Constructions and Performance of Asymmetric Quantum Codes

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Asymmetric Quantum Models – Biased Noise

An interesting observation

- In many physical systems the probabilities for X errors and Z errors are **not equal**. How can this be asymmetry be exploited for the design of efficient quantum codes?
- In particular, the relaxation time T_1 can be much longer than dephasing time T_2.

→ See, e.g., [Ioffe, Mézard 2007], [Evans et al 2007] [Aliferis, Preskill 2007]

→ See also QEC'07 talks by John Preskill and Panos Aliferis

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Talk Outline

- A motivating example
- Code constructions:
  - From families of nested CSS codes
  - From quantum LDPC codes
- Performance simulations
- Comparison with related work

Motivating Example

Combined amplitude damping and dephasing

Kraus operators:

$$A_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\lambda-\gamma} \end{bmatrix}, A_1 = \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{\lambda} \end{bmatrix}, A_2 = \begin{bmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{bmatrix},$$

where $\sqrt{1-\gamma-\lambda} = e^{-t/T_2}$ and $1-\gamma = e^{-t/T_1}$. Then

$$\begin{aligned} \mathcal{E}(\rho) &= \sum_k A_k \rho A_k^\dagger \\ &= \begin{bmatrix} 1-\rho_{11}e^{-t/T_1} & \rho_{01}e^{-t/T_2} \\ \rho_{10}e^{-t/T_2} & \rho_{11}e^{-t/T_1} \end{bmatrix}. \end{aligned}$$

Questions

- How to relate T_1 and T_2 to p_x, p_y, p_z?
- Perhaps more basic: how to get Pauli channel from this?

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Pauli Channels

One qubit

- Depolarizing channel:

$$\rho \mapsto (1-p)\rho + p/3(X\rho X + Y\rho Y + Z\rho Z)$$

- General Pauli channel:

$$\rho \mapsto (1-p_x-p_y-p_z)\rho + p_x X\rho X + p_y Y\rho Y + p_z Z\rho Z$$

Several qubits

- Combinations X, Z, and $Y = XZ$ of bit-flips and phase-flips cover all errors on a single qubit.
- We assume that errors affect qubits independently and are interested in correcting errors of bounded weight.

  $\rightarrow$ See also QEC'07 tutorial by Daniel Gottesman.

Pauli Twirling

Goal: Map an arbitrary channel to a Pauli channel. Recall that a general quantum channel is given by a CPTP map:

$$\mathcal{E} : \rho \mapsto \sum_k A_k \rho A_k^\dagger$$

Alternative representation is as $\rho \mapsto \sum_{k,\ell} \chi_{k,\ell} P_k \rho P_\ell$. Twirling with random Pauli operators maps this to

$$\overline{\mathcal{E}} : \rho \mapsto \sum_k \chi_{k,k} P_k \rho P_k$$

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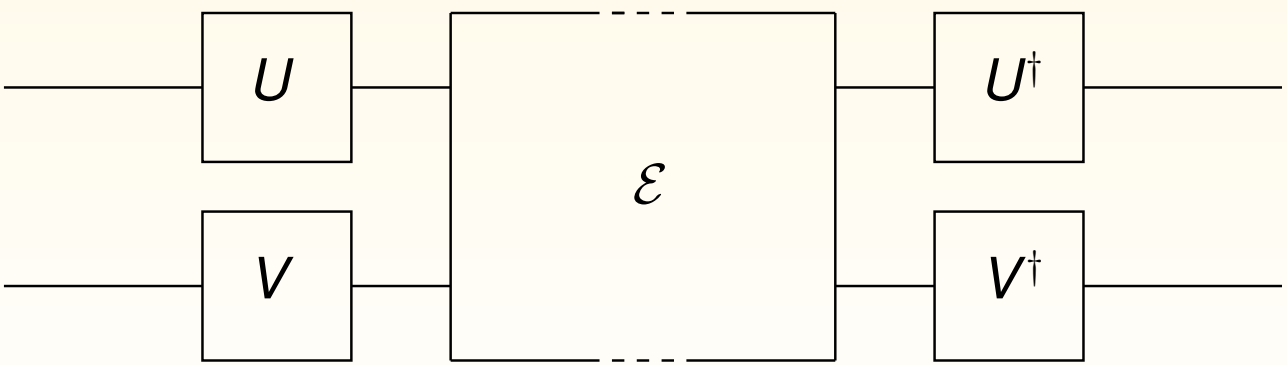

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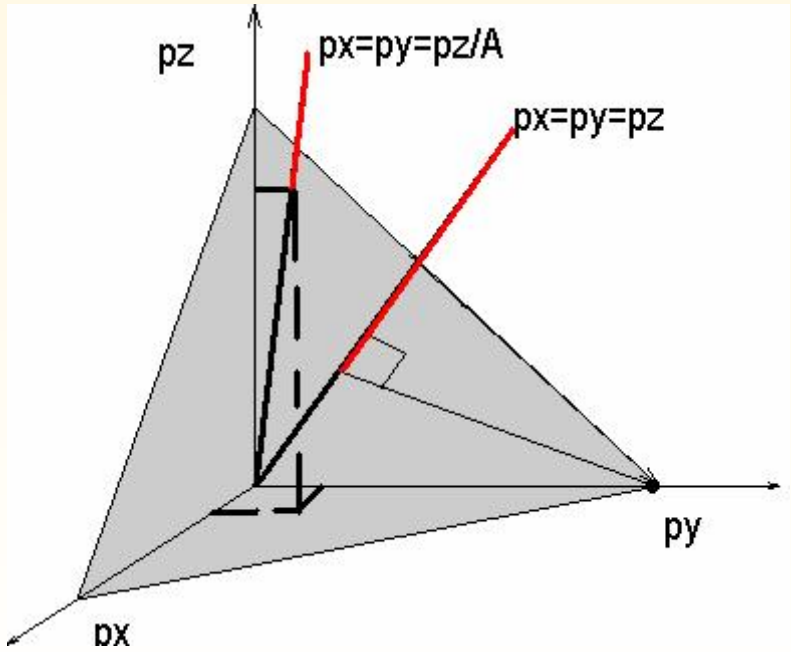
Amplitude Damping and Dephasing Redux

Associated Pauli-twirled channel:

$$\overline{\mathcal{E}}(\rho) = (1 - p_x - p_y - p_z)\rho + p_x X\rho X + p_y Y\rho Y + p_z Z\rho Z.$$

- Here $p_x = p_y = (1-e^{-t/T_1})/4$ and $p_z = 1/2 - p_x - \frac{1}{2}e^{-t/T_2}$.
- The asymmetry ratio is $\frac{p_z}{p_x} = 1 + 2\frac{1-e^{t/T_1(1-T_1/T_2)}}{e^{t/T_1}-1}$
- For $t \ll T_1$ this ratio is $2T_1/T_2 - 1$.



Code constructions

Defining Asymmetric Quantum Codes as CSS codes

CSS Construction

Let $C_x = [n, k_x]_q$, $C_z = [n, k_z]_q$ be codes in $\mathbb{F}_q^n$ with $C_x^\perp \subseteq C_z$.

- Corresponding stabilizer of the CSS code:

$$H = (X|Z) = \left(\begin{array}{c|c} H_x & 0 \\ 0 & H_z \end{array} \right)$$

  where H_x and H_z are parity check matrices of C_X, C_Z.

- This defines $[[n, k_x + k_z - n, d_x/d_z]]_q$ asymmetric quantum code, with $d_x = \mathrm{wt}(C_x \setminus C_z^\perp)$ and $d_z = \mathrm{wt}(C_z \setminus C_x^\perp)$.

Idea: Use CSS construction with C_x much stronger than C_z.
See also [Steane, PRA **54**, 4741 (1996)]

Singleton bound: Any pure asymmetric $[[n, k, d_x/d_z]]_q$ CSS code (more generally, any $\mathbb{F}_q$-linear CSS code) satisfies

$$k \leq n - d_x - d_z + 2.$$

Linear Programming Bounds

Theorem: If an $[[n, k, d_x/d_z]]_2$ asymmetric CSS stabilizer code with $k > 0$ exists, then there exists an integer solution to the problem of maximizing $\sum_{j=1}^{d_z-1} A_j$ subject to the constraints

- $A_0 = A_0^\perp = B_0 = B_0^\perp = 1$ and $A_j, A_j^\perp, B_j, B_j^\perp \geq 0$ $(1 \leq j \leq n)$,
- $\sum_{j=0}^{n} A_j = 2^{k'}, \sum_{j=0}^{n} B_j = 2^{n-k-k'}$, for some $0 < k' < n - k$,
- $A_j^\perp = 1/2^{k'} \sum_{r=0}^{n} K_j(r) A_r$ for all $0 \leq j \leq n$;
- $B_j^\perp = 1/2^{n-k-k'} \sum_{r=0}^{n} K_j(r) B_r$ for all $0 \leq j \leq n$;
- $A_j = B_j^\perp$ for all $0 \leq j < d_x$ and $A_j \leq B_j^\perp$ for all $d_x \leq j \leq n$;
- $B_j = A_j^\perp$ for all $0 \leq j < d_z$ and $B_j \leq A_j^\perp$ for all $d_z \leq j \leq n$;

where $K_j(r)$ denotes the Krawtchouk polynomial

$$K_j(r) = \sum_{s=0}^{j} (-1)^s \binom{r}{s} \binom{n-r}{j-s}.$$

Example: From this follows that a $[[15, 1, 3/7]]$ might exist.

