Improving the Fault Tolerance of Adiabatic Quantum Computers using Error Detecting Codes

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Adiabatic Quantum Computing

Hamiltonian:	H(0)	$\xrightarrow{\text{smoothly}}$	H(T)
ground state:	$\|000...\rangle$	$\longrightarrow$	$\|\text{answer}\rangle$

If $\left|\frac{dH}{dt}\right|$ is sufficiently small compared to the eigenvalue gap then the system will track the ground state.

Adiabatic Quantum Computing

- Originally proposed by Farhi *et al.* as a method for solving satisfiability problems (*e.g.* 3-SAT)
- Can be simulated by quantum circuits using standard Trotterization
- Can simulate quantum circuits (Aharonov *et al.*)

Adiabatic Architecture

Might one build an adiabatic quantum computer?

Possible advantages:

- No pulses: essentially DC
- Only endpoints of adiabatic path matter
- Intuition: if $k_b T$ is small compared to eigenvalue gap then system stays in ground state

Eigenvalue Gaps

- Gap usually shrinks with problem size:
  e.g. $\sim 1/g^2$ for simulating g gates

- Instead, look at behavior under *local* excitations

  - U is a 1-local operator

  - Does U couple the ground state to eigenstates with low energy compared to $k_b T$?

Local Operations

- Any operator on *n* qubits can be decomposed as a linear combination of *n*-fold tensor products of Pauli matrices $\{I,X,Y,Z\}$
- If each of these tensor products has at most *k* non-identity Pauli matrices then the operator is *k*-local
- This is not about spatial locality

Noise

- Noise comes from a Hamiltonian coupling the qubits to the environment

$$H = H_S + H_E + \lambda V$$

- *H* will usually be at most 2-local

- Any term in the Pauli expansion of *V* will be of the form $\sigma_a \sigma_b$

- Hence the qubits experience 1-local disturbances

Codes

- Can we engineer the adiabatic Hamiltonians so that 1-local operators don't couple the ground state to eigenstates below $k_b T$?
- Yes
  - use quantum error detecting codes
  - make energy penalty against 1-qubit errors
- This insulates against but does not correct errors

Encoded Hamiltonian

1) Write out the Pauli expansion of H

2) Replace each Pauli matrix by:

$$I \rightarrow I^{\otimes k} \qquad X \rightarrow X_L$$

$$Y \rightarrow Y_L \qquad Z \rightarrow Z_L$$

The resulting Hamiltonian H_L acts on k times as many qubits.

3) Add energy penalty term against leaving the codespace

Locality

- Challenge: encode Hamiltonian without requiring many-body interactions
- Example: Shor code
  - encodes 1 qubit in 9
  - 2-local Hamiltonian becomes 18-local once encoded!
- choose code carefully
- How close can we get to 2-local encoded operations?

4-qubit Code

- Use error detecting code
- Encode 1 qubit into 4:

$$|0_L\rangle = \frac{1}{2}(|0000\rangle + i|0011\rangle + i|1100\rangle + |1111\rangle)$$

$$|1_L\rangle = \frac{1}{2}(-|0101\rangle + i|0110\rangle + i|1001\rangle - |1010\rangle)$$

- This detects 1-local errors:

$$|\psi\rangle \in C \implies \sigma|\psi\rangle \in C^{\perp}$$

Encoded Operators

The following act as logical Pauli operators

$$X_L = Y \otimes I \otimes Y \otimes I$$

$$Y_L = -I \otimes X \otimes X \otimes I$$

$$Z_L = Z \otimes Z \otimes I \otimes I$$

That is,

$$X_L|0_L\rangle = |1_L\rangle$$

$$X_L|1_L\rangle = |0_L\rangle$$

etc.

These are all 2-local.

