Robust Quantum Error-Correction

via Convex Optimization

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Presented at QEC-07
University of Southern California, Los Angeles
Dec. 17-21, 2007

Research supported by DARPA QuIST Program (Quantum Information Science & Technology)

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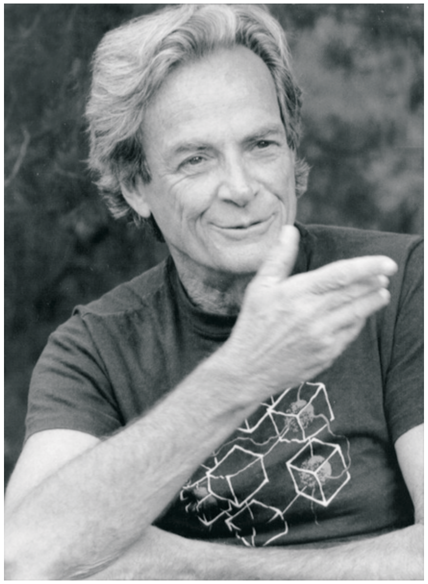
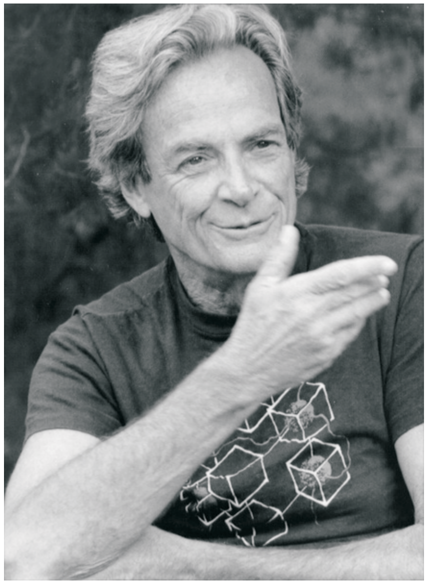


"Quantum Mechanical Computers,"
Optics News, February 1985.
Richard P. Feynman

"In a machine such as this there are very many other problems due to imperfections. . . . At least some of these problems can be remedied in the usual way by techniques such as error correcting codes . . . But until we find a specific implementation for this computer, I do not know how to proceed to analyze these effects. However, it appears that they would be very important in practice. This computer seems to be very delicate and these imperfections may produce considerable havoc."



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- Concerns

  - **Robustness** Perfect QEC schemes, as well those produced by optimization tuned to specific errors, are often not robust to even small changes in the noise channel.
  - **Cost** The encoding and recovery effort in perfect QEC typically grows exponentially with the number of errors in the noise channel.

- A robust optimization approach to QEC can address these.

  - **Robustness** Noise channels which do not satisfy the standard assumptions for perfect correction can be incorporated.
  - **Cost** If robust fidelity levels are sufficiently high, then no further increases in codespace dimension and/or levels of concatenation are necessary. This assessment, which is critical to any specific implementation, is not knowable without performing the robust optimization.

Outline

- Problem formulation
- Direct fidelity maximization
- Indirect fidelity maximization
- Numerical Examples
- Conclusions

Error correction model

OSR model

$$\rho_S \longrightarrow \boxed{\mathcal{C}} \xrightarrow{\rho_C} \boxed{\mathcal{E}} \xrightarrow{\sigma_C} \boxed{\mathcal{R}} \longrightarrow \hat{\rho}_S = \sum_{r,e,c} (R_r E_e C_c) \rho_S (R_r E_e C_c)^\dagger$$

$$\begin{aligned}
\mathcal{R} &= \left\{ R_r \sim n_S \times n_C,\ r = 1, \ldots, m_R \,\middle|\, \textstyle\sum_r {R_r}^\dagger R_r = I_{n_C} \right\} \\
\mathcal{E} &= \left\{ E_e \sim n_C \times n_C,\ e = 1, \ldots, m_E \,\middle|\, \textstyle\sum_e E_e^\dagger E_e = I_{n_C} \right\} \\
\mathcal{C} &= \left\{ C_c \sim n_C \times n_S,\ c = 1, \ldots, m_C \,\middle|\, \textstyle\sum_c {C_c}^\dagger C_c = I_{n_S} \right\}
\end{aligned}$$

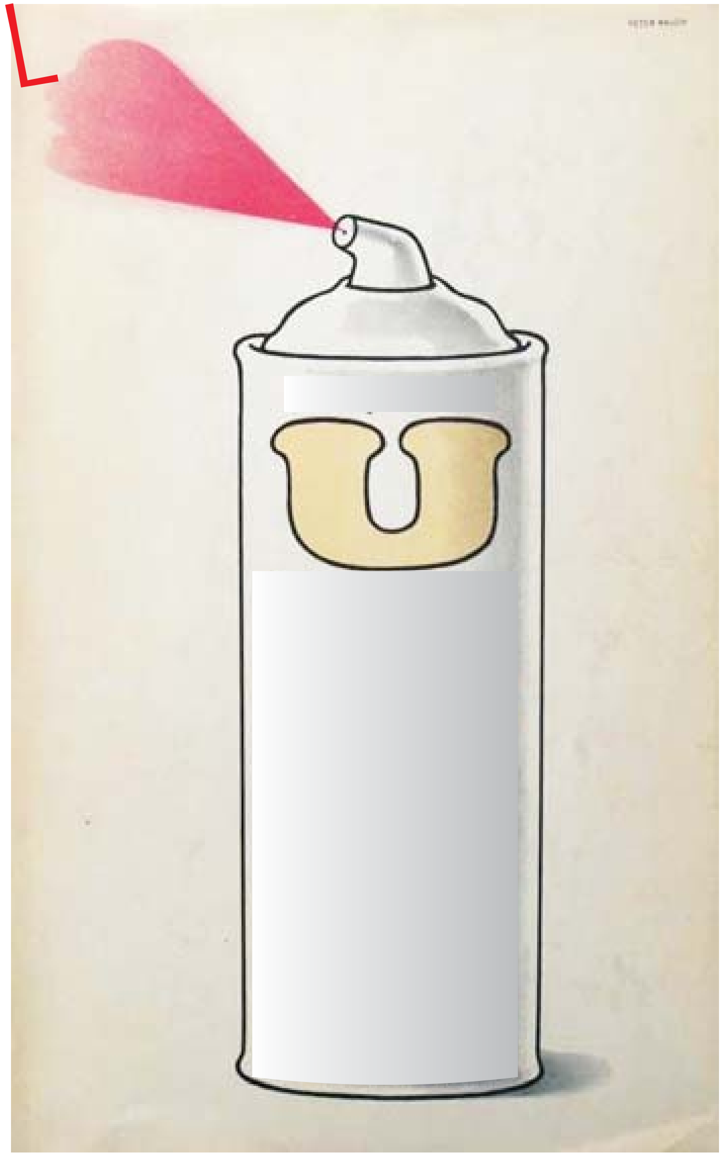
Design goal

Determine encoding $\mathcal{C}$ and recovery $\mathcal{R}$ to so that the noisy channel, $\mathcal{REC}$, is as close as possible to a desired unitary U_S.

CONTROL

UBIQUITOUS FOR QIP/QEC

CONTROL IS REQUIRED
EVERYWHERE
THERE IS A DESIRED UNITARY.



Source: 1969 Doubleday Book Cover
UBIK by Philip K. Dick

Performance measures [a] [b] [c]

Fidelity between $\mathcal{REC}$ and U_S

$$f = \frac{1}{n_s^2} \sum_{r,e,c} |\mathrm{Tr}\, {U_S}^\dagger R_r E_e C_c|^2$$

Perfect error correction, $f = 1$, if and only if there are constants α_{rec} such that

$$R_r E_e C_c = \alpha_{rec} U_S, \; \sum_{r,e,c} |\alpha_{rec}|^2 = 1.$$

This suggests the **indirect** measure of fidelity, the "distance-like" error,

$$d = \sum_{r,e,c} \| R_r E_e C_c - \alpha_{rec} U_S \|_{\mathrm{fro}}^2$$

[a] B. Schumacher, *Phys. Rev. A*, (1996)

[b] E. Knill, R. Laflamme, *Phys. Rev. A*, (1997)

[c] M.A. Nielsen, I.L. Chuang, *Quantum Computation and Quantum Information*, Cambridge (2000)

Direct Fidelity Maximization

$$\text{maximize} \quad f = \frac{1}{n_S^2} \sum_{r,e,c} |\mathrm{Tr}\, U_S{}^\dagger R_r E_e C_c|^2$$

$$\text{subject to} \quad \sum_r R_r{}^\dagger R_r = I_{n_C}, \quad \sum_c C_c{}^\dagger C_c = I_{n_S}$$

Indirect Fidelity Maximization

$$\text{minimize} \quad d = \sum_{r,e,c} \|R_r E_e C_c - \alpha_{rec} U_S\|_{\mathrm{fro}}^2$$

$$\text{subject to} \quad \sum_r R_r{}^\dagger R_r = I_{n_C}, \quad \sum_c C_c{}^\dagger C_c = I_{n_S}, \quad \sum_{r,e,c} |\alpha_{rec}|^2 = 1$$

Both are non-convex optimizations for which local solutions can be found from a bi-convex iteration between recovery and encoding. [a] [b] [c] [d] [e]

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[a] M. Reimpell & R. F. Werner, *Phys. Rev. Lett.* (2005)

[b] N. Yamamoto, S. Hara, & K. Tsumura, *Phys. Rev. A*, (2005)

[c] A. S. Fletcher, P. W. Shor, & M. Z. Win, arXiv: quant-ph/0606035, *Phys. Rev. A* (2007)

[d] R.L. Kosut & D.A. Lidar, arXiv: quant-ph/0606078

[e] R.L. Kosut, A. Shabani, & D.A. Lidar, arXiv:quant-ph/0703274, (to appear *Phys. Rev. Lett.*)