Reed-Muller Codes

A classic code construction

Boolean functions: $f := \{0,1\}^m \to \{0,1\}$, represented in algebraic normal form $f(x_1, \ldots, x_m) = \bigoplus_{v \in \{0,1\}^m} \alpha_v x_1^{v_1} \ldots x_m^{v_m}$. Degree of f is defined as $\max_{\alpha_v \neq 0}(\mathrm{wt}(v))$.

Reed-Muller code: $\mathrm{RM}(\mu, m)$ is generated by truth table of all Boolean functions in m variables of degree $\leq \mu$.

Example: RM(2,3)

$$\begin{pmatrix} 1 \\ x_1 \\ x_2 \\ x_3 \\ x_1x_2 \\ x_1x_3 \\ x_2x_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Asymmetric Quantum Codes From RM Codes

Properties of Reed-Muller codes

- The Reed-Muller code $\mathrm{RM}(r, m)$ of order r in m variables has length 2^m and parameters $[2^m, \sum_{j=0}^{r} \binom{m}{j}, 2^{m-r}]$.
- Fact: $\mathrm{RM}(r, m)^{\perp} = \mathrm{RM}(m - r - 1, m)$.
- Fact: if $r_1 \leq r_2$ then $\mathrm{RM}(r_1, m) \subseteq \mathrm{RM}(r_2, m)$.

Result (see also [Steane, IEEE-IT (1999)]):
There exist $[[2^m, \sum_{j=r_1+1}^{r_2} \binom{m}{j}, 2^{m-r_2}/2^{r_1+1}]]_2$ codes.

Examples:

r_1	r_2	Code $[[n, k, d_x/d_z]]_2$	Rate k/n	Asymmetry d_z/d_x
7	8	$[[1024, 45, 4/256]]_2$	0.044	64
6	8	$[[1024, 165, 4/128]]_2$	0.161	32
5	8	$[[1024, 375, 4/64]]_2$	0.366	16
6	7	$[[1024, 120, 8/128]]_2$	0.117	16

BCH Codes

Definition

- Given $\mathbb{F}_q$ and $n > 0$ such that $\gcd(n, q) = 1$. Let $r = \mathrm{ord}_n(q)$ and let α be a primitive nth root of unity. Then

$$H_{\delta,b} = \begin{bmatrix} 1 & \alpha^{b} & \alpha^{2b} & \cdots & \alpha^{b(n-1)} \\ 1 & \alpha^{b+1} & \alpha^{2(b+1)} & \cdots & \alpha^{(b+1)(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha^{(b+\delta-2)} & \alpha^{2(b+\delta-2)} & \cdots & \alpha^{(b+\delta-2)(n-1)} \end{bmatrix}.$$

  is a parity check matrix for a code $\mathrm{BCH}(\delta)$ with $d_{\min} \geq \delta$.

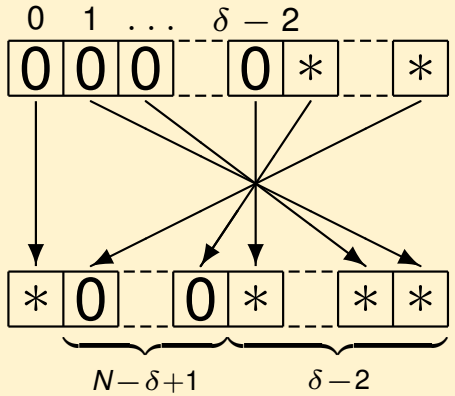
- Its defining set of zeros is $\mathcal{Z} = C_b \cup C_{b+1} \cup \cdots \cup C_{b+\delta-2}$, where $C_x = \{xq^i \bmod n \mid i \in \mathbb{Z}, i \geq 0\}$.
- Code is cyclic with generator polynomial $g(x)$ of the form

$$g(x) = \prod_{z \in \mathcal{Z}} (x - \alpha^z).$$

Asymmetric Quantum Codes From BCH Codes

Properties

- Dimension $k \geq n - (\delta - 1)\,\mathrm{ord}_n(q)$.
- Dual code:



- Nested codes: if $n = 2^m - 1$ and $\delta \leq \delta_{\max} = 2^{\lceil m/2 \rceil} - 1$ and δ odd, then $\mathrm{BCH}(\delta)^{\perp} \subseteq \mathrm{BCH}(\delta)$. [Steane, IEEE-IT (1999)], [Aly, Klappenecker, Sarvepalli, IEEE-IT (2007)]

Result

Let $m \geq 2$ and $2 \leq \delta_1 < \delta_2 < \delta_{\max}$, with δ_i odd. Then there exists an $[[2^m - 1, m(\delta_2 - \delta_1)/2, d_x/d_z]]_2$ asymmetric BCH stabilizer code, where $d_x \geq \delta_1$ and $d_z \geq \delta_{\max} + 1$.

Asymmetric BCH Quantum Codes

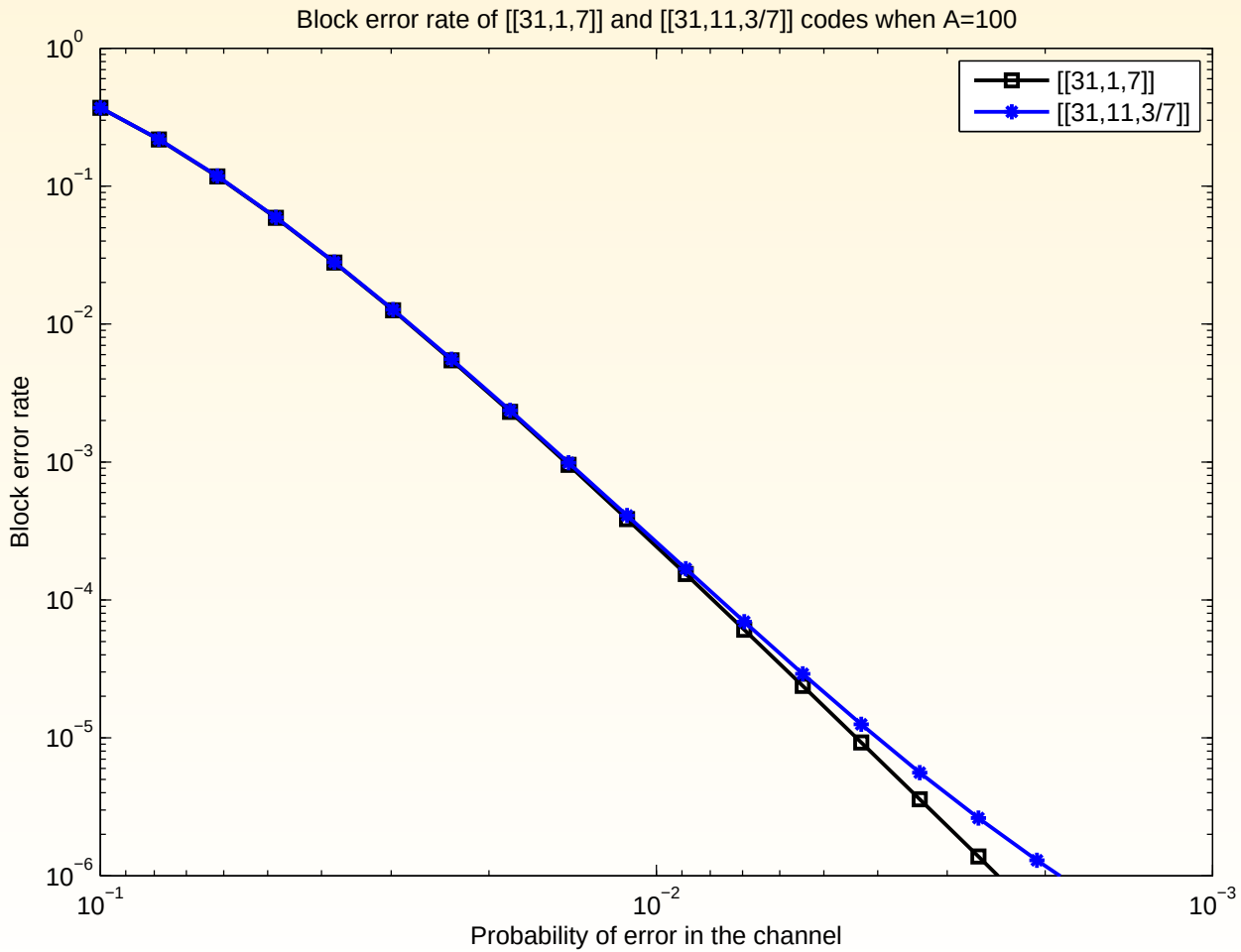
δ_1	δ_2	Code $[[n, k, d_x/d_z]]_2$	Rate k/n	Asymmetry d_z/d_x
29	31	$[[1023, 10, 29/32]]_2$	0.00978	≈ 1
17	31	$[[1023, 70, 17/32]]_2$	0.06843	
15	31	$[[1023, 80, 15/32]]_2$	0.07820	≈ 2
13	31	$[[1023, 90, 13/32]]_2$	0.08798	
11	31	$[[1023, 100, 11/32]]_2$	0.09775	≈ 3
9	31	$[[1023, 110, 9/32]]_2$	0.10753	
7	31	$[[1023, 120, 7/32]]_2$	0.11730	≈ 4
5	31	$[[1023, 130, 5/32]]_2$	0.12708	≈ 6
3	31	$[[1023, 140, 3/32]]_2$	0.13685	≈ 10
27	29	$[[1023, 10, 27/32]]_2$	0.00978	≈ 1
15	29	$[[1023, 60, 15/32]]_2$	0.05865	≈ 2
9	29	$[[1023, 100, 9/32]]_2$	0.09775	≈ 3
7	29	$[[1023, 110, 7/32]]_2$	0.10753	≈ 4
5	29	$[[1023, 120, 5/32]]_2$	0.11730	≈ 6
3	29	$[[1023, 130, 3/32]]_2$	0.12708	≈ 10