Energy Penalty

- We next add an energy penalty against going outside the codespace
- Our code is a stabilizer code

$$g_1 = X \otimes X \otimes X \otimes X$$
$$g_2 = Z \otimes Z \otimes Z \otimes Z$$
$$g_3 = X \otimes Y \otimes Z \otimes I$$

- One term of the form $-\frac{1}{2}E_p(g_1 + g_2 + g_3)$

for each encoded qubit provides the penalty.

Energy Penalty

Any 1-local operator anticommutes with at least one of the generators

$$g_1 = X \otimes X \otimes X \otimes X$$

$$g_2 = Z \otimes Z \otimes Z \otimes Z$$

$$g_3 = X \otimes Y \otimes Z \otimes I$$

hence 1-local operators couple the ground state only to eigenstates above E_p

Eigenstates

- If we write down the eigenstates of H (on n qubits)and encode them according to

$$|0\rangle \rightarrow |0_L\rangle \qquad |1\rangle \rightarrow |1_L\rangle$$

we get eigenstates of the encoded Hamiltonian

H_L on 4n qubits

- The penalty terms ensure that the ground state of the encoded Hamiltonian is the encoded ground state

2-local Noise

- The standard 5-qubit stabilizer code:

$$g_1 = X \otimes Z \otimes Z \otimes X \otimes I \qquad g_2 = I \otimes X \otimes Z \otimes Z \otimes X$$

$$g_3 = X \otimes I \otimes X \otimes Z \otimes Z \qquad g_4 = Z \otimes X \otimes I \otimes X \otimes Z$$

- Can have 3-local encoded Pauli operations:

$$X_L = -X \otimes I \otimes Y \otimes Y \otimes I$$

$$Y_L = -Z \otimes Z \otimes I \otimes Y \otimes I$$

$$Z_L = -Y \otimes Z \otimes Y \otimes I \otimes I$$

Noise Model

The qubits are spins weakly coupled to a bath of photons

$$V = \sum_i \int_0^\infty \mathrm{d}\omega \left[g(\omega) a_\omega \sigma_+^{(i)} + g^*(\omega) a_\omega^\dagger \sigma_-^{(i)} \right]$$

Resulting Davies master equation has noise terms like

$$\frac{\langle 0|\sigma_-^{(i)}|b\rangle\langle b|\sigma_+^{(i)}|0\rangle}{\exp[\beta(E_b - E_0)] - 1}$$

Noise Model

- $\langle 0|\sigma_{\pm}^{(i)}|b\rangle$ is zero unless $|b\rangle \in C^{\perp}$
- Hence $E_b - E_0 \geq E_p$
- So the terms

$$\frac{\langle 0|\sigma_{-}^{(i)}|b\rangle\langle b|\sigma_{+}^{(i)}|0\rangle}{\exp[\beta(E_b - E_0)] - 1}$$

  scale roughly as $\exp[-\beta E_p]$
- By choosing $T \sim E_p/\log(\epsilon)$ we can make noise terms $\sim \epsilon$

Limitations of Noise Model

- derived only to 2nd order in λ
- slightly arbitrary, e.g. $\sigma_\pm$ in z-direction
- however, in general expect lowest order noise terms to have $e^{-\beta E_p}$ scaling in any similar model
- current work: larger distance codes to protect against higher order errors

Unencoded Hamiltonians

- Codes ensure that 1-local operators have zero matrix elements coupling the ground state to eigenstates below E_p
- This allows exponential in T suppression of noise terms in master equation
- But what about unencoded Hamiltonians? Might these matrix elements be small anyway?