Direct fidelity maximization

$$\text{maximize} \quad f = \frac{1}{n_S^2} \sum_{r,e,c} \left| \text{Tr}\, U_S^\dagger R_r E_e C_c) \right|^2$$

$$\text{subject to} \quad \sum_c C_c{}^\dagger C_c = I_S, \ \sum_r R_r{}^\dagger R_r = I_C$$

Expand OSR elements in a fixed basis

$$C_c = \textstyle\sum_{i=1}^{n_S n_C} x_{ci} B_{Ci}, \quad B_{Ci} \sim n_C \times n_S$$

$$R_r = \textstyle\sum_{i=1}^{n_S n_C} x_{ri} B_{Ri}, \quad B_{Ri} \sim n_S \times n_C$$

Direct fidelity maximization is equivalent to

$$\text{maximize} \quad f = \textstyle\sum_{ijk\ell} (X_R)_{ij} (X_C)_{k\ell} F_{ijk\ell}(\mathcal{E})$$

$$\text{subject to} \quad \textstyle\sum_{ij} (X_R)_{ij} B_{Ri}^\dagger B_{Rj} = I_{n_C}, \ \sum_{k\ell} (X_C)_{k\ell} B_{Ck}^\dagger B_{C\ell} = I_{n_S}$$

$$(X_R)_{ij} = \textstyle\sum_r x_{ri} x_{rj}{}^*, \ (X_C)_{k\ell} = \sum_c x_{ck} x_{c\ell}{}^*$$

$$F_{ijk\ell}(\mathcal{E}) = \textstyle\sum_e (\text{Tr}\, U_S{}^\dagger B_{Ri} E_e B_{Ck})(\text{Tr}\, U_S{}^\dagger B_{Rj} E_e B_{C\ell})^* / n_S^2$$

Process matrices $X_R,\ X_C\ \sim n_S n_C \times n_S n_C$

$$(X_R)_{ij} = \sum_r x_{ri}\, x_{rj}{}^* \qquad X_R \geq 0$$

$$\left(\begin{array}{c} \text{relax quadratic constraints} \\ \text{to} \\ \text{semidefinite constraints} \end{array} \right) \Rightarrow$$

$$(X_C)_{k\ell} = \sum_c x_{ck}\, x_{c\ell}{}^* \qquad X_C \geq 0$$

Relaxed direct fidelity maximization

$$\boxed{\begin{array}{ll} \text{maximize} & f = \sum_{ijk\ell} (X_R)_{ij}\, (X_C)_{k\ell}\, F_{ijk\ell}(\mathcal{E}) \\ \text{subject to} & X_R \geq 0,\ \ \sum_{ij} (X_R)_{ij}\, B_{Ri}^\dagger B_{Rj} = I_{n_C}, \\ & X_C \geq 0,\ \ \sum_{k\ell} (X_C)_{k\ell}\, B_{Ck}^\dagger B_{C\ell} = I_{n_S} \end{array}}$$

- not a convex optimization jointly in X_R and X_C
- convex (SDP) in X_R for a given X_C and in X_C for a given $X_R \Rightarrow$ local solution obtained from a bi-convex iterative optimization – Semidefinite Programs (SDPs) – in encoding and recovery process matrices X_C and X_R

Given encoding $\mathcal{C}$, optimal recovery process matrix is solution of the SDP

$$\boxed{\begin{array}{ll} \text{maximize} & f = \mathrm{Tr}\; X_R W_R(\mathcal{E},\mathcal{C}) \\[2em] \text{subject to} & X_R \geq 0, \quad \sum_{i,j} (X_R)_{ij}\, B_{Ri}^\dagger B_{Rj} = I_{n_C} \end{array}}$$

OSR elements R_r obtained from X_R via the singular value decomposition

$$X_R = VSV^\dagger \;\Rightarrow\; R_r = \sum_{i=1}^{n_S n_C} \sqrt{s_r}\, V_{ir} B_{Ri}, \quad r = 1, \ldots, n_S n_C$$

Given encoding $\mathcal{C}$, optimal recovery process matrix is solution of the SDP

maximize	$f = \mathrm{Tr}\; X_R W_R(\mathcal{E},\mathcal{C})$
subject to	$X_R \geq 0, \quad \sum_{i,j} (X_R)_{ij}\; B_{Ri}^{\dagger} B_{Rj} = I_{n_C}$

OSR elements R_r obtained from X_R via the singular value decomposition

$$X_R = VSV^{\dagger} \;\Rightarrow\; R_r = \sum_{i=1}^{n_S n_C} \sqrt{s_r}\, V_{ir} B_{Ri}, \quad r = 1, \ldots, n_S n_C$$

Given recovery $\mathcal{R}$, optimal encoding process matrix is solution of the SDP

maximize	$f = \mathrm{Tr}\; X_C W_C(\mathcal{E},\mathcal{R})$
subject to	$X_C \geq 0, \quad \sum_{i,j} (X_C)_{ij}\; B_{Ci}^{\dagger} B_{Cj} = I_{n_S}$

OSR elements C_c obtained from X_C via the singular value decomposition

$$X_C = VSV^{\dagger} \;\Rightarrow\; C_c = \sum_{i=1}^{n_S n_C} \sqrt{s_c}\, V_{ic} B_{Ci}, \quad c = 1, \ldots, n_S n_C$$

Iterative direct fidelity maximization

$(X_R$-$X_C)$-SDP
Initialize OSR $\mathcal{C} \longrightarrow$ Process Matrix X_C
Repeat
1. Compute X_R (via SDP)
maximize Tr $X_R W_R(\mathcal{E}, X_C)$
subject to $X_R \geq 0, \quad \sum_{i,j} (X_R)_{ij} B_{Ri}^\dagger B_{Rj} = I_{n_C}$
2. Compute X_C (via SDP)
maximize Tr $X_C W_C(\mathcal{E}, X_R)$
subject to $X_C \geq 0, \quad \sum_{i,j} (X_C)_{ij} B_{Ci}^\dagger B_{Cj} = I_{n_S}$
Until
$f(X_R, X_C)$ stops increasing

- Each step can only increase $f(X_R, X_C)$.
- Compute OSRs $(\mathcal{R}, \mathcal{C})$ via SVDs at termination.

Error system uncertainty

- error system is one of a number of possible error systems:

$$\mathcal{E}_\beta = \left\{ E_{\beta e} \,|\, e = 1, \ldots, m_E \right\}, \; \beta = 1, \ldots, \ell$$

- sources of uncertainty

  - different runs of a tomography experiment can yield different error channels.
  - a physical model generated by a Hamiltonian $H(\theta)$ dependent upon an uncertain set of parameters θ – taking a sample $\{H(\theta_\beta)\}_{\beta=1}^{\ell}$ will result in a *set* of error systems

- errors can be mitigated by

  - fault-tolerant methods which rely on several levels of code concatenation

    and/or
  - robust error correction

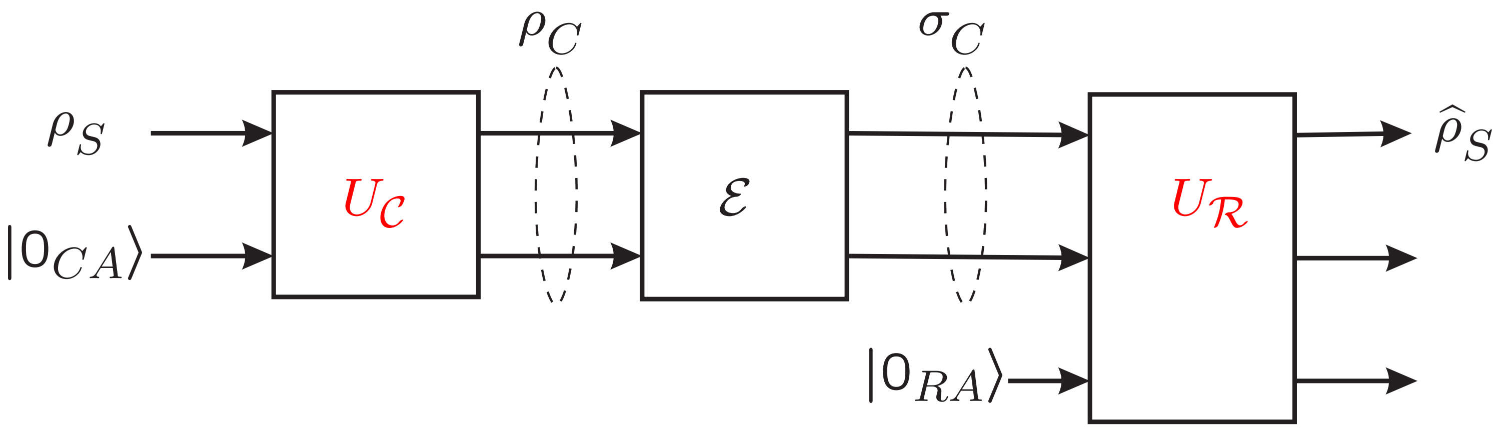
Robust error correction

Worst-Case	Average-Case
maximize $\min_\beta$ Tr $X_R W_R(\mathcal{E}_\beta, \mathcal{C})$ subject to $X_R \geq 0$, $\sum_{i,j} (X_R)_{ij}\, B^\dagger_{Ri} B_{Rj} = I_{n_C}$	maximize Tr $X_R \langle W_R(\mathcal{E}_\beta, \mathcal{C})\rangle_\beta$ subject to $X_R \geq 0$, $\sum_{i,j} (X_R)_{ij}\, B^\dagger_{Ri} B_{Rj} = I_{n_C}$
maximize $\min_\beta$ Tr $X_C W_C(\mathcal{E}_\beta, \mathcal{R})$ subject to $X_C \geq 0$, $\sum_{i,j} (X_C)_{ij}\, B^\dagger_{Ci} B_{Cj} = I_{n_S}$	maximize Tr $X_C \langle W_C(\mathcal{E}_\beta, \mathcal{R})\rangle_\beta$ subject to $X_C \geq 0$, $\sum_{i,j} (X_C)_{ij}\, B^\dagger_{Ci} B_{Cj} = I_{n_S}$

