Conditions for Approximate Quantum Error Correction

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Background :-

- Standard approach to quantum error correction: ideal notion of correctability -
  Codespace $\mathbb{C}$ is correctable under the action of a (*trace-preserving*) quantum channel $\mathbb{A}$ iff $\exists$ a trace-preserving quantum channel $\mathbb{R}$ such that $\forall\ \rho \in \mathbb{C}$,
  $\mathbb{R} \circ \mathbb{A}(\rho) = \rho$

- Condition for *perfect* error correction (Knill, Laflamme '97) -
  Channel $\mathbb{A}$ with Kraus operators $\{A_i\}$, is correctable on a codespace $\mathbb{C}$ with projector P iff $\exists$ scalars $\{\lambda_{ij}\}$ such that

$$PA_i^\dagger A_j P = \lambda_{ij} P$$

  - Independent of the Kraus representation; easily checkable once P is given.

Formulating the problem:-

- Define a notion of *approximate* correctability -

  $\exists$ a TP channel $\mathbb{R}$: $F[\mathbb{R}\circ\mathbb{A}(\rho), \rho] \geq 1 - \delta \;\; \forall \;\; \rho \in \mathbb{C}$

  (F: fidelity measure)

- Questions:

  - Characterizing approximate correctability : Can we find an easily checkable condition for approximate error correction?

  - Robustness of perfect error correction: What happens when the Knill-Laflamme condition is only approximately satisfied?

- Choice of the fidelity measure F :

  - Entanglement fidelity

  - Worst case fidelity: Every state in the codespace is recovered with high fidelity.

BK condition for approximate correction with high entanglement fidelity :-

- To correct the action of channel $\mathbb{A} = \{A_i\}$ on a codespace $\mathbb{C}$ (dimension d) with projector P, construct the recovery map $\mathbb{R}_{BK}$ with Kraus operators :

  $R_i^{BK} = \{ (P/d)^{1/2} A_i^\dagger \mathbb{A}(P/d)^{-1/2} \}$
- Barnum, Knill (J Math Phys, **43** 2097 [2002]) : For any TP recovery channel $\mathbb{R}$,

$$F_{ent}[P/d, \mathbb{R}_{BK} \circ \mathbb{A}] \geq (F_{ent} [P/d, \mathbb{R} \circ \mathbb{A}])^2$$

$\Rightarrow$ If $F_{ent} [P/d, \mathbb{R} \circ \mathbb{A}] \geq 1 - \delta$, then $F_{ent}[P/d, \mathbb{R}_{BK} \circ \mathbb{A}] \geq 1 - 2\delta$

- Easily checkable condition for approximate correctability: $\mathbb{A}$ is correctable on codespace $\mathbb{C}$ with high entanglement fidelity iff

$$F_{ent}[P/d, \mathbb{R}_{BK} \circ \mathbb{A}] \geq 1 - \delta$$

  - Recovery is optimal to within a factor of 2.
- Alternate approaches to find *the* optimal recovery map :

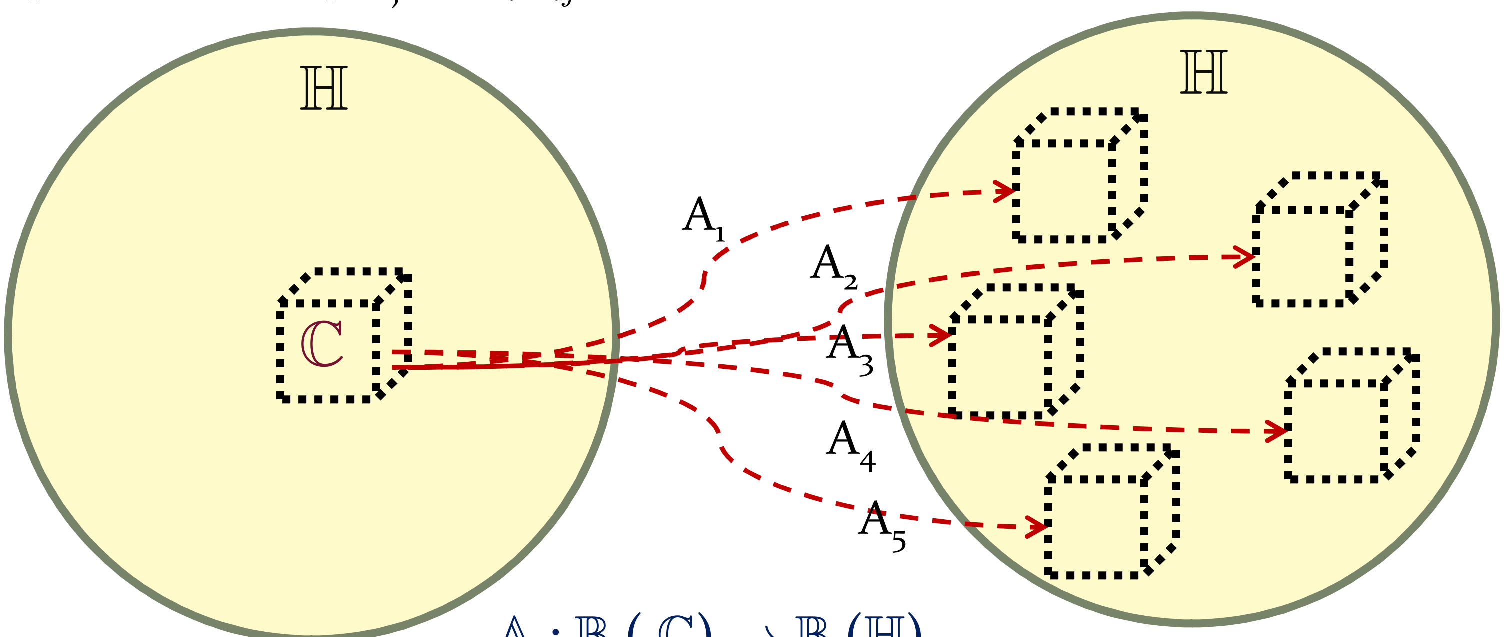
  A.Fletcher, P.W.Shor, MZ.Win quant-ph/0606035 (2006) using *Semi-Definite Programming*.