Numerics

- 2-local Hamiltonian H
- Compute corresponding ground state

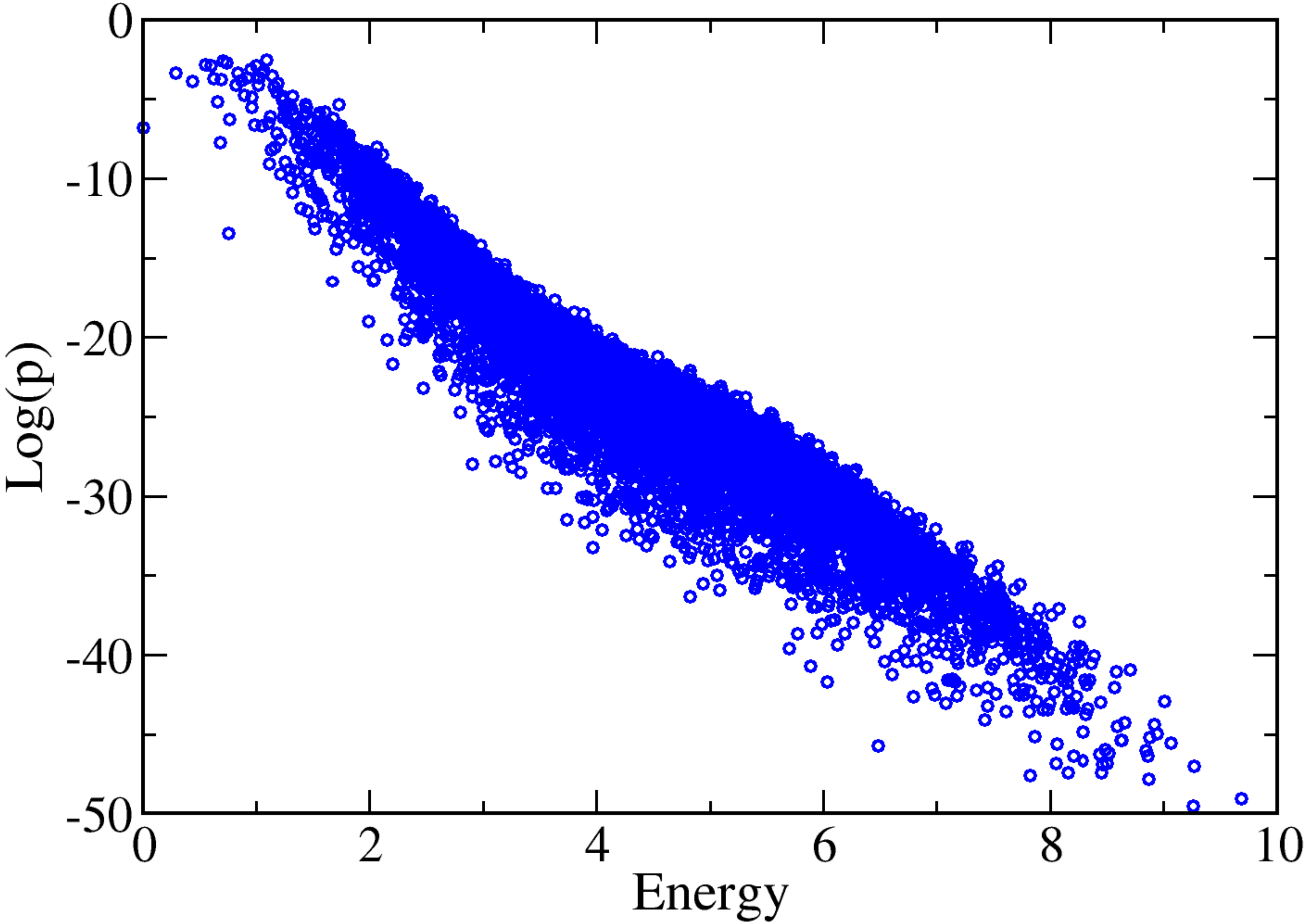
$$H|\psi_0\rangle = E_0|\psi_0\rangle$$

- Apply a π-rotation about a random axis

$$|\phi\rangle = U|\psi_0\rangle$$

- Compute the overlaps with the eigenstates of H :

$$|\langle\phi|\psi_n\rangle|^2$$



0
-10
-20
-30
-40
-50
Log(p)
0
2
4
6
8
10
Energy

Optimality

- Quantum singleton bound:

$$n - k \geq 2(d-1)$$

- 3-qubit error detecting stabilizer code is impossible

$$g_1 = \sigma_{11} \otimes \sigma_{12} \otimes \sigma_{13}$$

$$g_2 = \sigma_{21} \otimes \sigma_{22} \otimes \sigma_{23}$$

- 3-qubit distance two quantum code is impossible in general

Locality

- Each encoded Pauli operator is 2-local
- A 2-local universal Hamiltonian becomes 4-local once encoded
- It is impossible to make an error detecting code for 1-local errors which has 1-local encoded operators

Perturbative Gadgets

- Use perturbation theory to simulate 3-local Hamiltonians using 2-local Hamiltonians

  J. Kempe, A. Kitaev, O. Regev
  The complexity of the local Hamiltonian Problem
  quant-ph/0406180

- Can be generalized to simulate 4-local Hamiltonians with 3-local Hamiltonians

Perturbative Gadgets

- Problem: computational Hamiltonian is perturbation and gets scaled down to constant norm
- scaling undoes constant energy gap E_p !
- Solution: apply energy penalty against leaving the codespace in the non-perturbative part

Perturbative Gadgets

- our original energy penalty was expressed as $-\frac{1}{2}E_p(g_1 + g_2 + g_3)$ with:

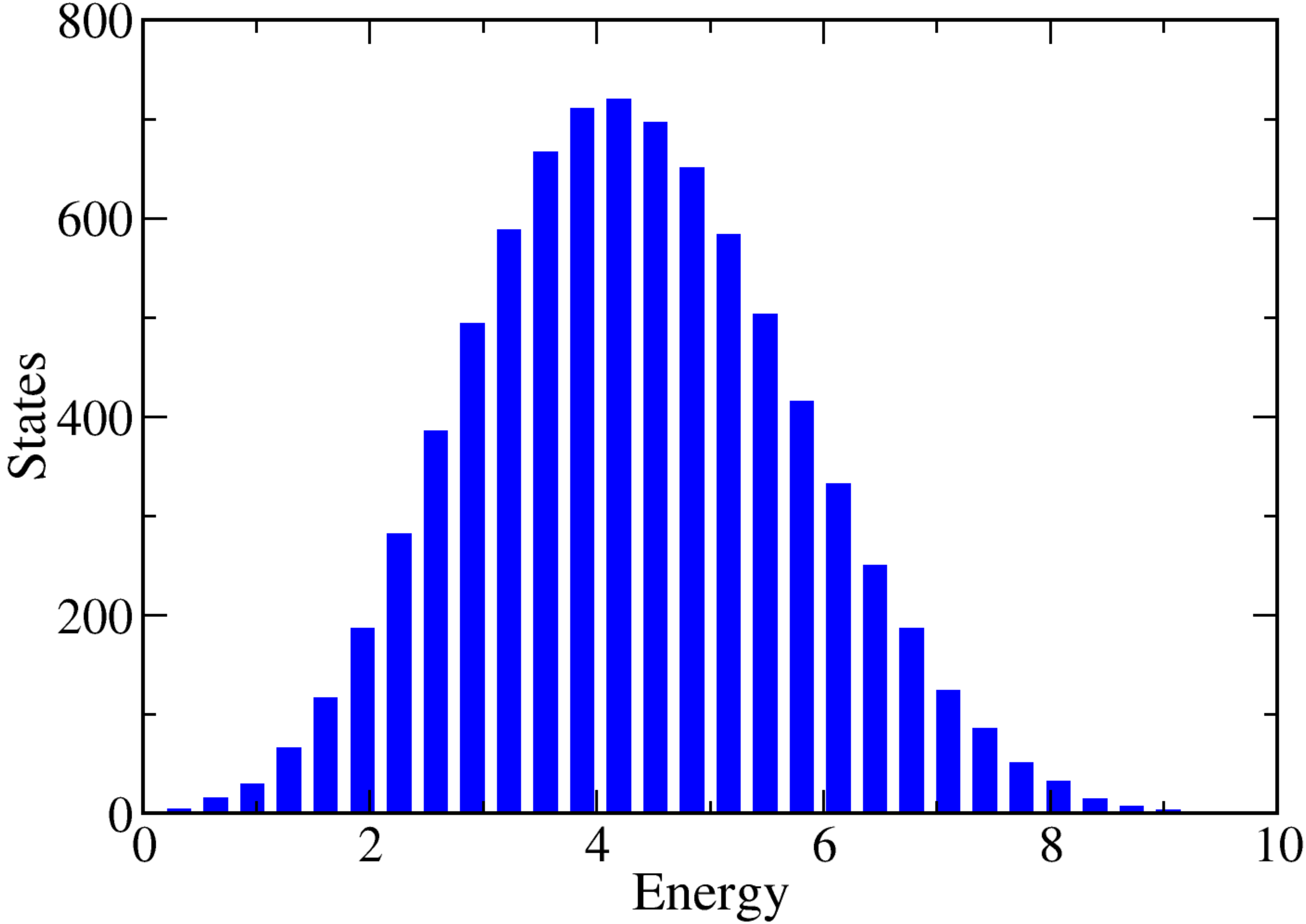
$$g_1 = X \otimes X \otimes X \otimes X$$
$$g_2 = Z \otimes Z \otimes Z \otimes Z$$
$$g_3 = X \otimes Y \otimes Z \otimes I$$