- Iterating between $\mathcal{R}$ and $\mathcal{C}$ results again in bi-convex (SDP) iterations

Indirect approach to fidelity maximization

System-ancilla model with unitary encoding



$$U_{\mathcal{C}} \;=\; [C \;\cdots], \; C \sim n_C \times n_S, \; C^\dagger C = I_{n_S} \; (n_C = n_S n_{CA})$$

$$U_{\mathcal{R}} \;=\; [R \;\cdots], \; R = \begin{bmatrix} R_1 \\ \vdots \\ R_{m_R} \end{bmatrix}, \; R_i \sim n_S \times n_C, \; m_R = n_{CA} n_{RA}, \; R^\dagger R = I_{n_C}$$

Indirect measure

$$\boxed{\begin{array}{c} d = \sum_{r,e} \|R_r E_e C - \alpha_{re} U_S\|_{\mathsf{fro}}^2 = \|RE(I_{m_E} \otimes C) - \alpha \otimes U_S\|_{\mathsf{fro}}^2 \\[2em] \alpha \sim m_R \times m_E, \quad \|\alpha\|_{\mathsf{fro}} = 1 \end{array}}$$

Indirect fidelity maximization

$$\boxed{\begin{array}{ll} \text{minimize} & d = \|RE(I_{m_E} \otimes C) - \alpha \otimes U_S\|_{\text{fro}}^2 \\ \text{subject to} & R^\dagger R = I_{n_C}, \quad C^\dagger C = I_{n_S}, \quad \text{Tr}\ \alpha^\dagger \alpha = 1 \end{array}}
\quad
\left\{ \begin{array}{ll} R & n_{CA} n_{RA} n_S \times n_C \\ E & n_C \times m_E n_C \\ C & n_C \times n_S \\ \alpha & n_{CA} n_{RA} \times m_E \end{array} \right.$$

Indirect fidelity maximization

$$\begin{array}{ll} \text{minimize} & d = \|RE(I_{m_E} \otimes C) - \alpha \otimes U_S\|_{\text{fro}}^2 \\ \text{subject to} & R^\dagger R = I_{n_C}, \quad C^\dagger C = I_{n_S}, \quad \text{Tr}\, \alpha^\dagger \alpha = 1 \end{array}$$

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Optimization over R gives the equivalent problem

$$\begin{array}{ll} \text{minimize} & n_S + \text{Tr}\, E(I_{m_E} \otimes CC^\dagger)E^\dagger - 2\text{Tr}\, \sqrt{E(\gamma \otimes CC^\dagger)E^\dagger} \\ \text{subject to} & C^\dagger C = I_{n_S}, \quad \text{Tr}\, \gamma = 1, \quad \gamma \geq 0 \end{array}$$

- optimization variables are C and $\gamma = \alpha^\dagger \alpha \sim m_E \times m_E$.
- optimizing R is

$$R = VU^\dagger$$

$$E(\alpha^\dagger \otimes CU_S) = USV^\dagger \quad \left\{ \begin{array}{l} U \sim n_C \times n_C \text{ unitary} \\ S = \text{diag}(s_1 \ \cdots \ s_{n_C}), \ s_1 \geq \cdots \geq s_{n_C} \geq 0 \\ V \sim n_C n_{RA} \times n_C, \ V^\dagger V = I_{n_C} \end{array} \right.$$

Indirect fidelity maximization

$$\begin{array}{ll} \text{minimize} & d = \|RE(I_{m_E} \otimes C) - \alpha \otimes U_S\|_{\text{fro}}^2 \\ \text{subject to} & R^\dagger R = I_{n_C}, \quad C^\dagger C = I_{n_S}, \quad \mathrm{Tr}\, \alpha^\dagger \alpha = 1 \end{array}$$

$$\left\{ \begin{array}{ll} R & n_{CA} n_{RA} n_S \times n_C \\ E & n_C \times m_E n_C \\ C & n_C \times n_S \\ \alpha & n_{CA} n_{RA} \times m_E \end{array} \right.$$

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- optimization variables are C and $\gamma = \alpha^\dagger \alpha \sim m_E \times m_E$.
- optimizing R is

$$R = \left(\alpha \otimes U_S C^\dagger\right) E^\dagger \sqrt{\left(E\left(\gamma \otimes CC^\dagger\right) E^\dagger\right)^{-1}}$$

Given C and γ, the following construction for α achieves the optimizing R

$n_{CA} \geq m_E$, $n_{RA} = 1$	$\Rightarrow \quad \alpha = \begin{bmatrix} \sqrt{\gamma} \\ 0_{n_{CA}-m_E \times m_E} \end{bmatrix}$	R is unitary $(n_C \times n_C)$
$n_{CA} < m_E$, $n_{RA}n_{CA} = m_E$	$\Rightarrow \quad \alpha = \sqrt{\gamma}$	R is tall $(m_E n_S \times n_C)$

- If $n_{CA} \geq m_E$, then R is unitary and no recovery ancillas are needed, *i.e.*, $n_{RA} = 1$, and $U_R = R$.[a]

- If $n_{CA} < m_E$, add recovery ancilla so that $m_E = n_{CA}n_{RA}$.
  - $n_{RA} = 2^{q_{RA}} \Rightarrow$ may require padding R with zeroes.

[a]Related results in D. Kribs & R. Spekkens, *Phys. Rev. A* (2006).

Iterative indirect fidelity maximization

Given encoding C, solve for γ from the SDP

$$\boxed{\begin{array}{ll} \text{maximize} & \mathrm{Tr}\,\sqrt{E(\gamma \otimes CC^\dagger)E^\dagger} \\ \text{subject to} & \gamma \geq 0, \quad \mathrm{Tr}\,\gamma = 1 \end{array}}$$

Obtain recovery $(R,\ \alpha)$ from the previous construction $\gamma \ \longrightarrow\ (R, \alpha)$

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Obtain recovery $(R,\ \alpha)$ from the previous construction $\gamma \ \rightarrow\ (R, \alpha)$

Given recovery $(R,\ \alpha)$ solve for C

$$\begin{array}{ll} \text{minimize} & d(R, C, \alpha) = \|RE(I_{m_E} \otimes C) - \alpha \otimes U_S\|_{\mathrm{fro}}^2 \\ \text{subject to} & C^\dagger C = I_{n_S} \end{array}$$

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Equivalent to **Constrained Least-Squares**

$$C = C_{\mathrm{ls}}\sqrt{\left(C_{\mathrm{ls}}^\dagger C_{\mathrm{ls}}\right)^{-1}}, \quad C_{\mathrm{ls}} = \sum_{r,e} \alpha_{re}\,(R_r E_e)^\dagger = \min_{C}\ d(R, C, \alpha)$$

Iterative indirect fidelity maximization

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$$C = C_{\text{ls}} \sqrt{\left(C_{\text{ls}}^\dagger C_{\text{ls}}\right)^{-1}}, \quad C_{\text{ls}} = \sum_{r,e} \alpha_{re} \left(R_r E_e\right)^\dagger = \min_{C} \ d(R, C, \alpha)$$

via SVD:

$$C_{\text{ls}} = U \begin{bmatrix} S \\ 0 \end{bmatrix} V^\dagger \ \rightarrow\ C = U \begin{bmatrix} I_{n_S} \\ 0 \end{bmatrix} V^\dagger$$

Iterative indirect fidelity maximization

(R-γ-C)-SDP

Initialize R and C

Repeat

1. Compute $\gamma \to \alpha$ (via SDP)

$$\begin{array}{ll} \text{maximize} & \mathrm{Tr}\,\sqrt{E(\gamma \otimes CC^\dagger)E^\dagger} \\ \text{subject to} & \gamma \geq 0, \quad \mathrm{Tr}\,\gamma = 1 \end{array}$$

2. Compute R (via SVD)

$$R = \left(\alpha \otimes U_S C^\dagger\right) E^\dagger \left(E\left(\alpha^\dagger \alpha \otimes CC^\dagger\right) E^\dagger\right)^{-1/2}$$

3. Compute C (via SVD)

$$C = C_{\mathsf{ls}} \left(C_{\mathsf{ls}}^\dagger C_{\mathsf{ls}}\right)^{-1/2}, \quad C_{\mathsf{ls}} = \sum_{r,e} \alpha_{re} \left(R_r E_e\right)^\dagger$$

Until

$d(R, C, \alpha)$ stops decreasing

- Each step can only decrease $d(R, C, \alpha)$.