  R.L.Kosut, D.A.Lidar quant-ph/0606078 (2006) using *Convex Optimization*

Approximate Correction with high worst case fidelity:-

- Attempt to construct approximate codes by relaxing the perfect error correction condition.
- Recall that for a perfectly correctable channel $\mathbb{A}$ on codespace $\mathbb{C}$, there exists a representation in which the Kraus operators map $\mathbb{C}$ to mutually *orthogonal* subspaces in a *unitary* fashion:

  $\exists \ \{A_i\}$ such that $PA_i^\dagger A_j P = \lambda_i \delta_{ij} P$

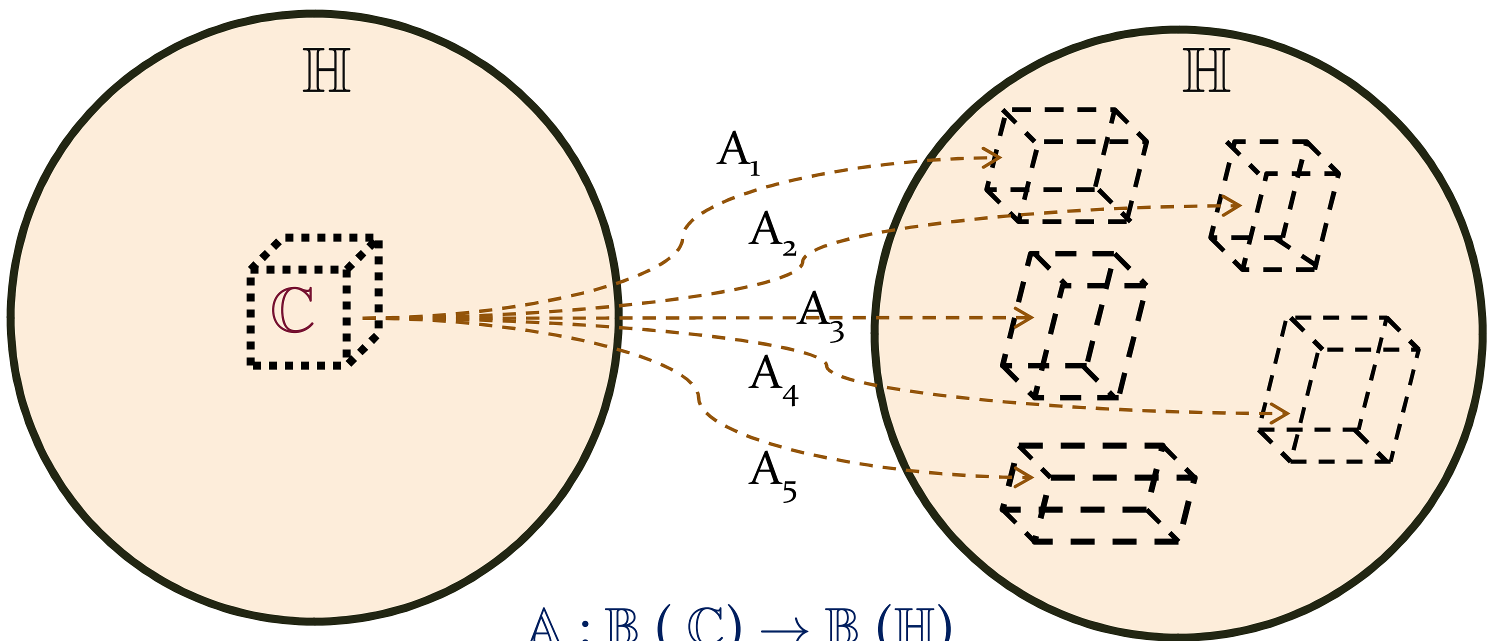


$\mathbb{A} : \mathbb{B}(\mathbb{C}) \rightarrow \mathbb{B}(\mathbb{H})$

The Leung *et al.* 4-qubit code:-

- Example of a code that corrects with high worst case fidelity: Leung *et al.* 4-qubit code (1997) for the Amplitude Damping Channel .

  - satisfies the Knill-Laflamme condition approximately.
- Kraus operators of the channel map $\mathbb{C}$ to mutually orthogonal subspaces which are *not unitary transforms* of the codespace.