- we can choose equivalent 3-local generators:

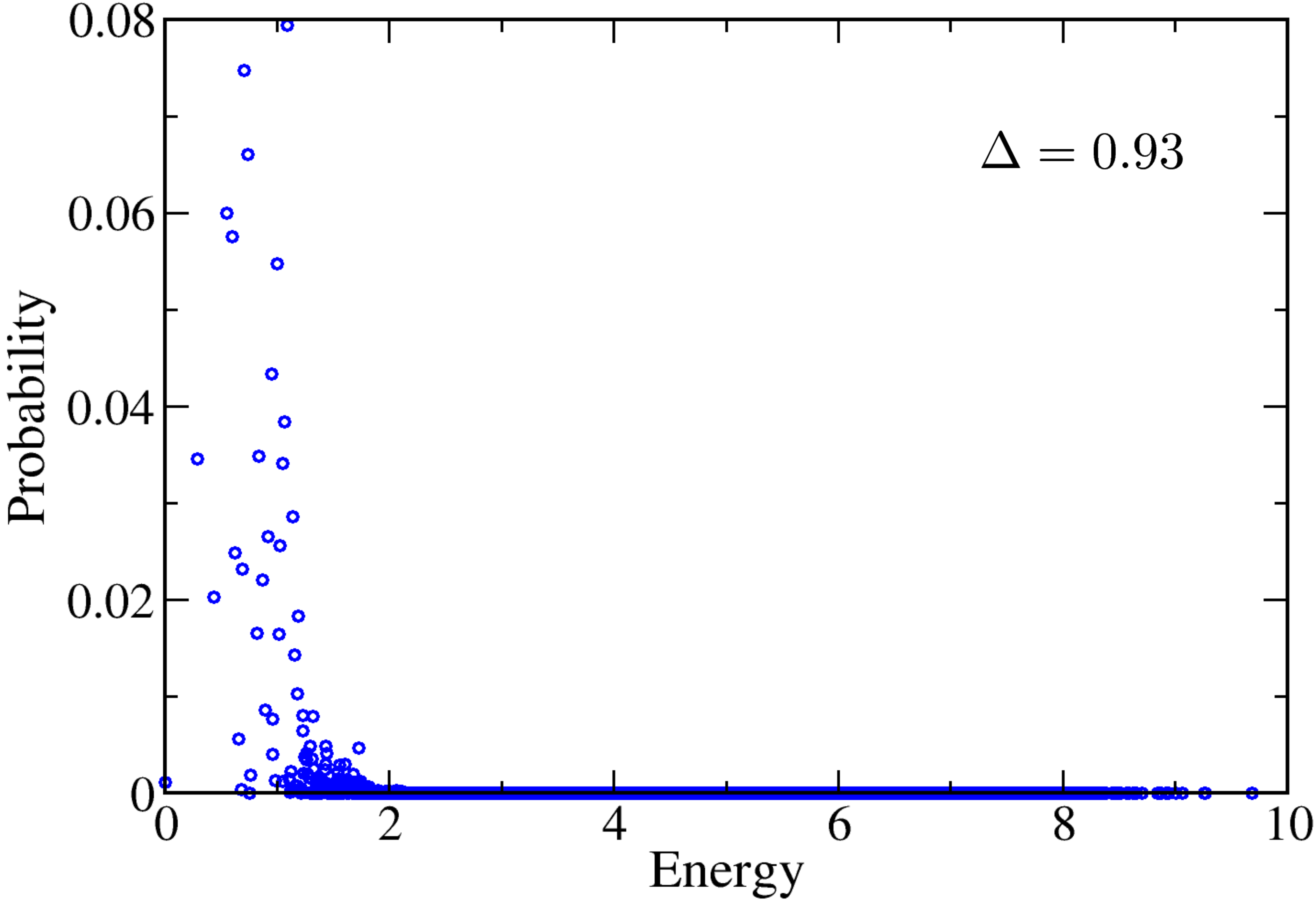
$$g_1 = X \otimes Y \otimes Z \otimes I$$
$$g_2 = I \otimes Z \otimes Y \otimes X$$
$$g_3 = Y \otimes X \otimes I \otimes Z$$