Alternate computation of indirect optimal recovery

Relation between fidelity $f(R,C)$ and distance $d(R,C,\alpha)$:

$$f(R,C) \geq (1 - d(R,C,\alpha)/2n_S)^2$$

Equality iff α solves the **constrained least-squares** problem:

$$\min_{\alpha,\alpha^\dagger\alpha=1} d(R,C,\alpha) \Rightarrow \alpha_{re} = \frac{\text{Tr } R_r E_e C}{n_S\sqrt{f(R,C)}} = \frac{\text{Tr } R_r E_e C}{\sqrt{\sum_{r,e} |\text{Tr } R_r E_e C|^2}}$$

Alternate computation of indirect optimal recovery

Relation between fidelity $f(R, C)$ and distance $d(R, C, \alpha)$:

$$f(R, C) \geq (1 - d(R, C, \alpha)/2n_S)^2$$

Equality iff α solves the **Constrained Least-Squares** problem:

$$\min_{\alpha, \alpha^\dagger \alpha = 1} d(R, C, \alpha) \Rightarrow \alpha_{re} = \frac{\mathrm{Tr}\ R_r E_e C}{n_S \sqrt{f(R, C)}} = \frac{\mathrm{Tr}\ R_r E_e C}{\sqrt{\sum_{r,e} |\mathrm{Tr}\ R_r E_e C|^2}}$$

(R-α)-LS
Initialize R, C
Repeat
1. Compute α from above
2. Compute $R = VU^\dagger$ from SVD:
$E(\alpha^\dagger \otimes C U_S{}^\dagger) = USV^\dagger$
Until
$d(R, C, \alpha)$ stops decreasing

- No convex optimization required: LS for α and SVD to get R
- **Conjecture** – Given C, returns a global d-optimal recovery equivalent to the previous (R-γ)-SDP

Iterative indirect fidelity maximization

(R-α-C)-LS

Initialize R and C

Repeat

1. Compute α

$$\alpha_{re} = (\mathrm{Tr}\ R_r E_e C) \left(\sum_{r,e} |\mathrm{Tr}\ R_r E_e C|^2 \right)^{-1/2}$$

2. Compute R (via SVD)

$$R = \left(\alpha \otimes U_S C^\dagger \right) E^\dagger \left(E \left(\alpha^\dagger \alpha \otimes C C^\dagger \right) E^\dagger \right)^{-1/2}$$

3. Compute C (via SVD)

$$C = C_{\mathsf{ls}} \left(C_{\mathsf{ls}}^\dagger C_{\mathsf{ls}} \right)^{-1/2}, \quad C_{\mathsf{ls}} = \sum_{r,e} \alpha_{re} \left(R_r E_e \right)^\dagger$$

Until

$d(R, C, \alpha)$ stops decreasing

- Each step can only decrease $d(R, C, \alpha)$.
- Each step is a form of Constrained-Least-Squares.

Structure of recovery

For $m_E > n_{CA}$, $\alpha \sim m_E \times m_E$ $(= \sqrt{\gamma})$ with SVD

$$\alpha = VSW^\dagger \quad \begin{cases} V,\ W \sim m_E \times m_E \text{ unitaries} \\ S = \mathrm{diag}(s_1\ \cdots\ s_k\ 0\ \cdots\ 0),\ s_k > 0 \end{cases}$$

Then,

$$\begin{aligned} R &= \left(\alpha \otimes U_S C^\dagger\right) E^\dagger \sqrt{\left(E\left(\gamma \otimes CC^\dagger\right) E^\dagger\right)^{-1}} \\ &= (V \otimes I_{n_S})(S \otimes U_S C^\dagger) F^\dagger \sqrt{\left(F(S^2 \otimes CC^\dagger) F^\dagger\right)^{-1}} \end{aligned}$$

with

$$F = E(W \otimes I_{n_C}) = [F_1\ \cdots\ F_{m_E}]$$

- W is unitary freedom in error OSR
- $V^\dagger$ is unitary freedom in recovery OSR $\Rightarrow$ reduces to k OSR elements

$$\boxed{\overline{R}_i = s_i U_S C^\dagger F_i^\dagger \sqrt{\left(\sum_{j=1}^{k} s_j^2 F_j C C^\dagger F_j^\dagger\right)^{-1}},\ i = 1, \ldots, k}$$

- more efficient then reduction via SVD of process matrix

Computational cost

Operation	Method		SDP Variables ($n_C = n_S n_{CA}$)
Recovery	Direct	Primal-SDP	$(n_S{}^2 - 1)n_C{}^2$
		Dual-SDP	$n_C{}^2$
	Indirect	SDP	$(n_{CA}n_{RA})^2$
		LS	0
Encoding	Direct	Primal-SDP	$(n_C{}^2 - 1)n_S{}^2$
		Dual-SDP	$n_S{}^2$
	Indirect	LS	0

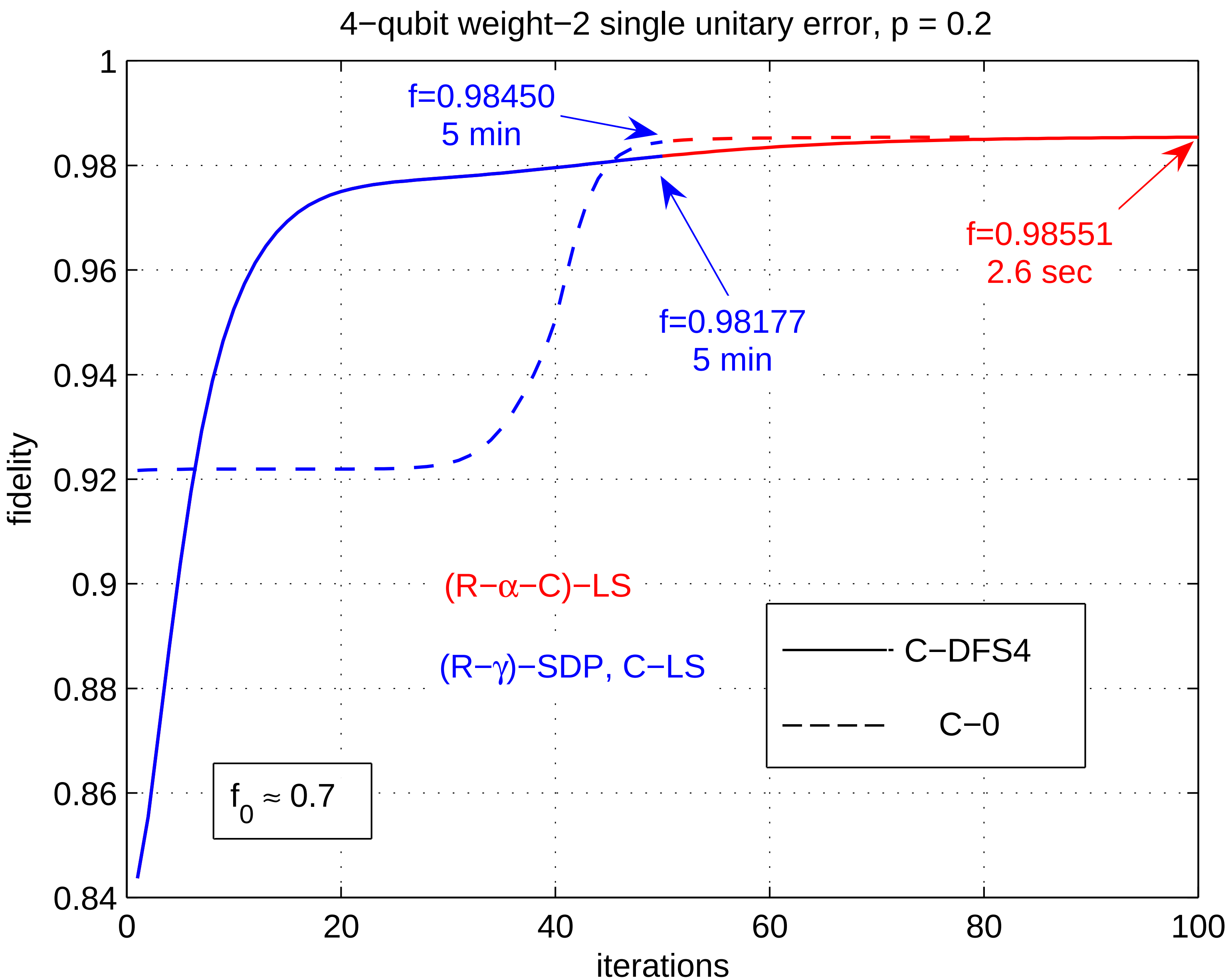
- Iterations to solve SDPs depend on number of variables, constraints, *etc.*
- Dual $\longrightarrow$ Primal requires solving linear equations
- Process matrix via SDP $\longrightarrow$ Recvery or Encoding OSR requires SVD.
- Recovery via (R-γ)-SDP $\longrightarrow$ Recovery OSR requires iterations + SVD
- Recovery via (R-α)-LS $\longrightarrow$ Recovery OSR requires SVD
- Encoding LS $\longrightarrow$ Encoding OSR requires SVD

- Exponential scaling with qubits seems unavoidable

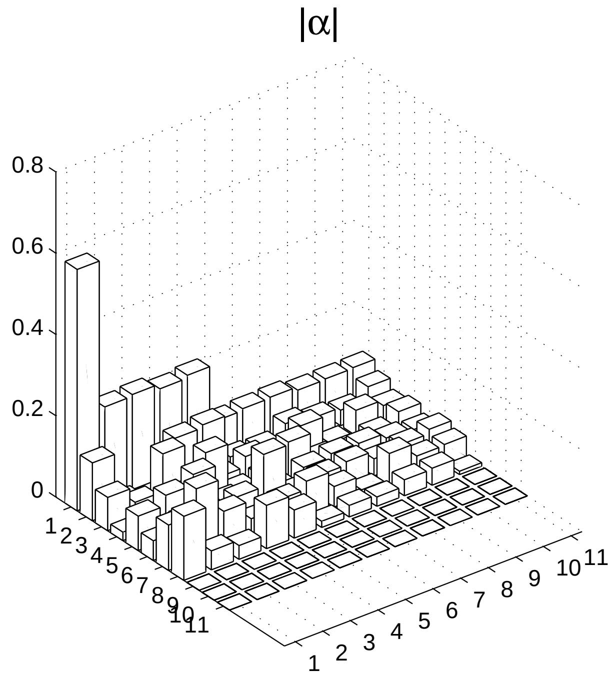
- The indirect methods also support both the robust worst-case & average-case criteria for the error system set