Performance simulations

Advantage I: Improvement of Data Rate

Property: Asymmetric codes can achieve a much higher data rate than symmetric codes with similar performance.

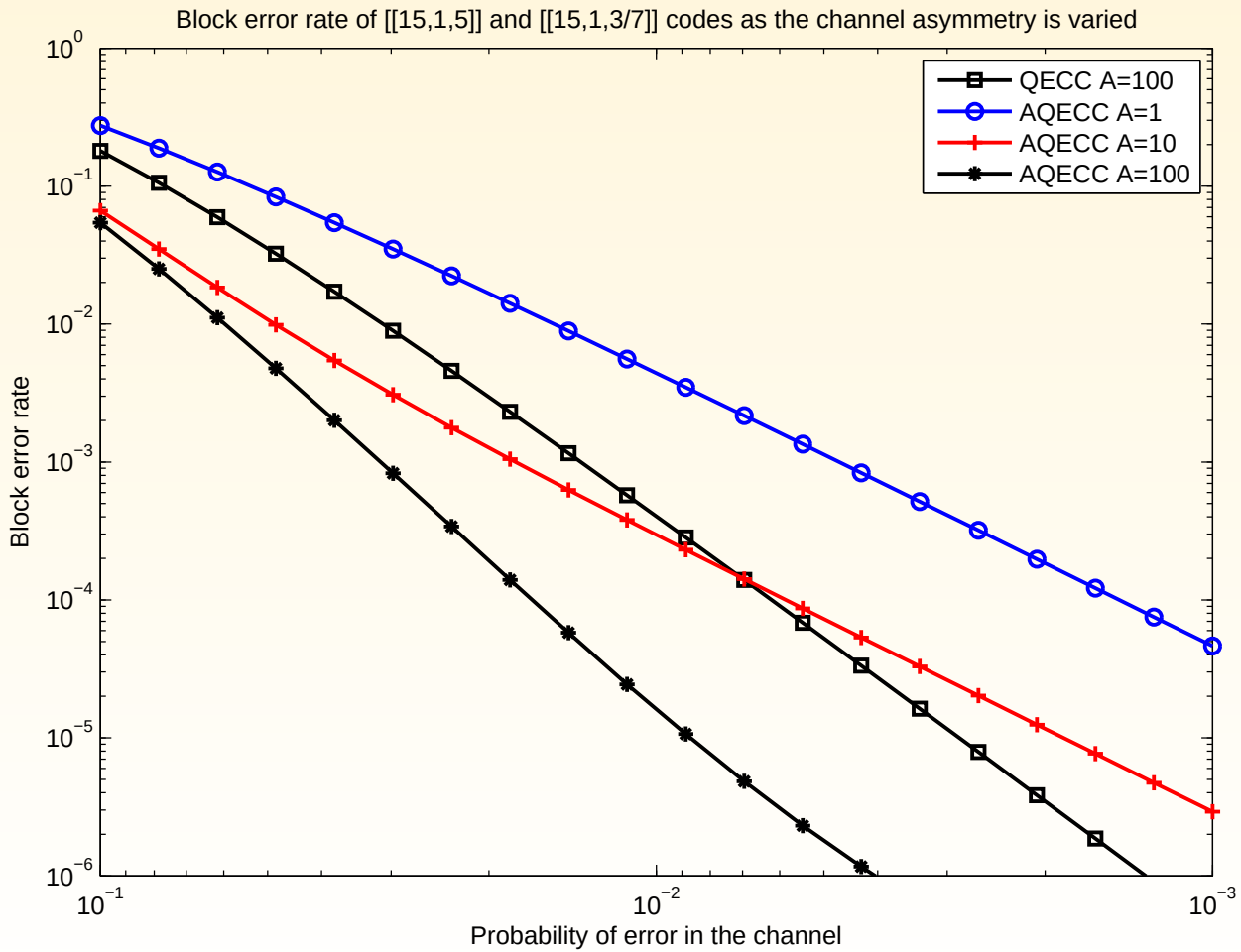
Example: Performances of $[[31, 1, 7]]$ and $[[31, 11, 3/7]]$.



Advantage II: Improvement of Performance

Property: Asymmetric codes perform better on asymmetric channels than symmetric codes with similar data rate.

Example: The performance of $[[15, 1, 3/7]]$.



Quantum LDPC Codes

Quantum LDPC Conundrum

Fact 1

LDPC codes do not perform well if Tanner graph has 4 cycles.

Fact 2

Tanner graph for stabilizer codes necessarily has 4 cycles.

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Tanner graph for stabilizer codes necessarily has 4 cycles.

Possible Ways Out Of This Dilemma

- Entanglement-enhanced codes
  $\longrightarrow$ See also QEC'07 talk by Todd Brun
- Subsystem codes
  $\longrightarrow$ [A. Klappenecker, MR, P. Sarvepalli] in preparation
- This talk takes yet another approach, the "blissfully ignorant" one. It is based on the fact that some LDPC codes can tolerate **enormous amounts** of 4 cycles without affecting performance.

Other approaches: [MacKey, Mitchison, McFadden '04], [Camera, Ollivier, Tillich '05], [Hagiwara, Imai '07].

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LDPC Codes 101

Parity check matrix

As an example take the $[7, 4, 3]$ Hamming code:

$$H = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

bit nodes $x_1, x_2, \ldots, x_7$

check nodes C_1, C_2, C_3.