Conclusion

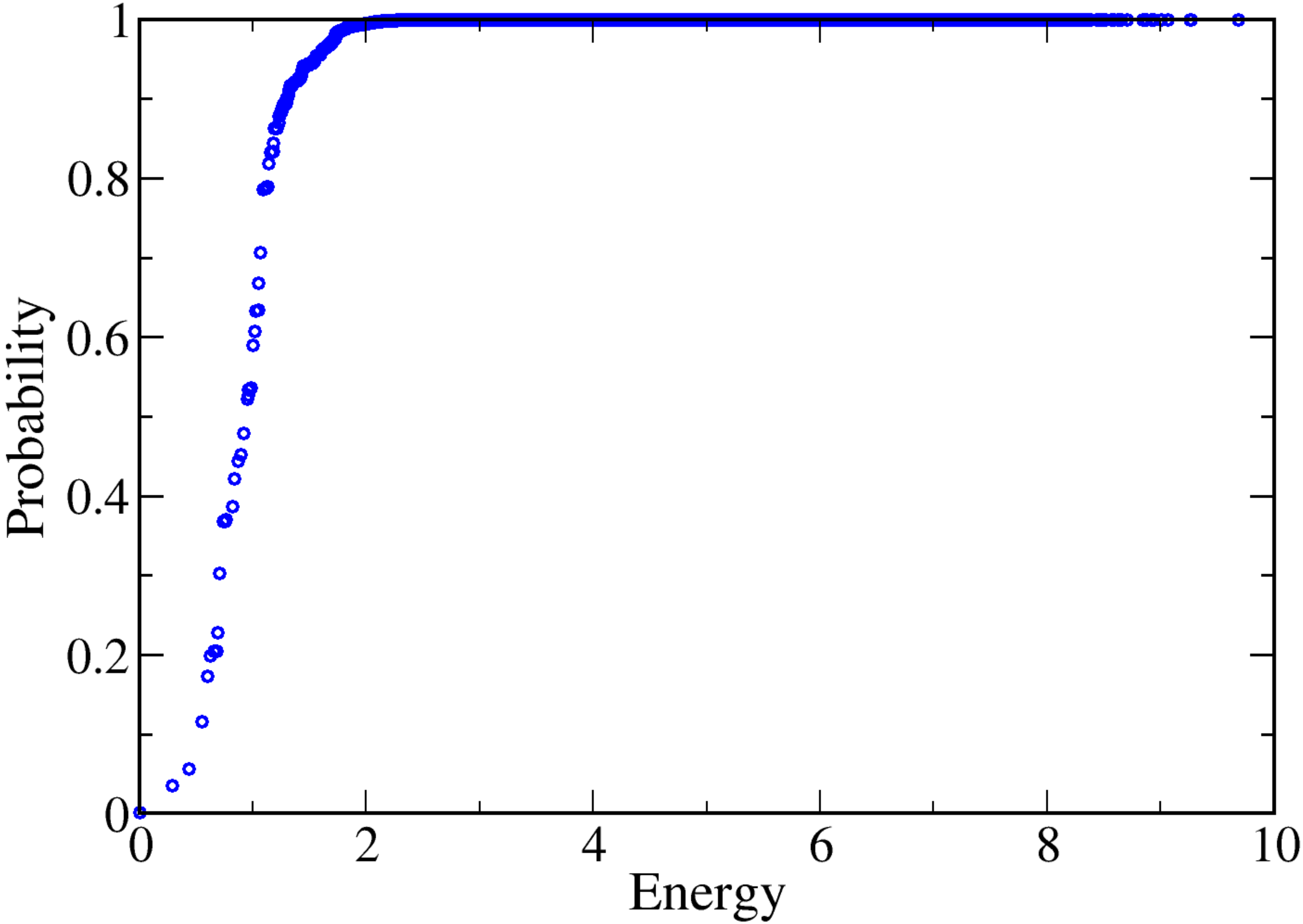
- Energy gap against local excitations can protect adiabatic quantum computers against noise
- Such gaps can be engineered into the Hamiltonians using error detecting codes
- The resulting universal Hamiltonians are 4-local
- Future work:
  - improve locality noise-resistant Hamiltonians
  - protect against higher weight errors
  - investigate control error
  - fault-tolerance threshold?



800
600
400
200
0
States
0
2
4
6
8
10
Energy



0.08
0.06
0.04
0.02
0
Probability
0
2
4
6
8
10
Energy
Δ = 0.93



Probability
Energy
0
0.2
0.4
0.6
0.8
1
0
2
4
6
8
10

Adiabatic Theorem

- Ambainis and Regev:

$$T \geq \frac{10^5}{\delta^2} \cdot \max\left\{ \frac{\|H'\|^3}{\gamma^4}, \frac{\|H'\| \|H''\|}{\gamma^3} \right\}$$

- Teufel:

$$T \geq \frac{4}{\epsilon} \left[\frac{\left\| \frac{\mathrm{d}H}{\mathrm{d}s}(0) \right\|}{\gamma(0)^2} + \frac{\left\| \frac{\mathrm{d}H}{\mathrm{d}s}(1) \right\|}{\gamma(1)^2} + \int_0^1 \mathrm{d}s \left(10 \frac{\left\| \frac{\mathrm{d}H}{\mathrm{d}s} \right\|^2}{\gamma^3} + \frac{\left\| \frac{\mathrm{d}^2 H}{\mathrm{d}s^2} \right\|}{\gamma} \right) \right]$$