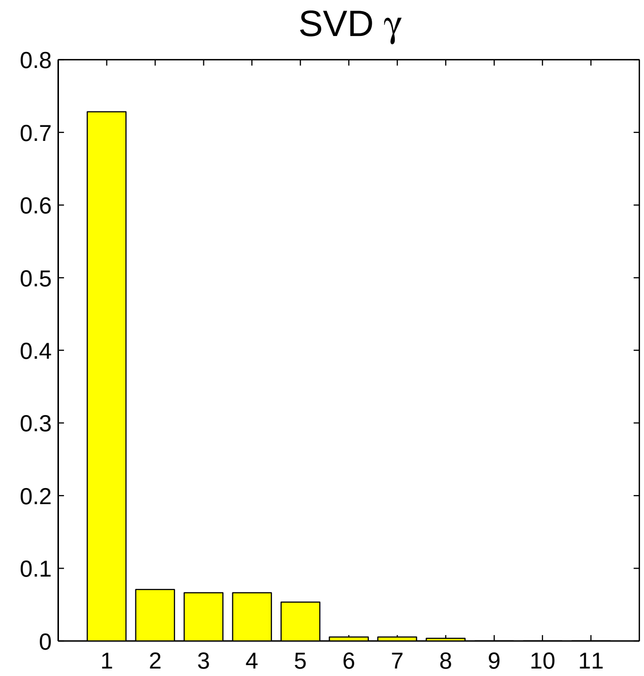
$$\mathcal{E} \in \left\{ \mathcal{E}_\beta \mid \beta = 1, \ldots, \ell \right\}$$

Recovery & Encoding via Iterative Optimization



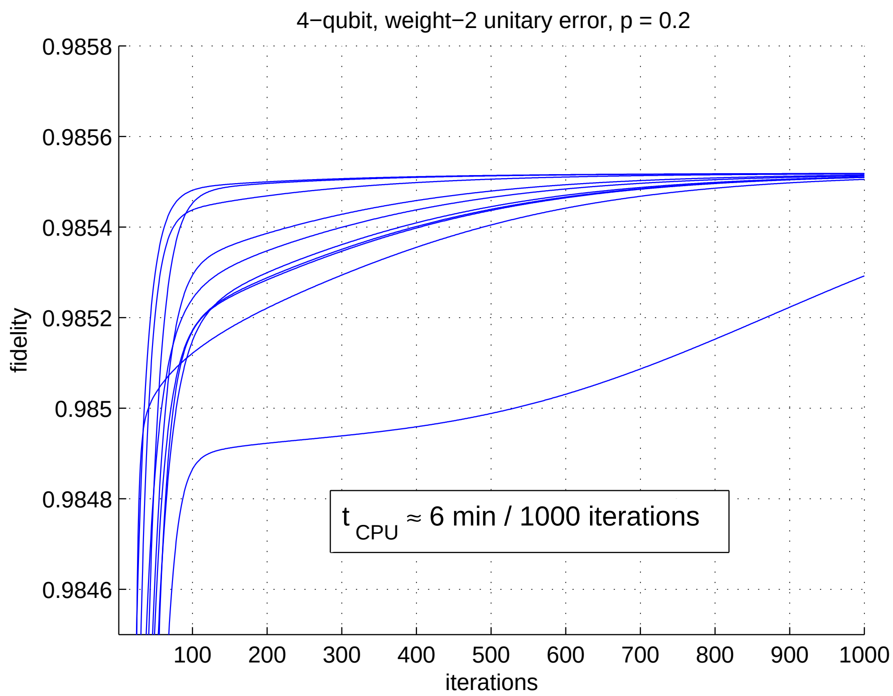
Iterated recovery $\Rightarrow$ $(\alpha,\ \gamma) \sim 11 \times 11$ for weight-2 unitary errors

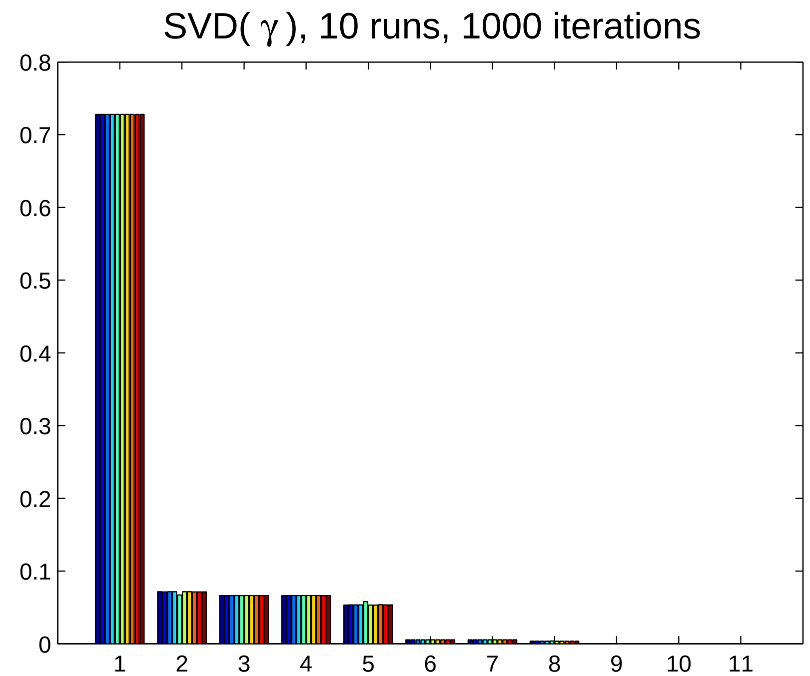




- α is a complex matrix – looks like an 8×11
- SVD of γ shows 8 significant terms (explains reduction from 11)
- 8 significant singular values in 32×32 recovery process matrix
- iterated encoding far from DFS4 encoding: $f(C_{\mathrm{DFS4}}, C) = 0.13$

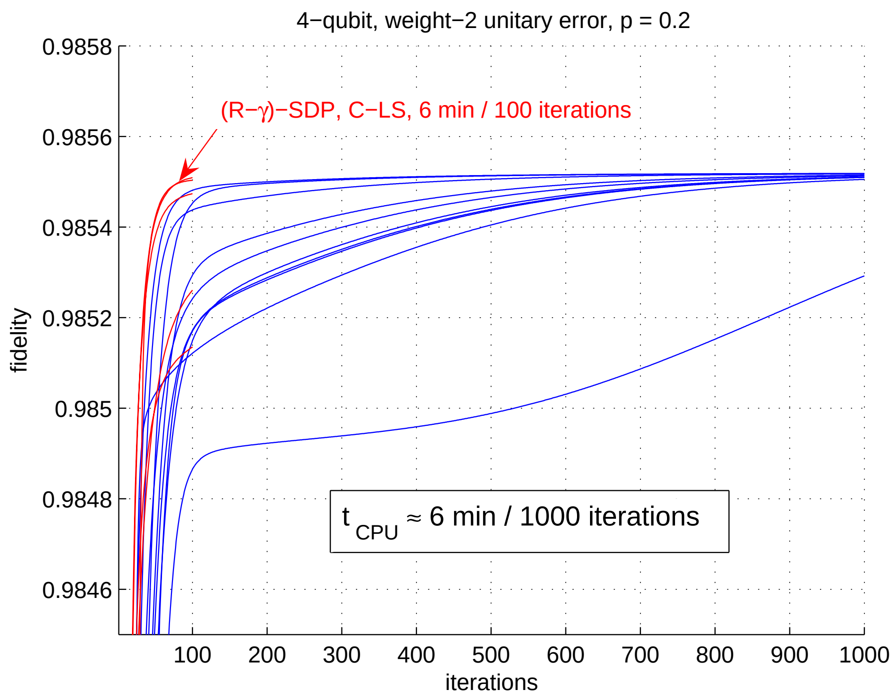
10 runs initlaized at random encodings ($\color{red}{C} \sim 16 \times 2$)

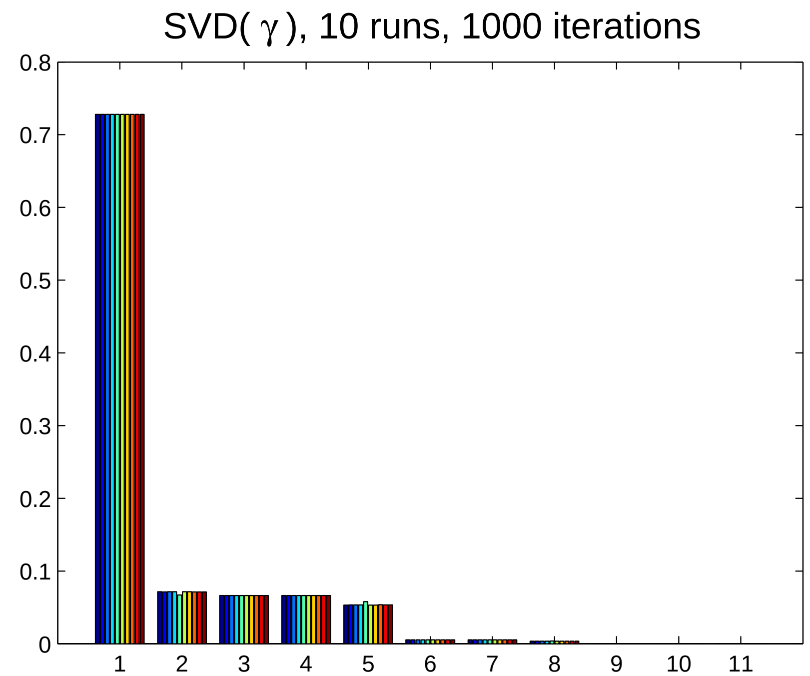




- final fidelities appear to be converging
- initial encodings are very different from each other
- final recoveries & encodings are **all** very different from each other
  - global optimum is not unique?