Tanner graph

LDPC Codes 101

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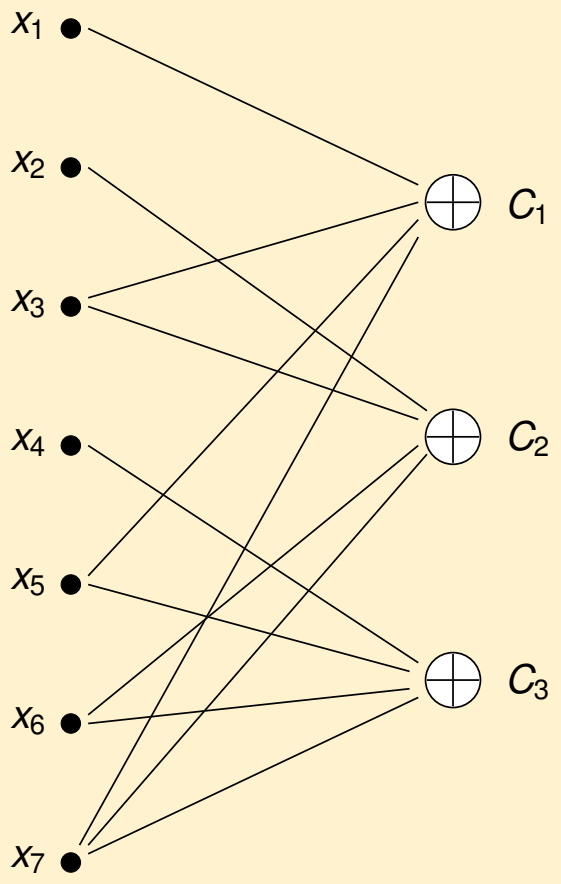
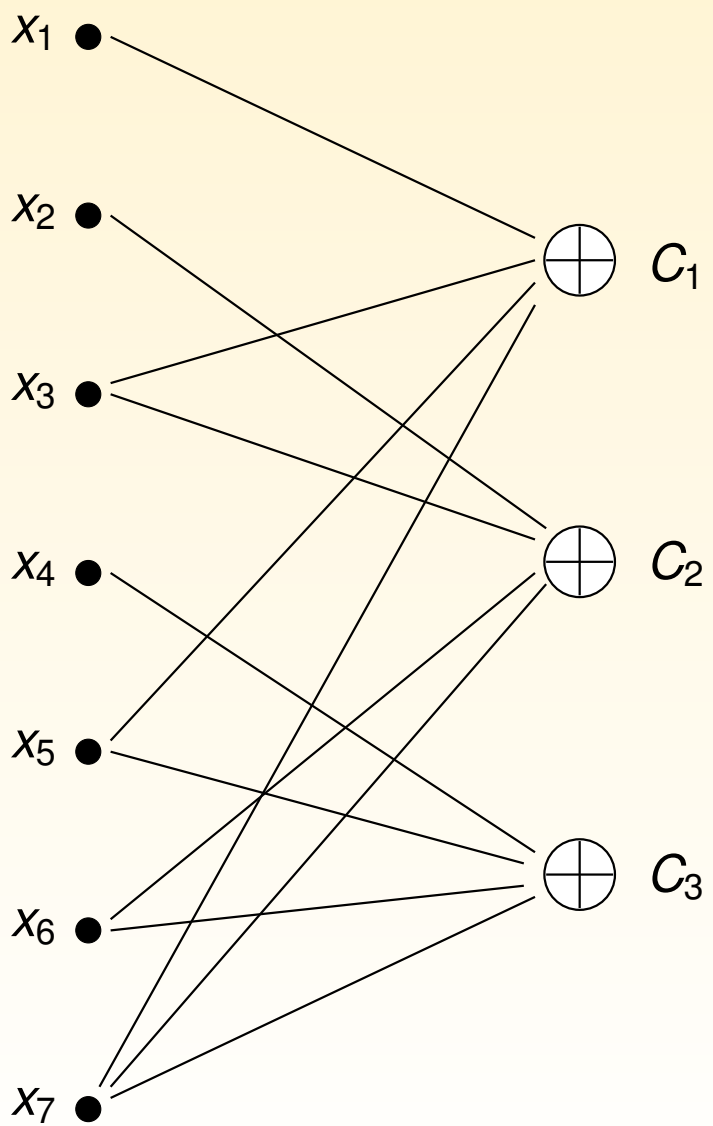
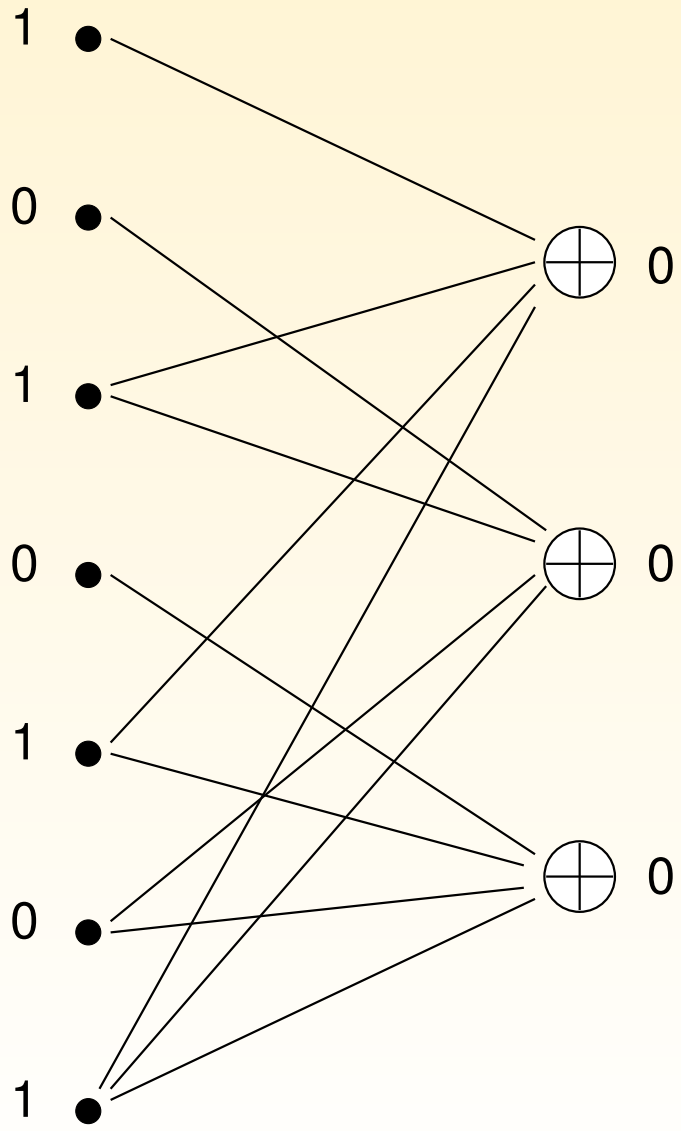


Iterative Decoding



Iterative Decoding – Example BF Algorithm



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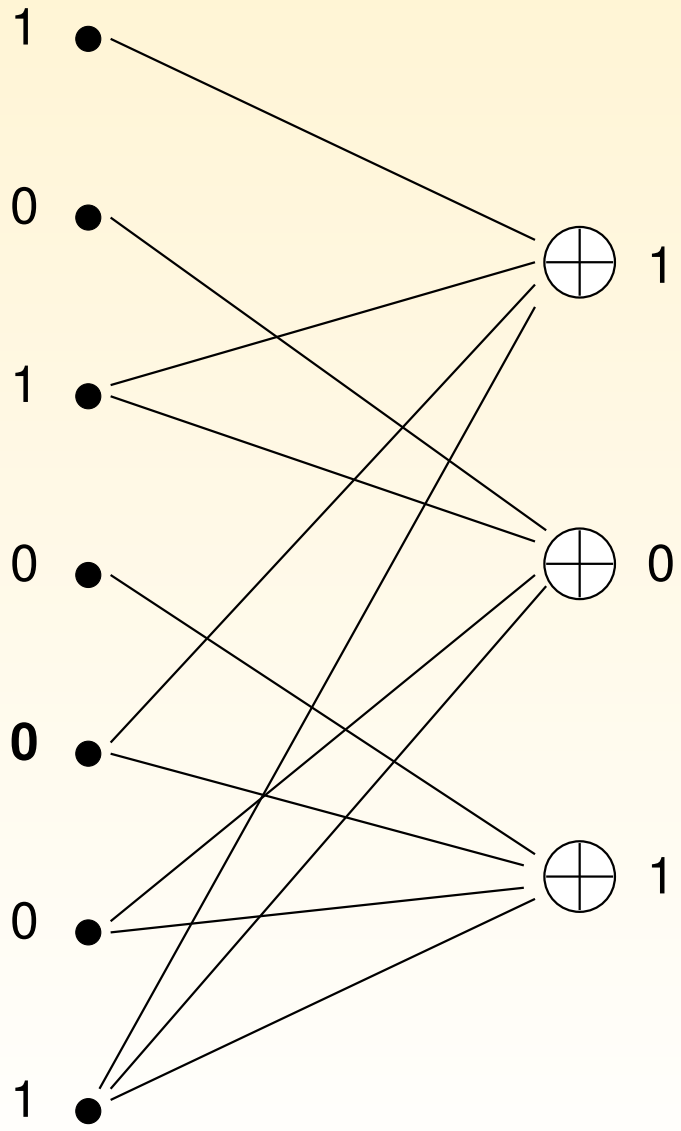

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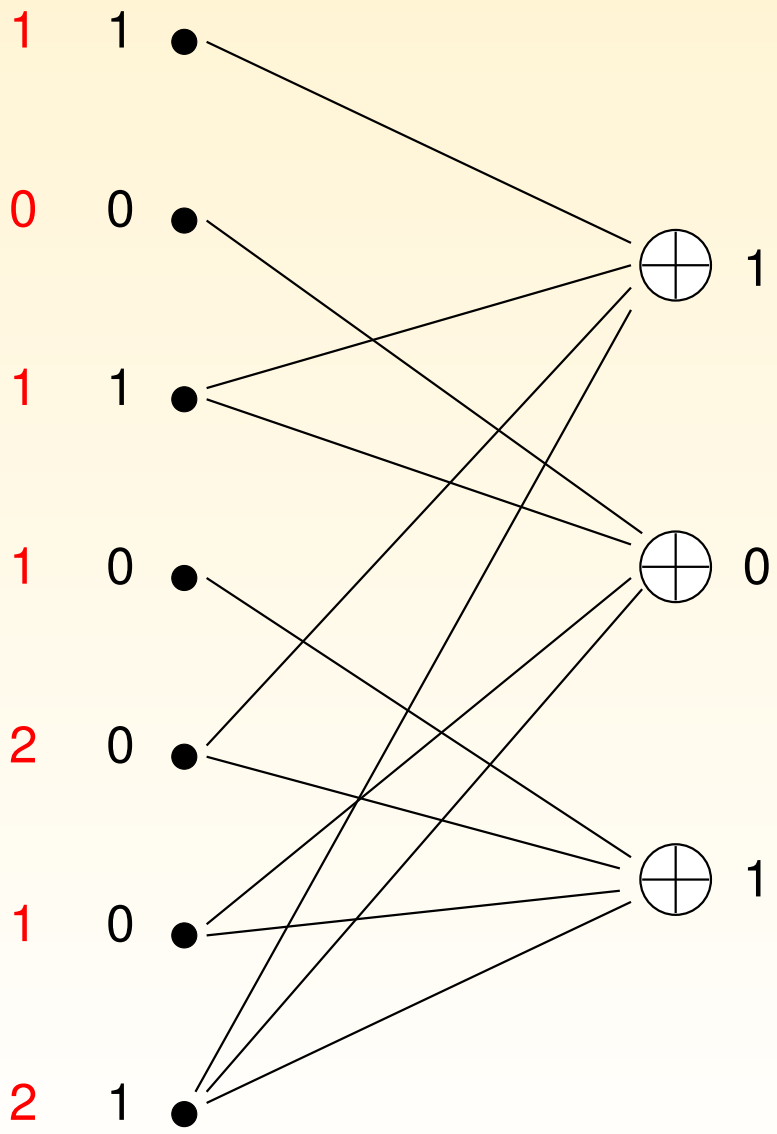

