

Approximate simulation of quantum channels

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Why approximate QEC?

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- The KL conditions could be far from necessary

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Crépeau, Gottesman, Smith (2005)

- However, one can correct $\sim n/2$ errors for n qubit, for large n

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Knill, Laflamme, Viola (2000)

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Tyson (2010)

- Bound similar to ours, different methods

Channel simulation

- **Error correction**: find \mathcal{C}, \mathcal{R} such that

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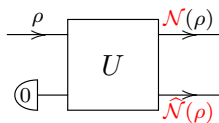
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- Equivalence relation

$$\mathcal{N} \sim \mathcal{M} \quad \text{if} \quad \mathcal{N} \succ \mathcal{M} \quad \text{and} \quad \mathcal{M} \succ \mathcal{N}$$

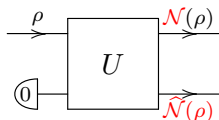
Complementary class

- $\hat{\mathcal{N}}$ complementary to \mathcal{N}



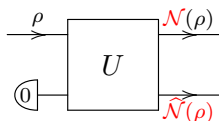
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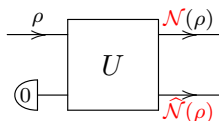
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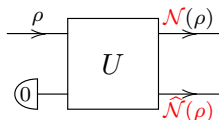


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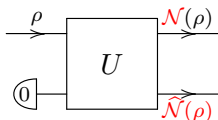
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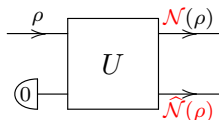
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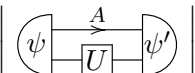
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- If $\mathcal{M} = \mathcal{P}_A$, can use $\widehat{\mathcal{M}} = \mathcal{P}_{A'}$ (commutant)
- E.g. we obtain the generalized KL conditions for subsystems codes when

$$\mathcal{A} = \{A \otimes \mathbf{1} : A\}$$

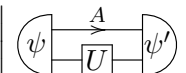
Approximate versions (fixed-state fidelity)

- Fidelity between states $\rho = \text{Tr}_B |\psi\rangle\langle\psi|$ and $\rho' = \text{Tr}_B |\psi'\rangle\langle\psi'|$

$$f(\rho, \rho') = \max_U \left| \left(\begin{array}{c} \text{---} \psi \text{---} \\ \text{---} U \text{---} \\ \text{---} \psi' \text{---} \end{array} \right) \right|$$
The diagram shows a quantum circuit with two horizontal lines representing qubits. The left qubit starts in state ψ and the right qubit starts in state ψ' . A unitary gate U acts on both qubits. A measurement A is performed on the right qubit. The circuit is enclosed in large vertical bars, indicating the absolute value of the expectation value.

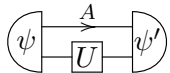
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A quantum circuit diagram enclosed in large vertical bars. On the left, a circle contains the symbol ψ . A horizontal line extends from this circle to the right, where it enters a square box labeled U . From the right side of the U box, the line continues to the right and enters another circle containing the symbol ψ' . Above the horizontal line, between the ψ circle and the U box, there is a double-headed arrow labeled A .

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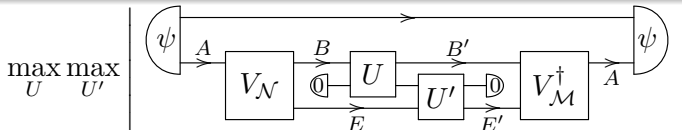
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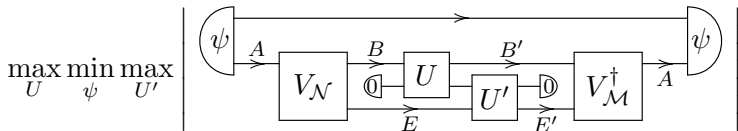
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Proof



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- When $\mathcal{M} = \mathcal{P}_{\mathcal{A}}$: approx. subsystem QEC.

Near-optimal decoding channel

If we found σ such that

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and for any CP map \mathcal{S} ,

$$\overline{\mathcal{S}(\rho)} = \mathcal{S}(\mathcal{S}^{\dagger}(\mathbf{1})^{-\frac{1}{2}} \rho \mathcal{S}^{\dagger}(\mathbf{1})^{-\frac{1}{2}}).$$

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- Answers Leung, Nielsen, Chuang, Yamamoto, 1997

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- Weaker than Knill, Laflamme, Viola 2000

General theory: 1103.0649

Perturbative QEC: 1102.3809