10 runs initlaized at random encodings ($C \sim 16 \times 2$)





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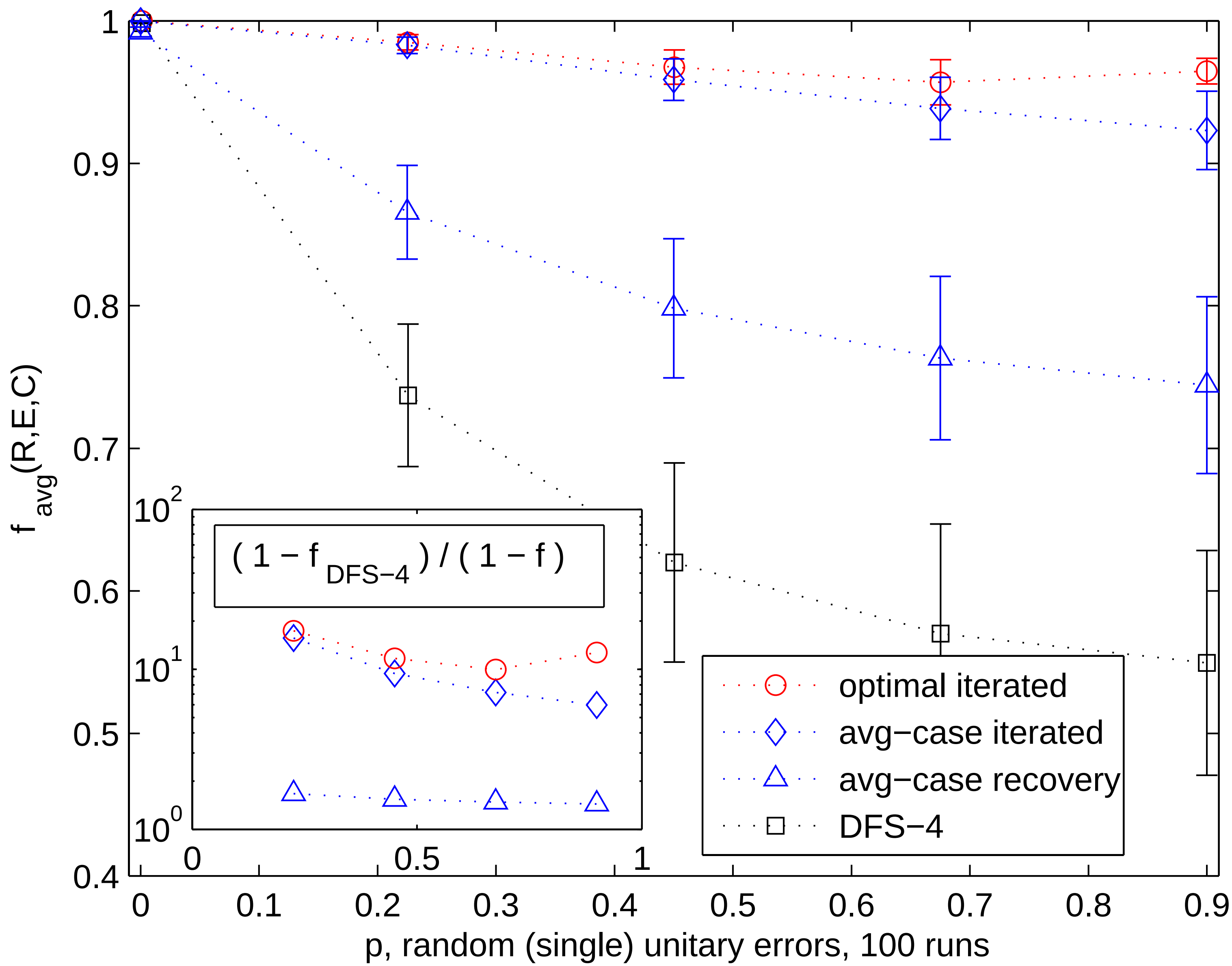
Weight-2 single-random unitary errors for 4-qubit codes.



(DFS-4 – perfect against collective errors)

Optimal Recovery

- Preserve single qubit ($n_S = 2^1,\ U_S = I_2$) in 5-qubit codespace ($n_C = 2^5$)
- $[5,1,3]$ encoding – perfect against arbitrary weight-1 errors and against weight-2 bit-flip errors *with ideal recovery*

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Weight-2 bit-flip errors $p = 0.2,\ \mathcal{E} \sim 32 \times 32 \times 16$			
Method	Fidelity	m_E ($\mathcal{R} \sim 2 \times 32 \times m_E$)	CPU (sec)
Direct-SDP	1.0	$64 \longrightarrow 16$	38.8
Indirect (R-γ)-SDP	1.0	16	12.8
Indirect (R-α)-LS	1.0	16	0.063 (5$\times$)

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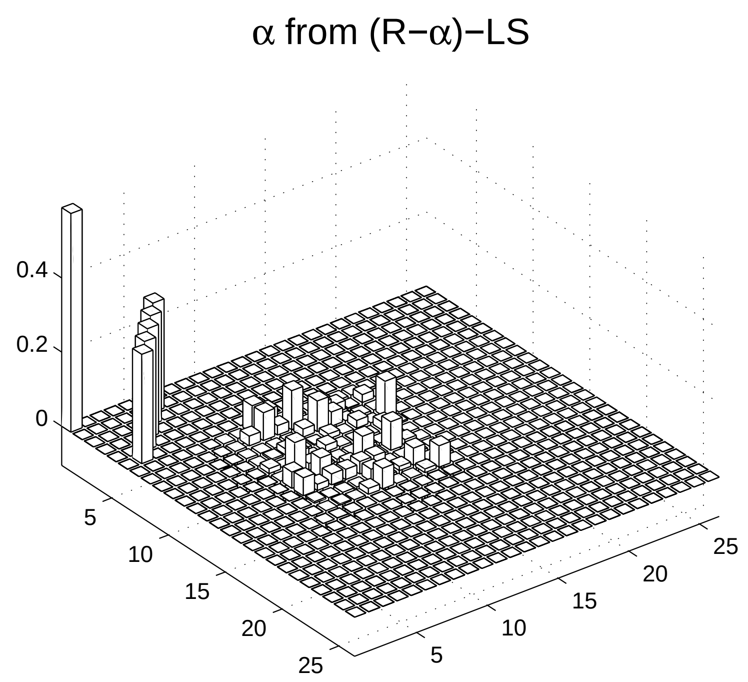
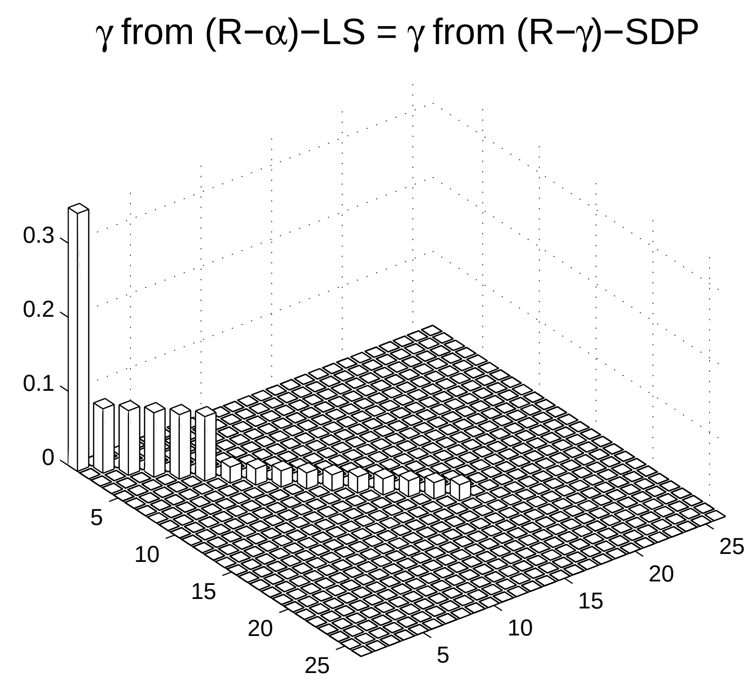
Weight-3 bit-flip errors $p = 0.2,\ \mathcal{E} \sim 32 \times 32 \times 26$			
Method	Fidelity	m_E ($\mathcal{R} \sim 2 \times 32 \times m_E$)	CPU (sec)
Direct-SDP	0.94845	$64 \longrightarrow 16$	40.6
Indirect (R-γ)-SDP	0.94845	$26 \longrightarrow 16$	31.6
Indirect (R-α)-LS	0.94845	$26 \longrightarrow 16$	0.41 (10$\times$)