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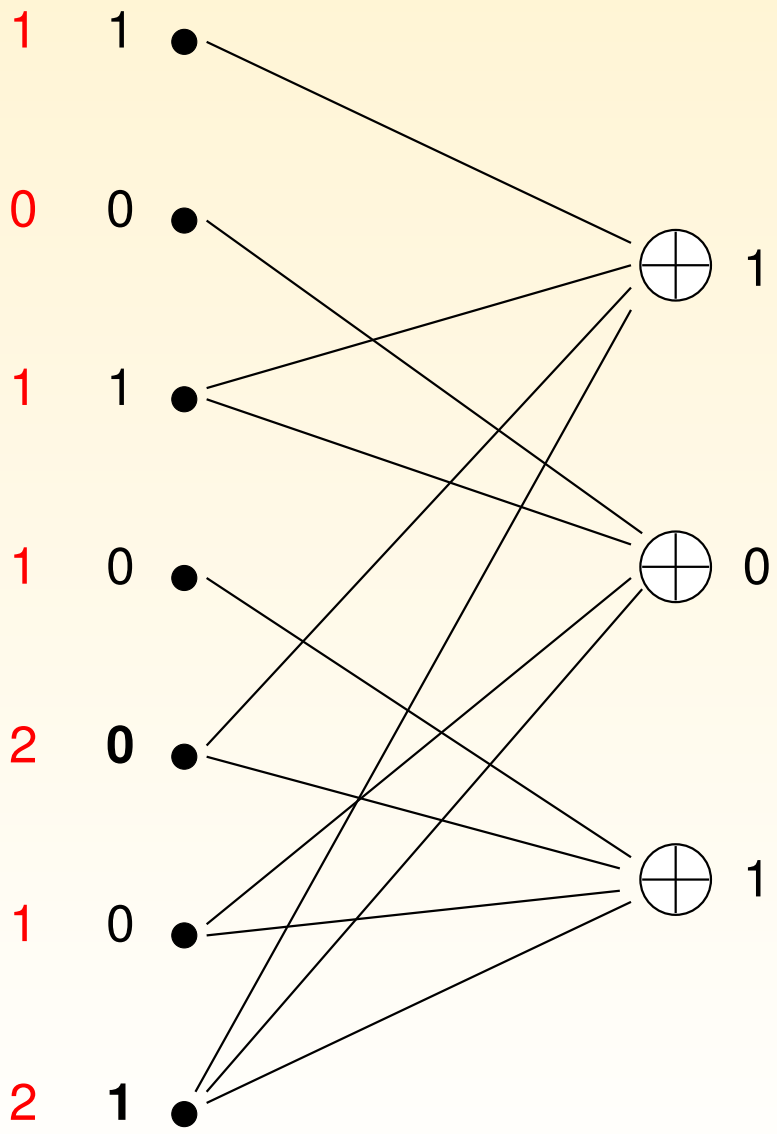

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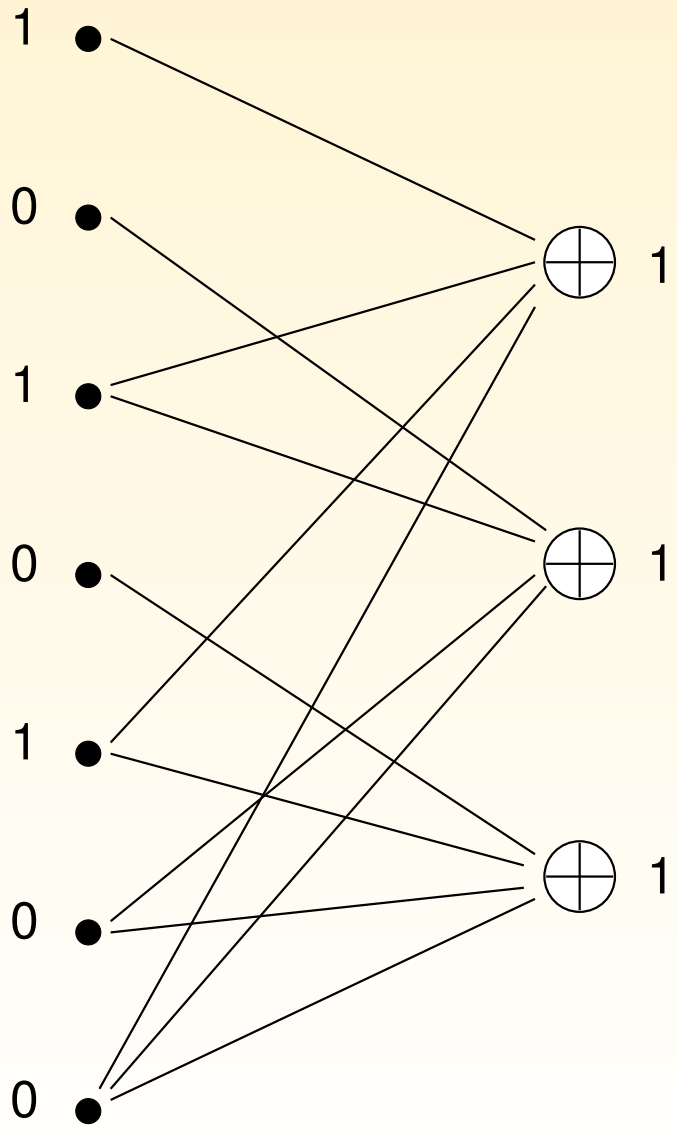

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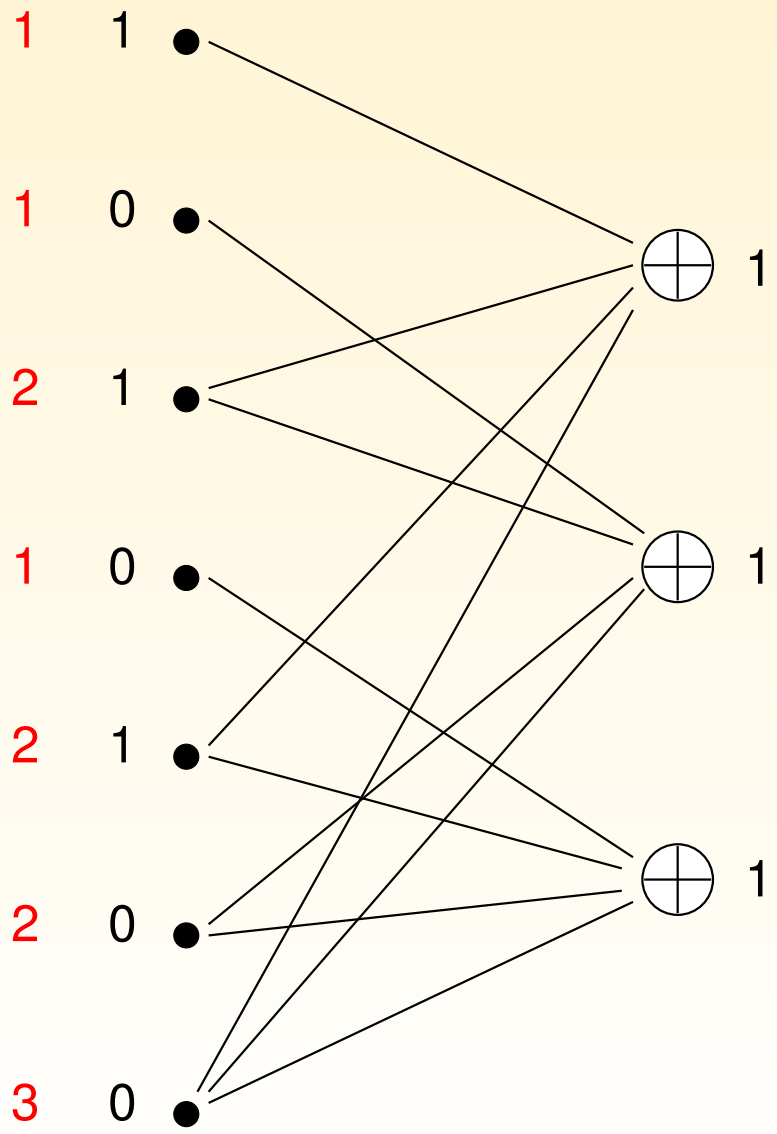