$\mathbb{A} : \mathbb{B}(\mathbb{C}) \rightarrow \mathbb{B}(\mathbb{H})$

Algebraic condition motivated by the 4-qubit code:-

- If the action of channel $\mathbb{A}$ = { A_i , i $=$ 1,2…N } on codespace $\mathbb{C}$ is such that $\exists$ scalars { λ_i > 0 } and positive operators {B_i } such that the Kraus operators in the *canonical representation* (Tr[$A_i^\dagger A_j P$] = 0 $\forall$ i $\neq$ j) satisfy

$$PA_i^\dagger A_j P = \lambda_i P \delta_{ij} + PB_i P \delta_{ij}$$

with

$$\max_{|\psi\rangle \in \mathbb{C}} \sum_{i=1}^{N} \langle\psi| PB_iP |\psi\rangle \quad \leq \quad \epsilon$$

then, there exists a TP recovery $\mathbb{R}$ such that

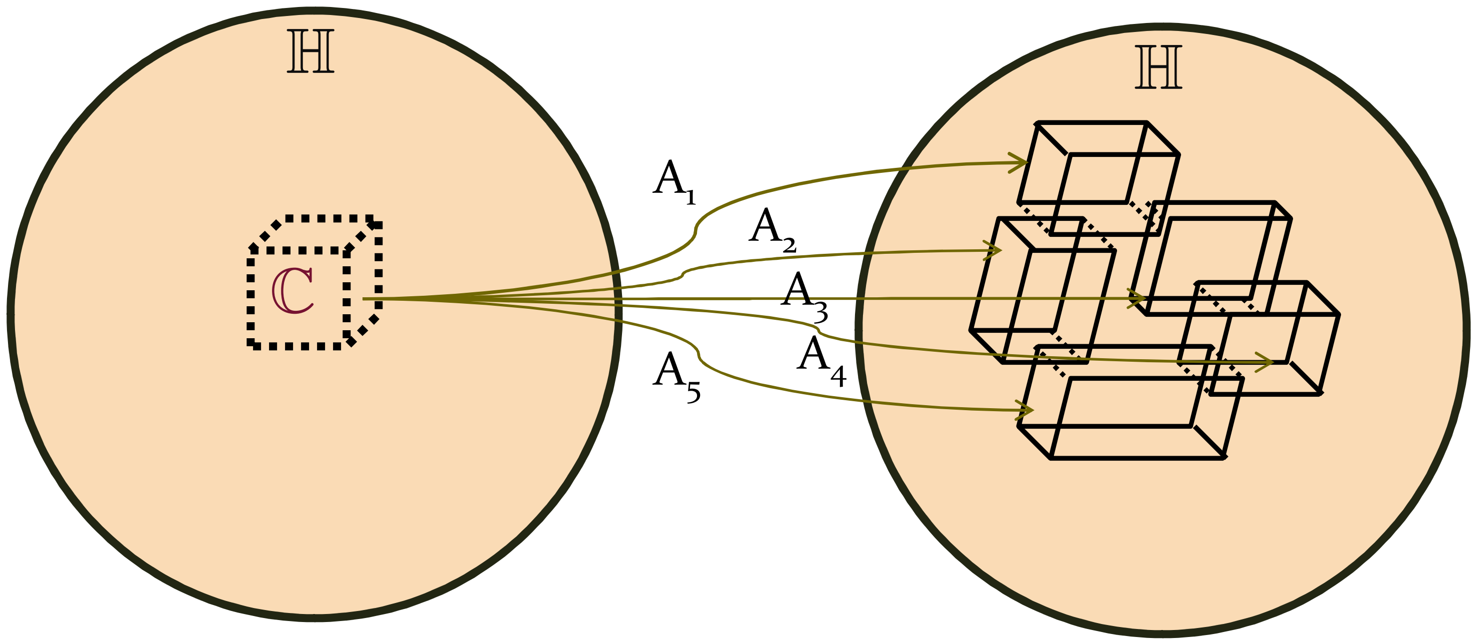
$$F_{pure}(\mathbb{R} \circ \mathbb{A}, \mathbb{C}) = \min_{|\psi\rangle \in \mathbb{C}} Tr[|\psi\rangle\langle\psi| \mathbb{R} \circ \mathbb{A}(|\psi\rangle\langle\psi|)] \geq 1 - \epsilon$$