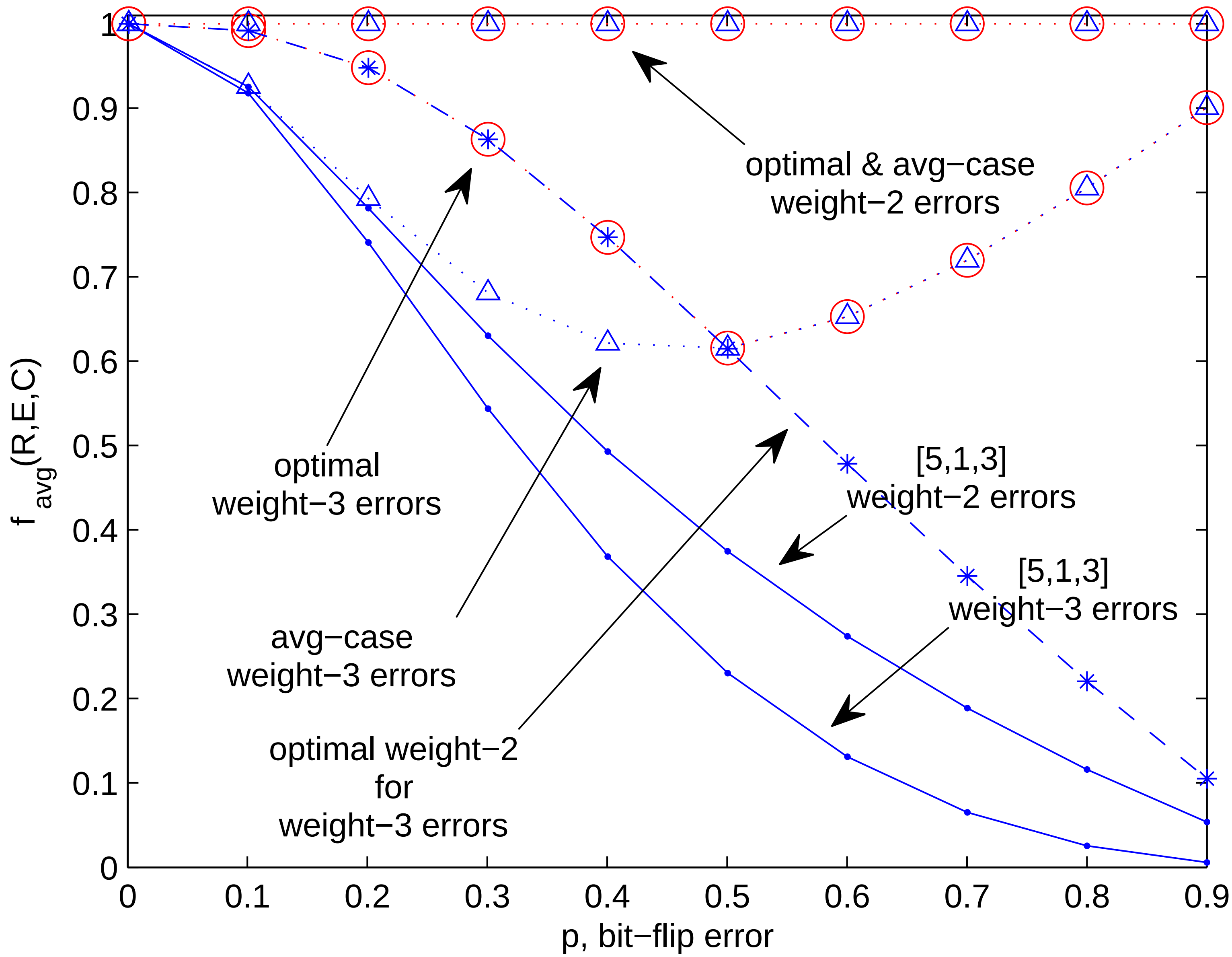
Recovery $(\alpha,\ \gamma) \sim 26 \times 26$ for weight-3 bit-flip errors





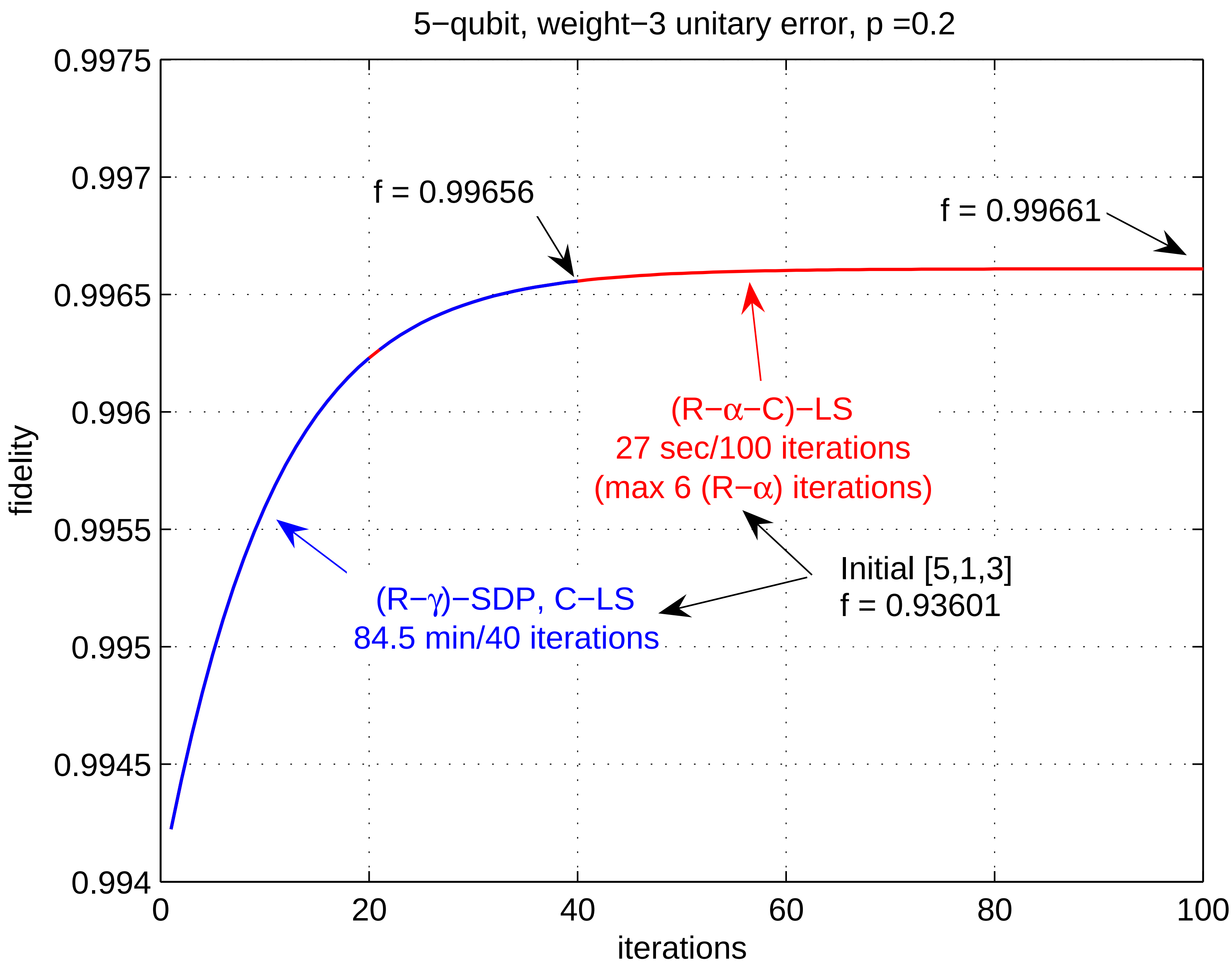
- all elements of α and γ are real (in this example)
- γ is diagonal with 16 non-zero elements
  - explains reduction of recovery to 16 OSR elements

Weight-2 & 3 bit-flip errors, $[5,1,3]$ encoding, optimal & average-case recoveries

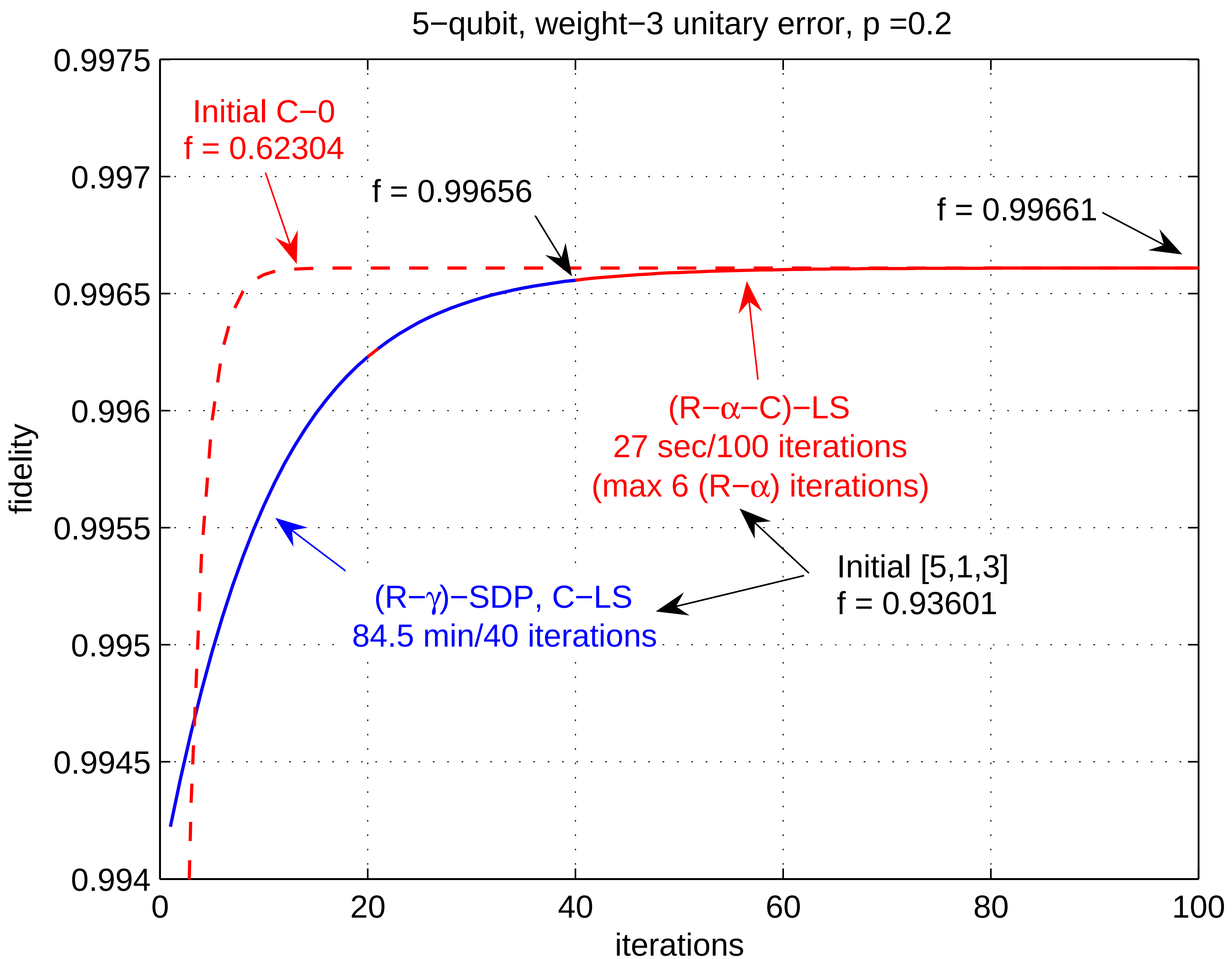


O – optimal for each p △ – avg-case all p • – [5,1,3]