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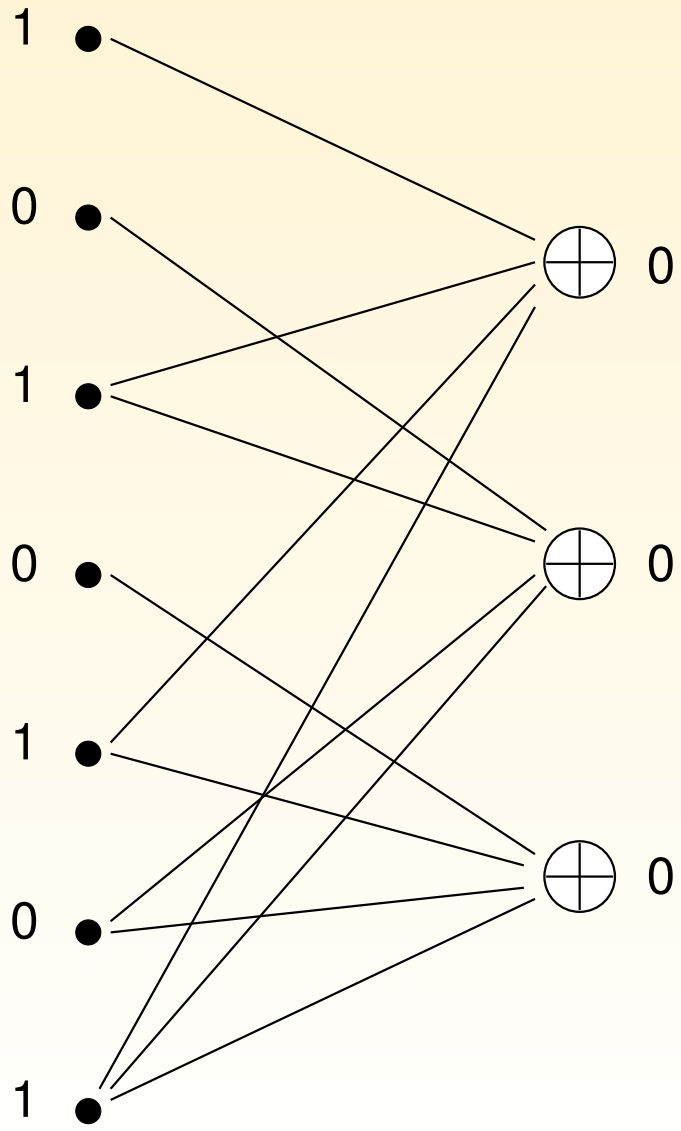
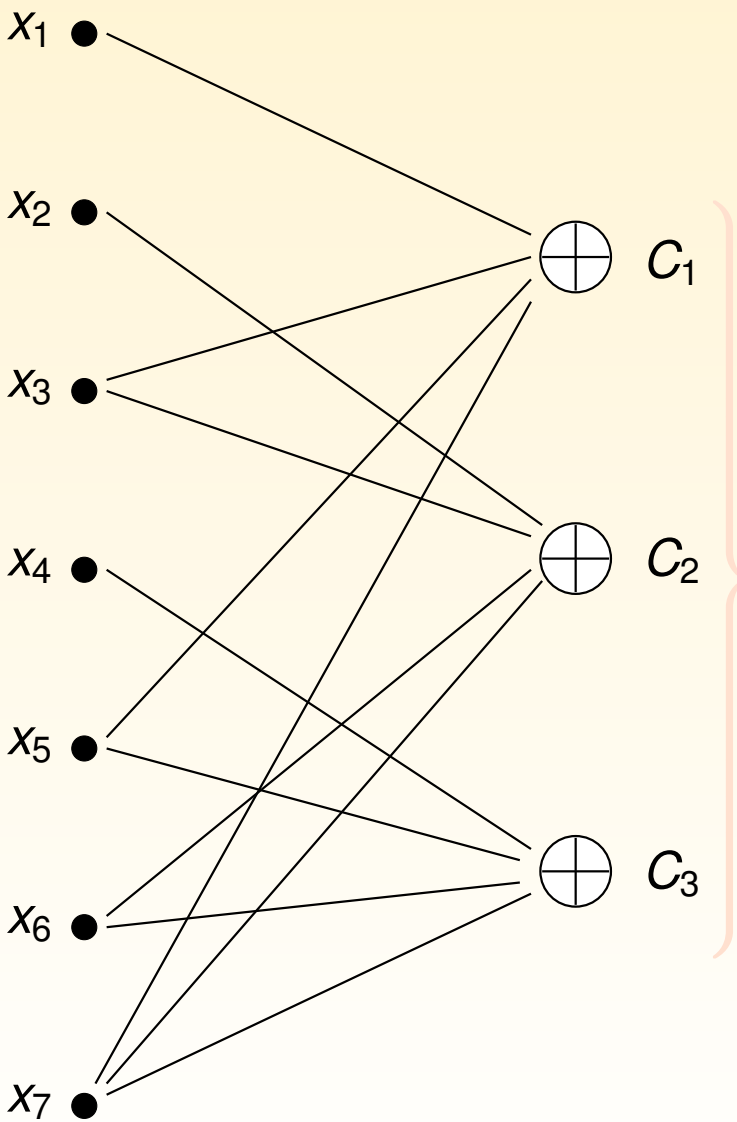


Iterative Decoding: The Quantum Case

All zeros goes here



Syndrome goes here

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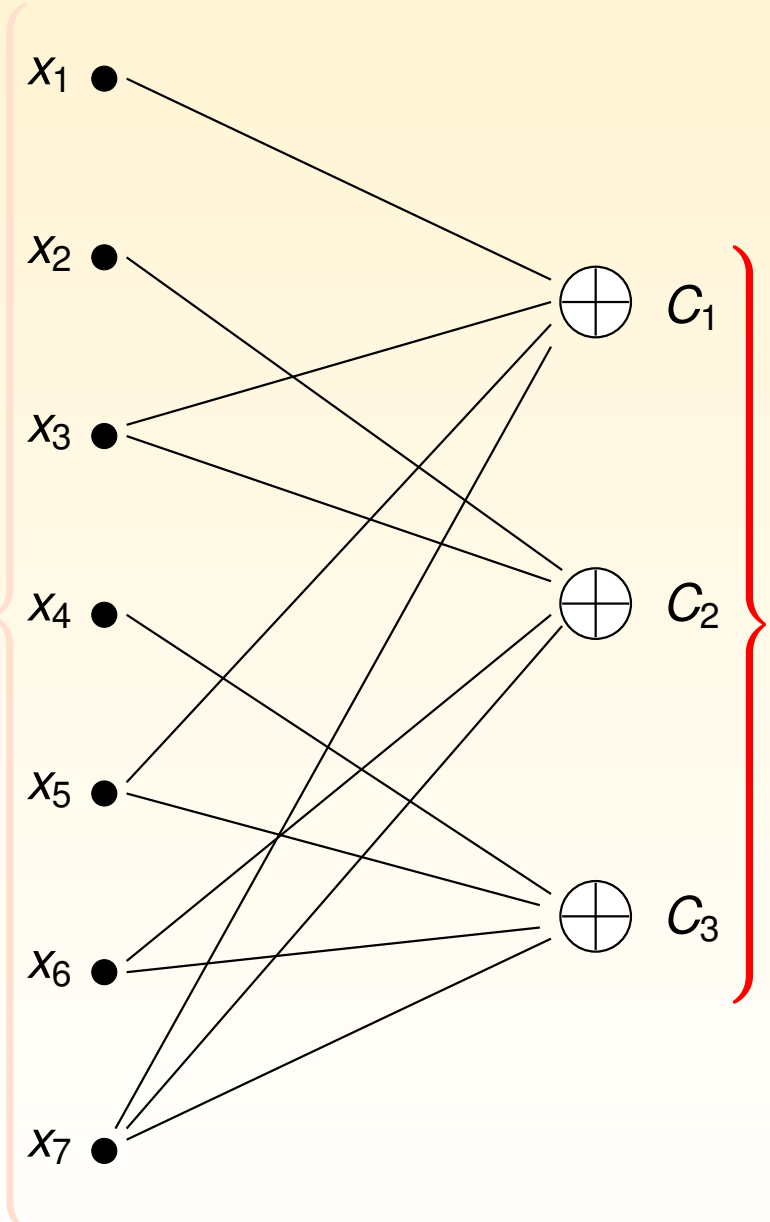