- The map $\mathbb{R}$ is constructed with Kraus operators { $R_i = PU_i^\dagger$ } where

$A_i P = U_i \sqrt{\lambda_i P + PB_i P}$ [U_i : unitary that approximates the action of A_i]

Approximate correctability of a general channel:-

- Algebraic condition stated above characterizes approximately correcting codes for a restricted class of channels.
- For an arbitrary channel : Codespace gets *distorted* under the action of different Kraus operators and is mapped to subspaces that are *not necessarily mutually orthogonal.* ($PA_i^\dagger A_j P \neq 0 \; \forall \; i \neq j$)



$\mathbb{A} : \mathbb{B}(\mathbb{C}) \rightarrow \mathbb{B}(\mathbb{H})$

A general condition for approximate correctability:-

- If $\mathbb{A} = \{ A_i , i = 1,2\ldots N \}$ acts on $\mathbb{C}$ such that

  (i) $\exists$ real numbers $\lambda_i > 0$, and positive operators $\{B_i\}$ satisfying

$$PA_i^\dagger A_i P \quad = \quad \lambda_i P + PB_iP$$

with

$$\max_{|\psi\rangle \in \mathbb{C}} \sum_{i=1}^{N} \langle\psi|\, PB_iP \,|\psi\rangle \quad \leq \quad \epsilon$$

(ii) For $i \neq j$

$$\sum_{i \neq j} \frac{\| PA_i^\dagger A_j P \|^2}{\lambda_i \lambda_j} \leq \delta$$

($\| M \|$= Spectral norm = Max. singular value of M)

then, $\exists$ TP $\mathbb{R}$ such that

$$F_{pure} = \min_{|\psi\rangle \in \mathbb{C}} Tr[|\psi\rangle \langle\psi|\, \mathbb{R} \circ \mathbb{A}(|\psi\rangle \langle\psi|)] \geq 1 - \epsilon - \delta$$

Constructing the optimal recovery channel:-

- Construct *partial isometries* $\{W_i\}$ by a Gram-Schmidt like procedure:

$$
\begin{aligned}
W_1 &= U_1 \\
W_2 &= U_2 - W_1 P (P W_1^\dagger W_1 P)^{-1} P W_1^\dagger U_2 \\
W_3 &= U_3 - \left[W_2 P (P W_2^\dagger W_2 P)^{-1} P W_2^\dagger \right] U_3 - \left[W_1 P (P W_1^\dagger W_1 P)^{-1} P W_1^\dagger \right] U_3 \\
W_i &= U_i - \left[\sum_{k=1}^{i-1} W_k P (P W_k^\dagger W_k P)^{-1} P W_k^\dagger \right] U_i
\end{aligned}
$$

where $A_i P = U_i \sqrt{\lambda_i P + P B_i P}$

- $(PW_k^\dagger W_K P)^{-1}$ exist on the codespace since

$$P \geq P W_i^\dagger W_i P \geq (1-\delta) P - O(\delta^2) P$$

- Orthogonality: $PW_i^\dagger W_j P = 0 \;\; \forall \;\; i \neq j$

Properties of the recovery map:-

- By constructing $\{W_i\}$, we have generated *a set of mutually orthogonal subspaces* that approximate the action of the channel with high enough fidelity.
- $\{M_i = W_i P W_i^\dagger\}$: positive operators that have disjoint support
  $\{Supp(M_i)\}$: a set of mutually orthogonal subspaces of $\mathbb{H}$
- Recovery map with Kraus operators $R_i = PW_i^\dagger$ recovers with high worst case fidelity.

To conclude…

- We have outlined a *sufficient* condition for approximate correctability that characterizes a general class of approximately correcting codes.

  In the process, we have shown that the perfect error correction condition is robust.

- Can we use this condition to obtain trade-offs between code lengths and fidelity for non-Pauli error channels?

- Is this condition also *necessary* for approximate correctability? This would imply that every approximately correcting code can be visualized as a perturbation of the Knill-Laflamme condition!

- Is the Barnum-Knill recovery close to optimal for worst case fidelity as well? This would lead to a characterization of approximate codes, independent of the Knill-Laflamme condition.