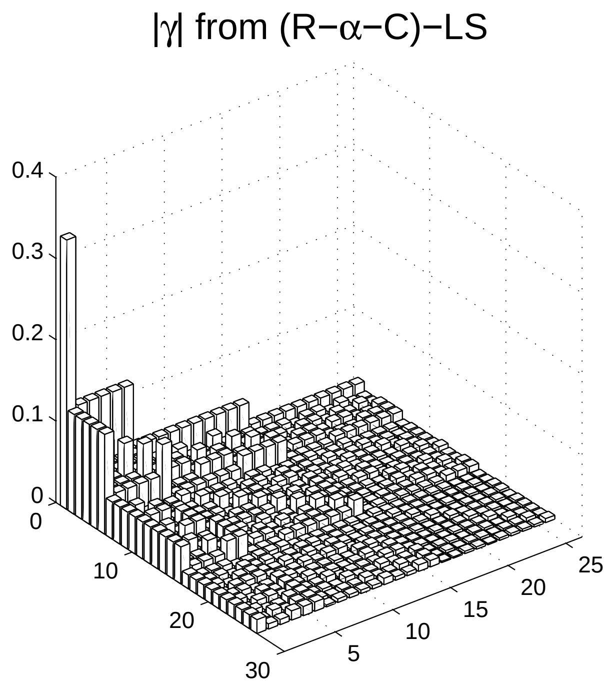
Recovery & Encoding via Iterative Optimization

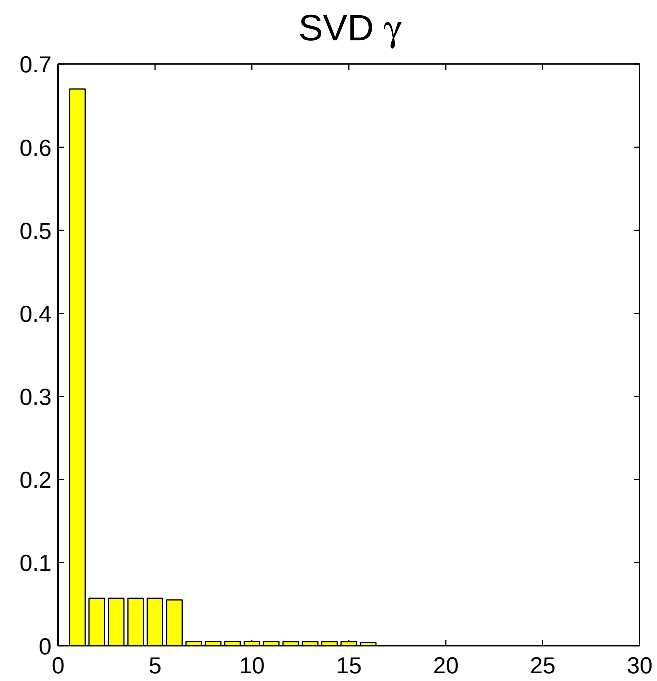


Recovery & Encoding via Iterative Optimization



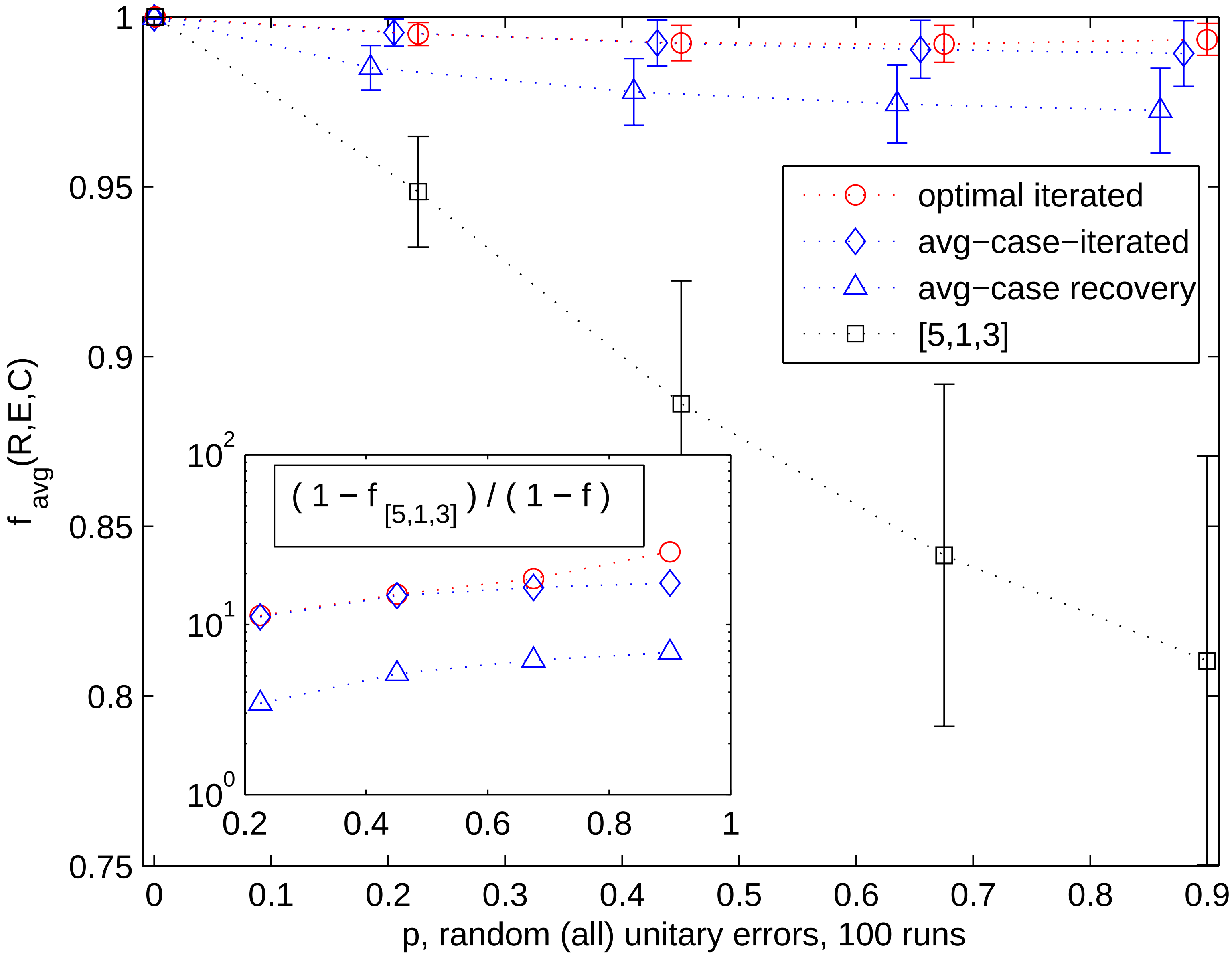
Iterated recovery $\Rightarrow$ $(\alpha,\ \gamma) \sim 26 \times 26$ for weight-3 unitary errors





- γ is a complex matrix (in this example)
- SVD of γ shows 16 significant terms (explains reduction from 26)
- iterated encoding similar to [5,1,3] encoding: $f(C_{[5,1,3]}, C) = 0.95$
- iterated recovery far from [5,1,3] recovery: $f(R_{[5,1,3]}, R) = 0.05$

Weight-2 all-random unitary error for 5-qubit codes.

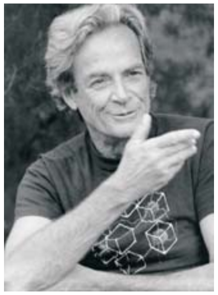


Conclusions

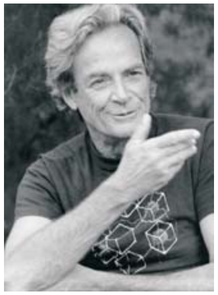
- optimal and/or robust codes potentially mitigate the need for higher dimensional codes or further levels of concatenation.
  - codes can be tuned for a range of error models
- optimal and robust designs via indirect approach is more efficient than direct approach
  - least-squares (SVD) vs. convex optimization (SDP)
- indirect approach reveals some structure of the recovery

Issues & extensions

- incorporate structure, *e.g.*, fault-tolerant and concatenation architectures
  - non-unique global optimum may provide this design freedom
- incorporate specific knowledge, *e.g.*, specific inputs, specific physical models, specific knowledge about the “imperfections”
- how to optimize – one module at a time, a robust design over a large set of modules, ...
- implementation & control – “comprehensibility often overides optimiality”
- “black box” error correction: use state/process tomography to identify errors for optimal/robust encoding/recovery
- what is the character of the encoding/recovery landscape



“ ... and these imperfections may produce considerable havoc.”



“ ... and these imperfections may produce considerable havoc.”

“It ain’t what you don’t know
that gets you into trouble.
It’s what you know for sure
that just ain’t so.”

– Mark Twain

Oxford, 1907