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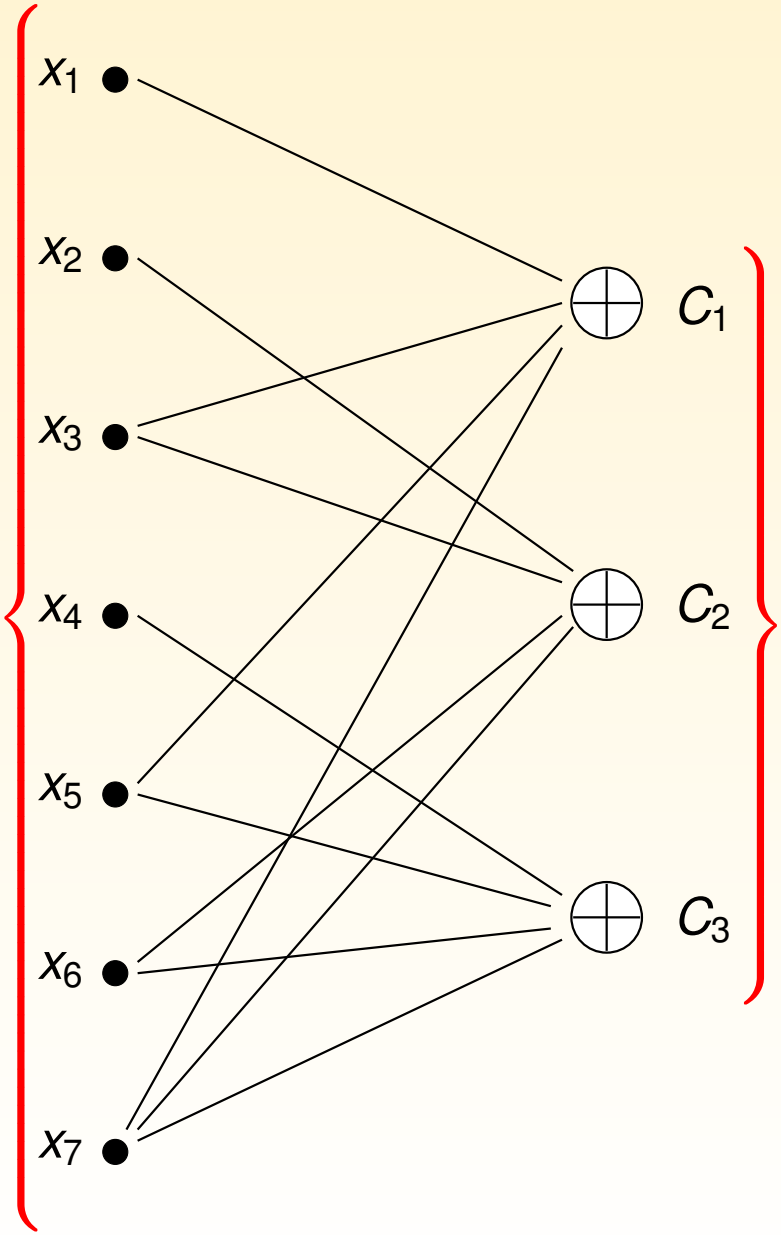


Syndrome goes here

Finite Geometry LDPC Codes

Finite Geometries – Axioms

Finite geometry is a collection of points and lines such that

- Every line has ρ points
- Every point is on γ lines
- Any two lines intersect only at one or no points
- Any two points are connected by only one line

Finite Geometries – Realizations

There are (at least) two geometries that satisfy these axioms.

- Euclidean geometry $\mathrm{EG}(m, p^s)$ corresponding to $\mathbb{F}_{p^s}^m$
- Projective geometry $\mathrm{PG}(m, p^s)$ corresponding to $\mathbb{F}_{p^s}^{(m+1)}$

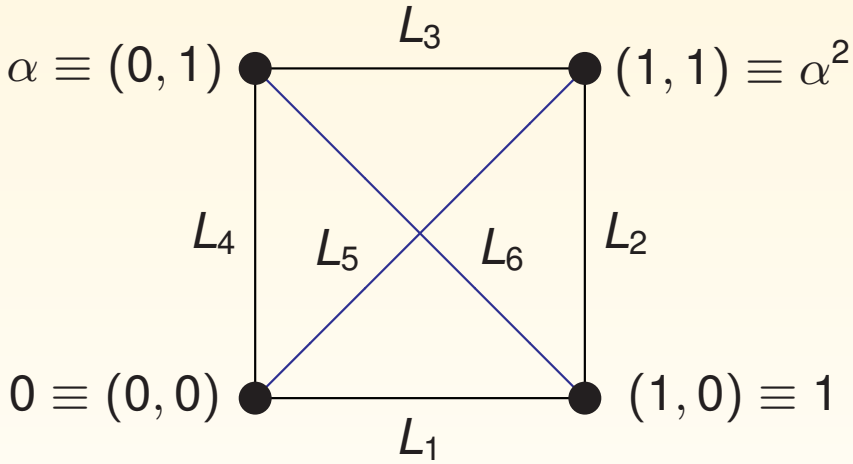
Finite Geometries – A Simple Example

- Identify $\mathbb{F}_2^2$ with the elements $\{0, 1, \alpha, \alpha^2\}$ of $\mathbb{F}_4$, i. e., $\{(0,0), (1,0), (0,1), (1,1)\}$
- Lines $= \{\{0,1\}, \{0,\alpha\}, \{0,\alpha^2\}, \{1,\alpha\}, \{1,\alpha^2\}, \{\alpha,\alpha^2\}\}$

Assign incidence vector to lines

For instance L_1 corresponds to

Points	0	1	α	α^2
Line	1	1	0	0

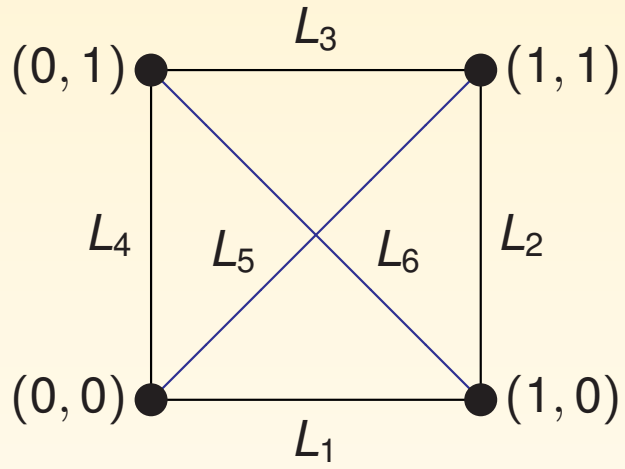


- More generally, this can be applied to $\mathbb{F}_{p^s}^m$ and leads to a finite Euclidean geometry $\mathrm{EG}(m, p^s)$.

Finite Geometry LDPC Codes

Now, list all the incidence vectors for all the lines:

$$H_{\mathrm{EG}}^{(1)} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$



- Gives the $[4,1,4]$ code
- Note the redundancy in the parity checks

Define $C_{\mathrm{EG}}^{(1)}(m,s,p)$ code as null space of the incidence vectors of the lines in $\mathrm{EG}(m,p^s)$.

Parameters of Finite Geometry LDPC Codes

- The associated parity check matrices have precisely ρ many 1s in each row and γ many 1s in each column
- Somewhat involved expressions are known for the dimension k of these codes
- Excluding the origin gives cyclic codes
- Including the origin gives extended cyclic codes

Cyclic

n	$p^{ms} - 1$
ρ	p^s
γ	$\frac{p^{ms}-1}{p^s-1} - 1$
d	$\geq \gamma + 1$
k	$\ldots$

Extended

n	p^{ms}
ρ	p^s
γ	$\frac{p^{ms}-1}{p^s-1}$
d	$\geq \gamma + 1$
k	$\ldots$

Generalizing This Simple Idea

Hyperplanes and general FG codes

- We consider now lines to be bit nodes and planes to be check nodes. Or more generally hyperplanes (flats) of different dimensions.
- Start with the Euclidean geometry $\mathrm{EG}(m, p^s)$. Consider all the flats of dimension μ_1 and μ_2, where $\mu_2 > \mu_1$.
- The incidence vector of a μ_2 flat is an n-tuple with ith component 1 if and only if ith μ_1-flat is contained in it
- The incidence vectors of all μ_2-flats gives the parity check matrix of a $C_{\mathrm{EG}}^{(1)}(m, \mu_2, \mu_1, s, p)$ code
- If $\mu_2 - \mu_1 > 1$, then these codes have many 4-cycles, yet they do not show any error floor.