- runtime scales polynomially with inverse gap

Control Error

- Suppose final Hamiltonian is slightly off:

$$H(T) = H_0(T) + \lambda V$$

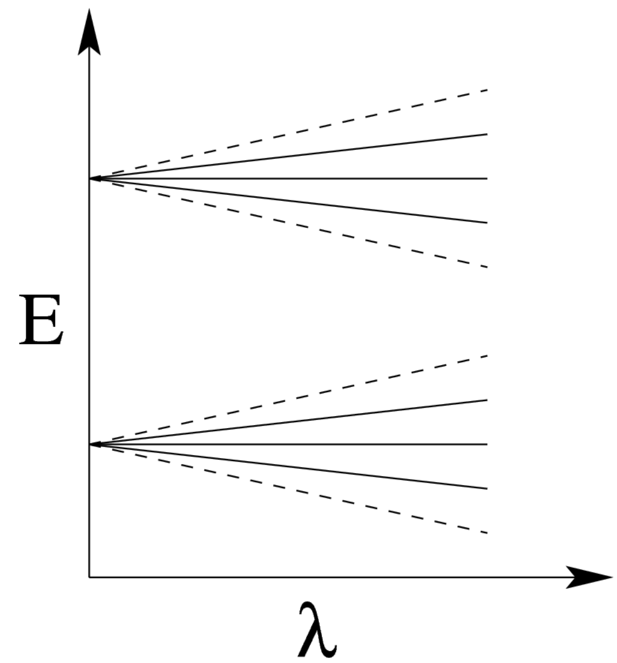
- Resulting ground state is:

$$|\tilde{\psi}_0\rangle = |\psi_0\rangle + \lambda \sum_{k \neq 0} |\psi_k\rangle \frac{\langle \psi_k | V | \psi_0 \rangle}{E_0 - E_k} + \mathcal{O}(\lambda^2)$$

- Gap against local operators ensures denominator $\geq E_p$

Degenerate Perturbation Theory

$$H = H_0 + \lambda V$$



$$H_{\text{eff}}(\lambda) \equiv \sum_{j=1}^{d} E_j(\lambda) |\psi_j(\lambda)\rangle\langle\psi_j(\lambda)|$$

Perturbation to All Orders

$$H_{\text{eff}} = E_0^{(0)} I + \sum_{m=1}^{\infty} \lambda^m H_{\text{eff}}^{(m)}$$

$$H_{\text{eff}}^{(m)} = (-1)^{m-1} \sum_{k_1 \ldots k_{m+1}} S^{k_1} V S^{k_2} V \ldots V S^{k_{m+1}}$$

$$S^k = \begin{cases} \sum_{j \neq 0} \frac{P_j}{(E_j^{(0)} - E_0^{(0)})^k} & \text{if } k > 0 \\ -P_0 & \text{if } k = 0 \end{cases}$$

converges if $\|\lambda V\| < \frac{\gamma}{2}$

2^{nd} Order Gadget

- To simulate $Z \otimes Z \otimes Z \otimes Z$ add an ancilla qubit and choose $H_0 = -E_p|0\rangle\langle 0|$
- choose $$V = Z \otimes Z \otimes I \otimes I \otimes X$$ $$+ I \otimes I \otimes Z \otimes Z \otimes X$$

$$H_{\text{eff}}^{(1)} = S^0 V S^0 = P_0 V P_0$$

$$H_{\text{eff}}^{(2)} = S^0 V S^0 V S^1 + S^0 V S^1 V S^0 + S^1 V S^0 V S^0$$

$$S^k = \begin{cases} \sum_{j \neq 0} \frac{P_j}{(E_j^{(0)} - E_0^{(0)})^k} & \text{if } k > 0 \\ -P_0 & \text{if } k = 0 \end{cases}$$

2^{nd} Order Gadget

$$V = Z \otimes Z \otimes I \otimes I \otimes X + I \otimes I \otimes Z \otimes Z \otimes X$$

$$H_{\text{eff}}^{(2)} = P_0 V \frac{P_1}{E_1^{(0)} - E_0^{(0)}} V P_0 = \frac{1}{E_p} P_0 V^2 P_0$$

$$= \frac{1}{E_p} \left(2Z \otimes Z \otimes Z \otimes Z \otimes I + 2I\right)$$

In general, simulates 2k-local Hamiltonian with (k+1)-local Hamiltonian

Penalty term breaks this degeneracy