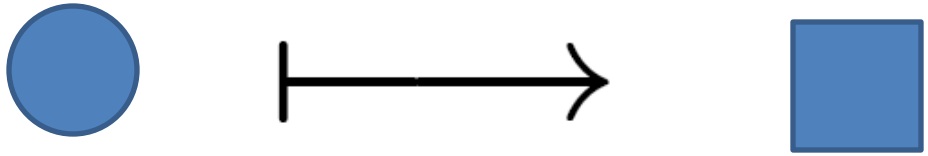
$$\theta : C_{\mathcal{G}} \longrightarrow \pm 1$$

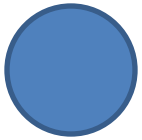
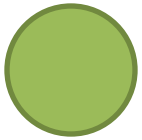
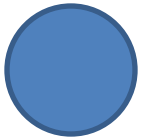
$$\kappa : C_{\mathcal{G}} \times C_{\mathcal{S}} \longrightarrow \pm 1$$

charge morphism

$$\phi : \mathcal{G} \longrightarrow \pm 1 \quad \Rightarrow \quad \phi|_{\mathcal{S}} : \mathcal{S} \longrightarrow \pm 1$$

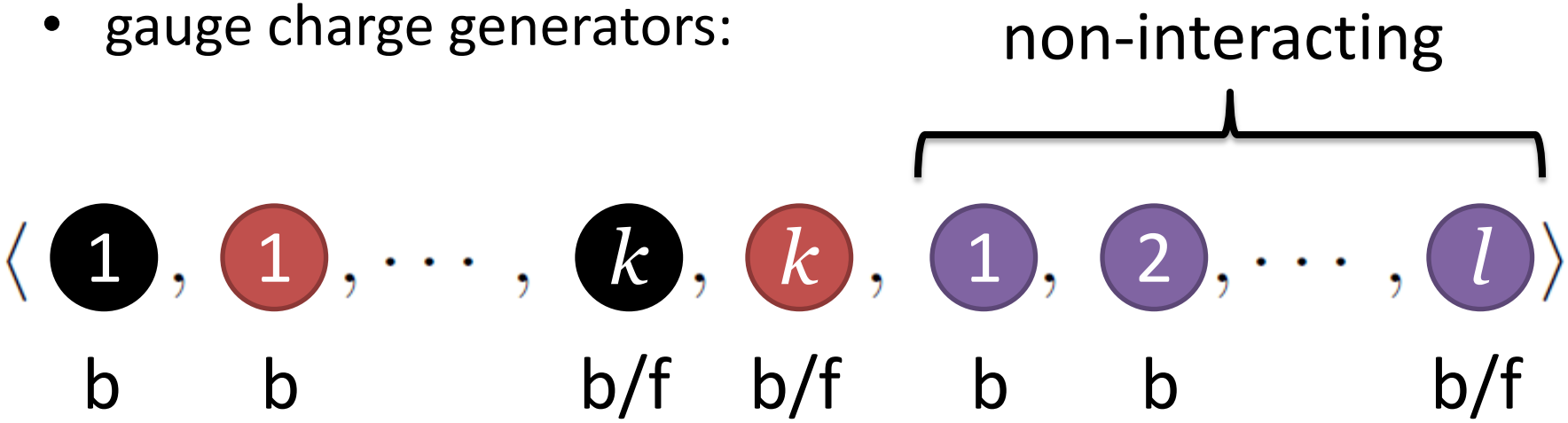
$$C_{\mathcal{G}} \longrightarrow C_{\mathcal{S}}$$



$\kappa($  $,$  $) = \kappa($  $,$  $)$

canonical generators

- gauge charge generators:



- **dual** stabilizer charge generators:

canonical generators

- all possible anyon models are **combinations** of

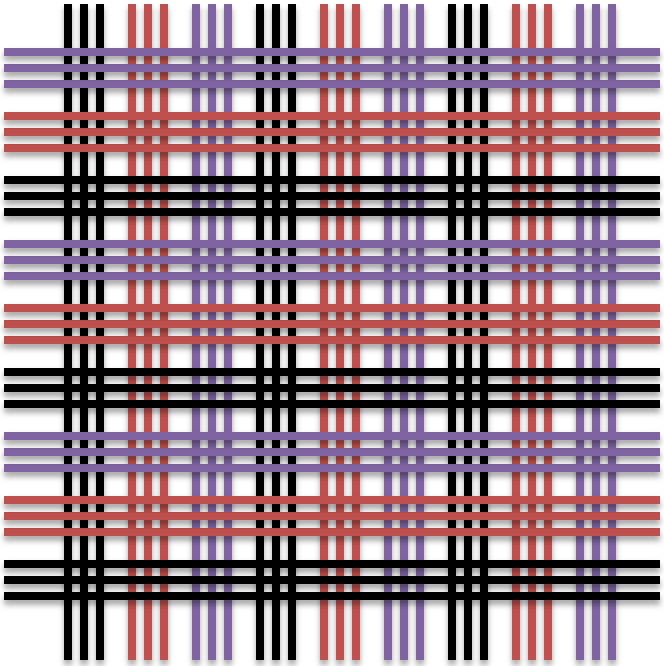
toric code: b b

topological subsystem color codes: f f

subsystem toric code: b

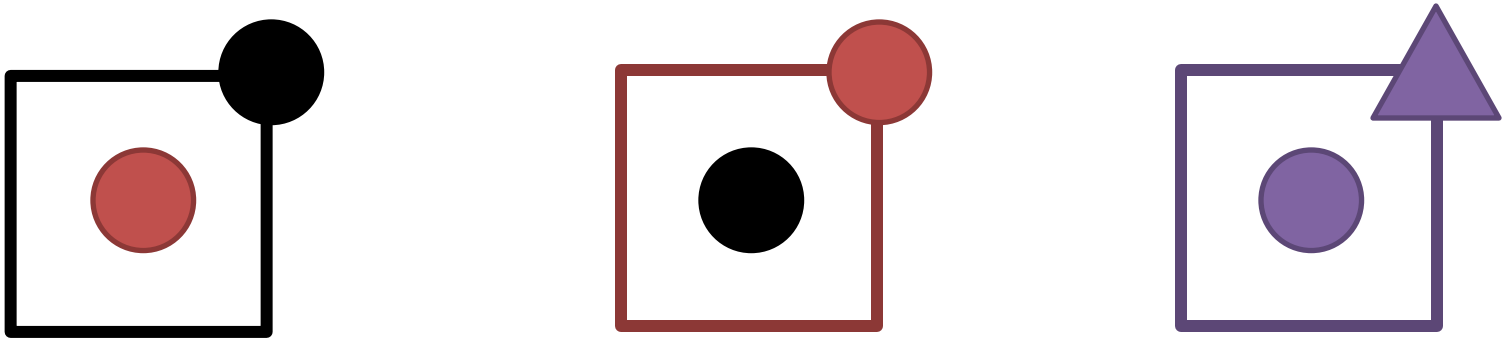
honeycomb subsystem code: f

string framework

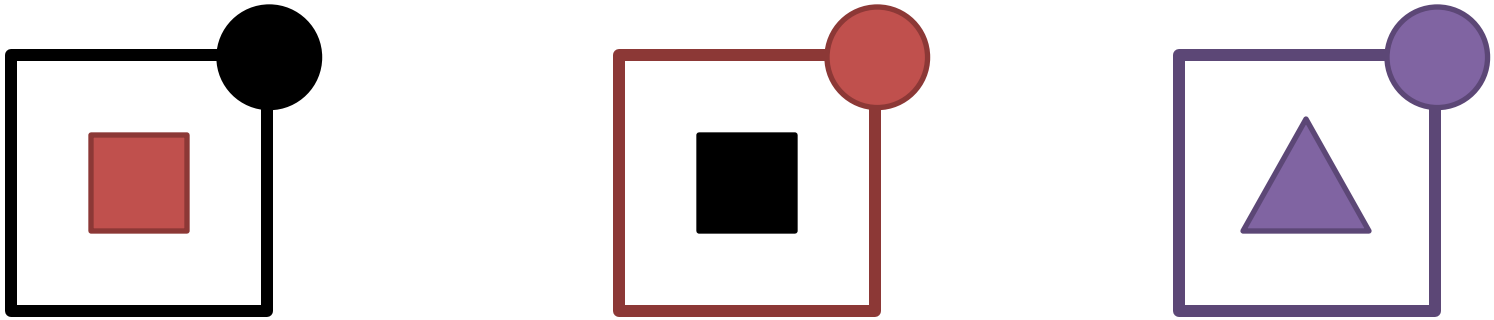


string framework

- gauge generators (non-trivial charge)



- stabilizer generators (non-trivial charge)



string framework

every 2D TSC has a structure based on an anyon model

conclusions & questions

- the long-range entanglement pattern of toric codes is universal for 2D topological stabilizer models
- all 2D TSCs are anyon based
- the same approach could be used for boundaries or point defects…
- more general 2D models?
- what about 3D?