Cyclic $C_{\mathrm{EG}}^{(1)}(3, 2, 0, 3, 2)$ code $[511, 448]$ has > 3.6 million 4-cycles, yet no error floors for BER around 10^{-6}

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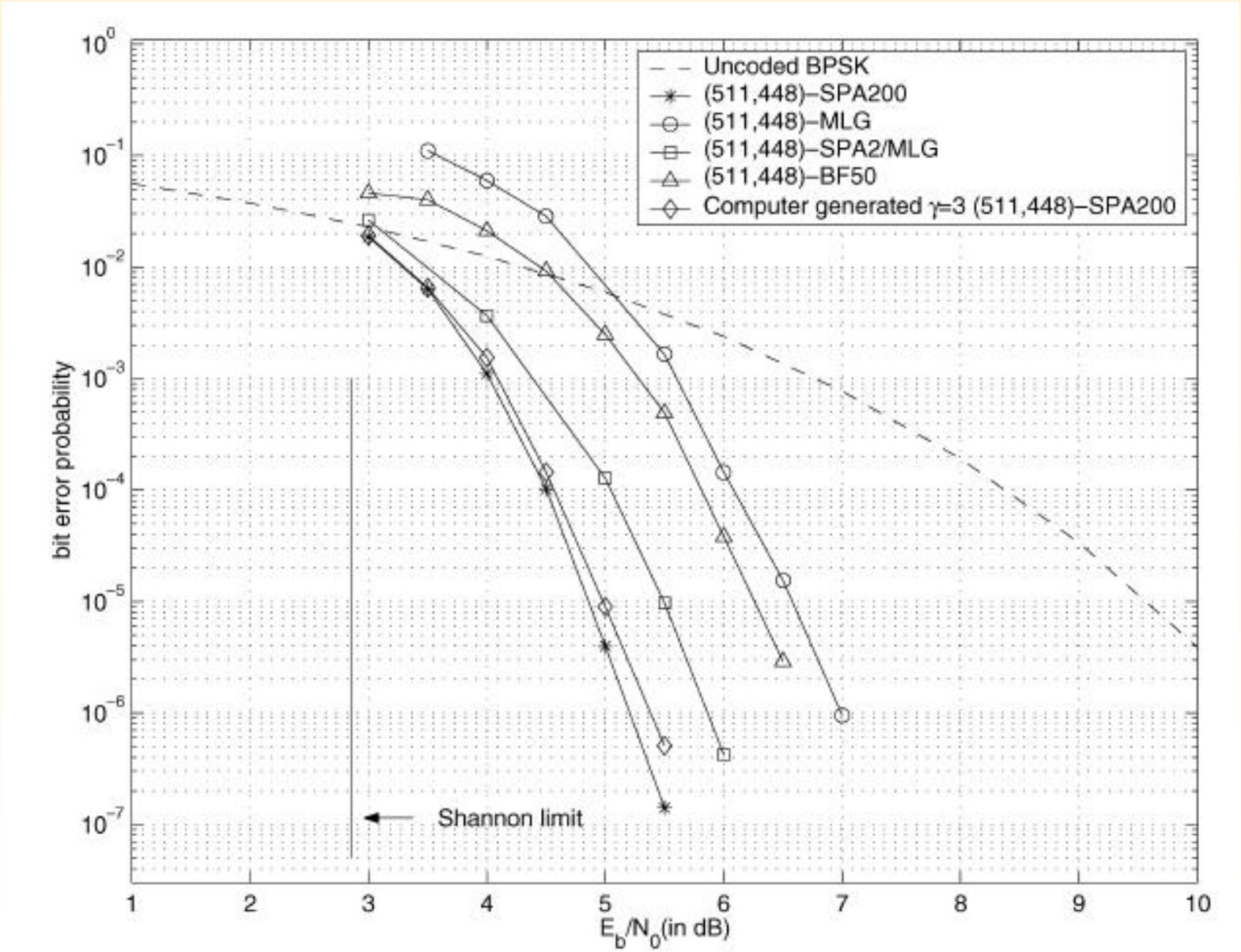
Classical Performance in Presence of Many 4 Cycles

Example: $C_{\mathrm{EG}}^{(1)}(3,2,0,3,2) = [511, 448]$



Finally: Quantum LDPC codes

Connection to Generalized Reed-Muller Codes

- We consider the special case $\mu_1 = 0$, *i.e.*, $C^{(1)}_{\mathrm{EG},c}(m, \mu, 0, s, p)$ codes. These are cyclic.
- Recall that the subfield subcode of $C \subseteq \mathbb{F}_{q^s}^n$ is given by $C|_{\mathbb{F}_q} = \{\text{All codewords of } C \text{ such that they are in } \mathbb{F}_q^n\}$.
- It turns out that the dual $C^{(1)}_{\mathrm{EG},c}(m, \mu_2, 0, s, p)^\perp$ is the subfield subcode of a generalized Reed-Muller code.
- Is is possible to embed this generalized Reed-Muller code into a BCH code.

Theorem: Let $C = C^{(1)}_{\mathrm{EG},c}(2, 1, 0, s, 2)$ and $\delta = 2t + 1 \leq 2^s - 1$. Then an $[[2^{2s} - 1, 2^{2s} - 3^s - s(\delta - 1), \delta/2^s + 1]]_2$ exists.

Compare with [Ioffe/Mézard]: they start with BCH code, pick low weight codewords to define sparse parity checks. We start with LDPC code and search for a BCH code containing its dual.

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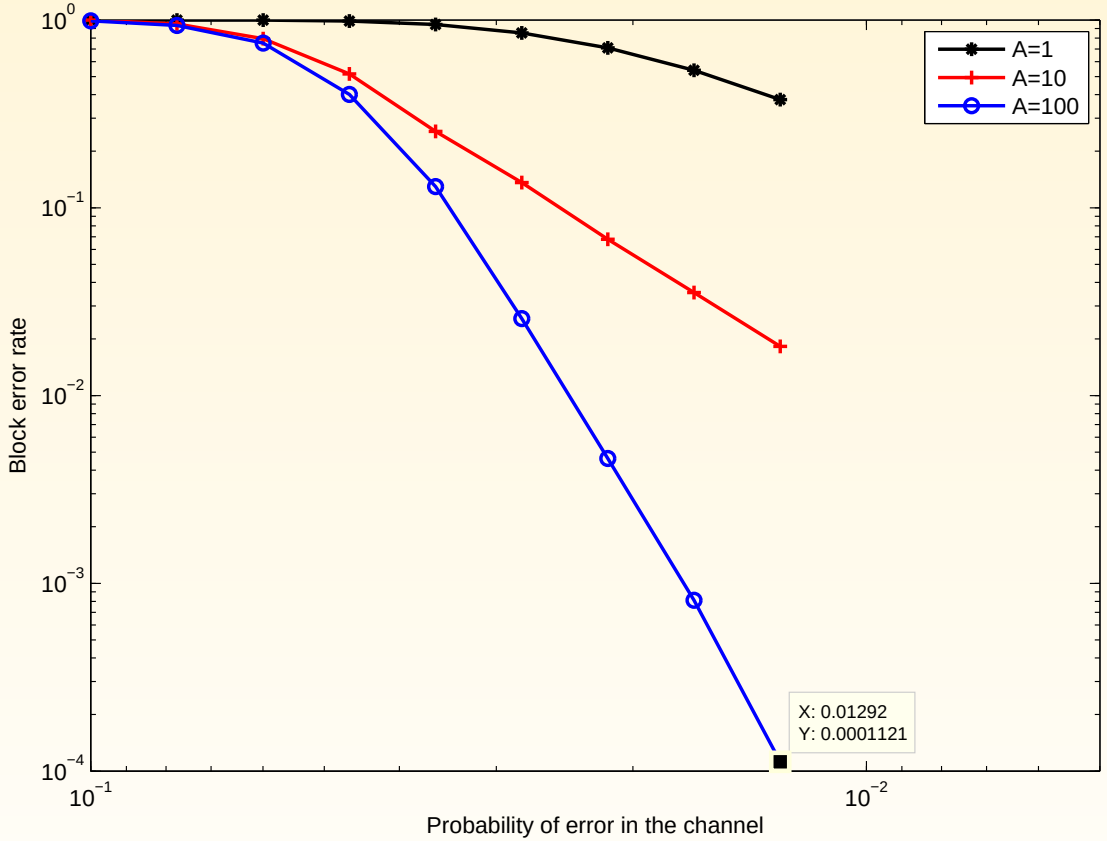
Quantum LDPC Codes: Performance

Example: Performance of a $[[255, 159, 5/17]]$ quantum LDPC code for $A = 1, 10, 100$. Maximum no. of iterations set to 50.



- Hard decision bit flipping algorithm [Gallager], [Kou et al]
- Many other ways of message passing: MLG, BF, weighted BF, APP, BP.

Summary

Wrap up:

- Asymmetric error models / biased noise
- Constructions of asymmetric codes from nested codes
- Construction of asymmetric quantum LDPC codes
- Performance simulations

Take home:

- The constructed codes are good for memory
- Achieve higher rate while similar performance
- Achieve better performance while similar rate
- Don’t know yet if they are also useful for FTQC

Comparison With Related Work

- [Rahn, Doherty, Mabuchi, PRA **66**, 032304 (2002)] studied performance of block codes such as [[5,1,3]] or [[7,1,3]] over arbitrary (not necessarily symmetric) Pauli channels.
- [Ioffe, Mézard, PRA **75**, 032345 (2007)] closest to our work regarding methods. Use LDPC codes with irregular degree profiles. Harder to analyze but performance seems ok.
- [Aliferis, Preskill, quant-ph/0710.1301] use concatenation of repetition code with any other code to get asymmetric codes for biased noise. Universal QC possible!
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